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The Finite Radon Transform

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abstract

In a previous study[1], the author has given several inversion formulas for the Radon transform on the Euclidean space and the Damek-Ricci space via the heat kernel type. In this paper, we continue the same approach by establishing the inversion theorems for the finite Radon transform and its finite dual Radon transform. We show that the Radon's operators R and its adjoint R^* may be inverted by the same kernel $G(y,x)$ and we study its basic properties. We construct the range-characterizing operator for the finite Radon transform.
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I. INTRODUCTION AND PRELIMINARIES

The Radon transform is defined in $\mathbb{R}^2$ by John Radon[17] and generalized in $\mathbb{R}^n$ by several authors particularly[4,10] and[14]. The Radon transform in Euclidean space $\mathbb{R}^n$ associates to a function f on $\mathbb{R}^n$ a function Rf on $\mathbb{P}^n$ ($\mathbb{P}^n$ denotes the space of all hyperplanes $H(t,\omega)$ in $\mathbb{R}^n$) by the formula

$$Rf(t,\omega) = \int_{H(t,\omega)} f(x)\, d\mu(x),$$

where $d\mu(x)$ is the Euclidean measure on the hyperplane $H(t,\omega)$.
In the case of the finite set, the analogue of this definition consists to make the average of function f over the subsets of a finite set X (see also the works[3,8,18,19]). In the sequel of this work, we adopt the definition of ref[18] for two reasons:
-The Strichartz's definition[18] is more natural and similar to the classical case. Indeed, in the study[18], the author has used the finite plane geometry to define the finite Radon transform.
-This definition can be extended (see[18]) to the finite k-plane transform R_k with $k \in \{2,3,...,card(X)\}$. More precisely, let X be a finite set of points and N be the cardinal of X ($N = card(X)$). Let Y be the set of lines of X, each line $y \in Y$ being a subset of X subject to the single axiom "two points determine a unique line", which is equivalent to:
(A) "For any two points $x_1, x_2 \in X$, there exists a unique $y \in Y$ such that $x_1 \in y$ and $x_2 \in y$."
We say that Y is simple if for all lines $y \in Y$, $card(y) = 2$. Y is not simple if there exists $y_0 \in Y$ such

that $card\,(y_0) > 2$. Let $l^2\,(X)$ (resp.$l^2\,(Y)$) be the space of all complex-valued functions on X (resp.on Y).
The finite Radon transform is defined[18] as the operator R on $l^2\,(X)$ to $l^2\,(Y)$ given by the formula

$$Rf\,(y) = \sum_{x\in y} f\,(x)\,, y\epsilon Y. \tag{1.1}$$

For each $x\epsilon X$, we denote by G_x the set of all $y\epsilon Y$ which contained the element x. We define also the scalar product on $l^2\,(X)$ by

$$\langle \phi, \psi\rangle = \sum_{a\in X} \phi\,(a)\,\psi\,(a)\,, \text{for all } \phi, \psi \in l^2\,(X)\,.$$

Then $l^2\,(X)$ becomes a Hilbert space. For any finite set A,write $card\,(A)$ for the cardinality of A and χ_A for the characteristic function of A. If A is reduced to a point $x \in X$, the function χ_A is just $\chi_{\{x\}}$. Let $\beta_{y_0}(y_0 \in Y)$be the function defined on Y by $\beta_{y_0}\,(y) = 0$ if $y \neq y_0$ and $\beta_{y_0}\,(y_0) = 1$. Then it follows from (1) that for any $A \subset X$, $R\chi_A(y) = card(A \cap F_y)$,where $F_y = \{x \in X\ /\ x \in y\}$. When $A = \{x\}$, we have $R\chi_x = \chi_{G_x}$.
Let $\mathcal{P}\,(X)$ be the set of all parts of X. For $x, x' \in X$, let $\eta\,(x, x')$ be the number of lines containing x and x'. From the axiom (A) above, it is clear that $\eta\,(x, x') = 1$ if $x \neq x'$, while $\eta\,(x, x) = \eta\,(x)$ the number of lines containing x. We suppose in the sequel that $\eta\,(x) > 1$ for all $x \in X$.
The paper is organized as follows.

In section 2, we study the properties of finite Radon transform R and also its dual transform R^*. We show that the axiom (A) implies the Bolker's condition (see[3], p.29). Consequently, $R : l^2\,(X) \longrightarrow l^2\,(Y)$ is injective. In the next paragraph, we identitify R with a rectangular matrix $(a_{ij})_{i,j}$, where $1 \leq i \leq N$, $1 \leq j \leq N'$ and $N' > N$, written in the natural coordinate systems $(\chi_x)_{x\in X}$, $(\beta_y)_{y\in Y}$ of $l^2\,(X)$ and $l^2\,(Y)$ respectively. When Y is simple, we have $N' = \frac{N(N-1)}{2}$.

In section 3, we calculate explicitly a kernel $G\,(y, x)$ ($y \in Y$ and $x \in X$). More precisely, let $G\,(.,.)$ be the kernel defined on $Y \times X$ by the formula

$$RG\,(y_0, .) = \beta_{y_0},\ \ (y_0 \in Y)\,, \tag{1.2}$$

with $RG\,(y_0, .)\,(y) = \sum_{x\in y} G\,(y_0, x)$ for all $y \in Y$. The solution of the equation (1.2) is given by

$$G\,(y_0, x) = \frac{1}{(\eta\,(x) - 1)}\left[\chi_{y_0}\,(x) - \frac{(card(y_0) - 1)}{(card\,(X) - 1)}\right],\ \ (x \in X) \tag{1.3}$$

(see Lemmas (12), (13), (14) and formula (20)).
We obtain from the duality formula and the equation (2) the following theorem.

Let F be an element of $l^2\,(Y)$, then the finite dual Radon transform R^* is inverted as follows:

$$F\,(y_0) = \sum_{x\in X} G\,(y_0, x)\,R^*F\,(x)\,,\ y_0 \in Y. \tag{1.4}$$

We obtain from (4) the analogue of Calderon's identity

$$\beta_{y_0} = \sum_{x\in X} G\,(y_0, x)\,\chi_{G_x}. \tag{1.5}$$

The operators R and R^* are inverted by the same remarkable kernel $G\,(y, x)$ (see formulas (30), (33)).
In the case of Euclidean space $\mathbb{R}^n$ and Damek -Ricci space[1], the author has established the inversion formulas (for the Radon transform and its dual) which are analogous to the formulas (4), (33). But the

corresponding kernels in these cases are not equal as for the finite case.
We finish this section by proving the following theorem :
Let f be an element of $l^2(X)$, then for all $x_0 \in X$ we have

$$f(x_0) = \sum_{y \in Y} G(y, x_0) Rf(y). \tag{1.6}$$

As precedently, we obtain the development of Dirac's " function" χ_x into plane waves :

$$\chi_x(.) = \sum_{y \ni x} G(y, .), \ x \in X. \tag{1.7}$$

The last formula is analogous to the Calderon's formula ([15],[16]).
For $x, x_0 \in X$, let $A(x, x_0) = \sum_{y \in Y} G(y, x) G(y, x_0)$ and define the operator $\mathbb{B}$ on $l^2(X)$ by the formula

$$\mathbb{B}f(x') = \sum_{x \in X} A(x, x') f(x), (x' \in X). \tag{1.8}$$

While combining the formulas (4) and (6), we obtain for all $x_0 \in X$

$$f(x_0) = \sum_{x \in X} \left(\sum_{y \in Y} G(y, x) G(y, x_0) \right) R^* Rf(x). \tag{1.9}$$

This equality can be written in the form

$$\mathbb{B}RR^* f = f \text{ for all } f \in l^2(X). \tag{1.10}$$

Using the theorems (16) and (22), we get the following formula

$$F(y_0) = \sum_{y \in Y} \left(\sum_{x \in X} G(y, x) G(y_0, x) \right) RR^* F(y), \text{ with } F \in l^2(Y). \tag{1.11}$$

Let $\mathbb{D}$ be the kernel defined on $Y \times Y$ by:$\mathbb{D}(y, y_0) = \sum_{x \in X} G(y, x) G(y_0, x)$, and Λ the operator on $l^2(Y)$ given by: $\Lambda F(y_0) = \sum_{y \in Y} \mathbb{D}(y, y_0) F(y)$, then the formula (11) can be written as follows

$$F = \Lambda RR^* F, \text{ for all } F \in l^2(Y), \tag{1.12}$$

Λ and $\mathbb{B}$ are the Calderon -Zygmund operators type (see[11],[18]). The inversion formulas (10) and (12) are analogous to the formulas obtained in the case of Euclidean space $\mathbb{R}^n$ and symmetric space of noncompact type (see[6],[10],[11]).

In the last section we characterize the image of finite Radon transform via an operator P (matrix) such that $Im\, R = \ker P$.

A. Motivation

In the literature, there are several methods for inverting the Radon transform, particularly Gelfand's and Helgason's methods.
The fundamental goal of this paper is to describe a new method for inverting the finite Radon transform.This method is based on the resolution of certain functional equations. More precisely, we seek a kernel $G(y, x)$ on $Y \times X$ such that

$$(M_1) \qquad R^* G(., x_0) = \chi_{x_0}, \ x_0 \in X$$
$$(M_2) \qquad RG(y_0, .) = \beta_{y_0}, \ y_0 \in Y.$$

The solution of (M_1) (resp. of (M_2)) gives an inversion formula for R (resp. for R^*).The operator R and its dual R^* are inverted by the same kernel $G(y, x)$. For the case of the Euclidean space, the author[1] has used the same method to calculate the kernels of the equations corresponding to (M_1) and (M_2). But these kernels are different.
The finite Radon transform theory has many applications in chemestry and physics.

II. PROPERTIES OF OPERATORS R AND R^*

Definition 1 *The finite dual Radon transform of R is the operator R^* of $l^2(Y)$ into $l^2(X)$ given by the formula*

$$R^*F(x) = \sum_{y \ni x} F(y), \text{ for all } F \in l^2(Y) \text{ and } x \in X. \tag{2.1}$$

As in the classical case, we have the duality formula

$$\sum_{y \in Y} Rf(y)\,F(y) = \sum_{x \in X} R^*F(x)\ f(x). \tag{2.2}$$

The proof of this equality is exactly as in Euclidean case.

Lemma 2 *Let $f \in X$, then for all $x \in X$ we have*

$$f(x) = \frac{1}{(\eta(x)-1)}\left[R^*Rf(x) - \sum_{x' \in X} f(x')\right] \tag{2.3}$$

Proof. From definition of R^*,we have

$$R^*Rf(x) = \sum_{y \ni x} Rf(y).$$

Replacing R by its expression (see formula (1.1)), we obtain

$$\begin{aligned} R^*Rf(x) &= \sum_{y \ni x'}\left(\sum_{x' \in y} f(x')\right) \\ &= \eta(x)\,f(x) + \sum_{\substack{x' \in X \\ x' \neq x}} f(x'). \end{aligned}$$

Thus $R^*Rf(x) = \sum_{x' \in X} f(x')\,\eta(x,x')$. Since $\eta(x,x') = 1$ if $x \neq x'$,the above equation becomes

$$\begin{aligned} R^*Rf(x) &= \eta(x)\,f(x) - f(x) + \sum_{x' \in X} f(x') \\ &= (\eta(x)-1)f(x) + \sum_{x' \in X} f(x'). \end{aligned}$$

Whence

$$f(x) = \frac{1}{(\eta(x)-1)}\left[R^*Rf(x) - \sum_{x' \in X} f(x')\right].$$

Remark 3 *Let ϕ be an element of $l^2(X)$, we can write ϕ in the following form*

$$\phi = \sum_{x \in X} \phi(x)\,\chi_x$$

Thus $R\phi = \sum_{x \in X} \phi(x)\,R(\chi_x)$. Replacing $R(\chi_x)$ by χ_{G_x} in the above equality (see introduction), we obtain

$$R\phi = \sum_{x \in X} \phi(x)\,\chi_{G_x}. \tag{2.4}$$

The family $\chi_{x_1}, \chi_{x_2}, ..., \chi_{x_N}$ is the canonical base of the N-dimentional vector space $l^2(X)$ and also the family $\beta_{y_1}, \beta_{y_2}, ..., \beta_{y_{N'}}$ is the canonical base of the N'-dimentional vector space $l^2(Y)$. If Y is simple, $N' = \frac{N(N-1)}{2}$. Let $\beta(y,\phi)$ be the expression defined by

$$\beta(y,\phi) = \sum_{x \in X} \phi(x) \chi_{G_x}(y).$$

If we write χ_{G_x} with respect to the base $\beta_{y_1}, \beta_{y_2}, ..., \beta_{y_{N'}}$, we have

$$\chi_{G_x} = \sum_{y \in Y} \beta_y . \chi_{G_x}(y).$$

Consequently the function $R\phi$ can be written in the form

$$R\phi = \sum_{y \in Y} \beta(y,\phi) \chi_y.$$

Hence $R : l^2(X) \longrightarrow l^2(Y)$ is a rectangular matrix $(a_{ij})_{\substack{1 \le i \le N \\ 1 \le j \le N'}}$, with $a_{ij} \in \{0,1\}$

In the next paragraph, we give the expression of the matrix R when Y is simple. In order to invert the finite Radon transform, we first study its injectivity. We begin by proving that the axiom (A) implies the Bolker's condition.

■

Proposition 4 *Let x and x' be two distinct elements of X, then $card(G_x \cap G_{x'}) = 1$.*

Throughout this paper, we denote by $l(x_1,x_2)$ the line of Y which contained x_1 and x_2 with $x_1 \neq x_2$. Recall that $l(x_1,x_2)$ comes down to $\{x_1,x_2\}$ if Y is simple, and $card(l(x_1,x_2)) > 2$ if Y is not simple.

For $X = \{a,b,c,d\}$ we shall denote by (D) (resp.(D')) the hypergraphe (resp.the graphe) of X

Proof of the above proposition:

Let $y \in G_x \cap G_{x'}$, then $y \ni x$ and $y \ni x'$. From the axiom (A) $l(x,x') = y$. It is clear that $y = \{x,x'\}$ if Y is simple.

Proposition 5 *The following two statements are equivalent*
(a) Y is simple
(b)For any $x \in X$, $card(G_x) = \eta(x) = card(X) - 1$, and for all $x, x' \in X$ $(x \neq x')$ $card(G_x \cap G_{x'}) = 1$.

Proof. Show that (a) implies (b). Let $x \in X$, then there exists $j_0 \in \{1,2,....,N\}$ such that $x = x_{j_0}$. The set $G_{x_{j_0}}$ is formed by the lines $\{x_{j_0},x_1\}$, $\{x_{j_0},x_2\}, \ldots, \{x_{j_0},x_{j_0-1}\}$, $\{x_{j_0},x_{j_0+1}\}, \ldots, \{x_{j_0},x_{N-1}\}, \{x_{j_0},x_N\}$. Then $card(G_{x_{j_0}}) = \eta(x_{j_0}) = card(X) - 1$. The rest of the assertion (b) is already proved in proposition 4.
Suppose that Y is not simple ,then there exists $l(x,x') \in Y$ for which $card(l(x,x')) > 2$,let be $x_1 \in l(x,x')$ with $x_1 \neq x$ and $x_1 \neq x'$.It is clear that $\eta(x_1) \neq \eta(x_j)$ or $\eta(x') \neq \eta(x_j)$ for a certain j for which $x_j \notin l(x,x')$ .Then Y is simple. ■

Remark 6 *If Y is simple then $card(G_x \cap G_{x'}) = 1$ and the function $x \to \eta(x)$ is constant on X. Consequently, Y is simple implies the Bolker's condition ([3],page 29).*

Theorem 7 *The operator R of $l^2(X)$ into $l^2(Y)$ is injective.*

We suppose that Y is simple. When Y is not simple we use the same techniques as the demonstrations in the case where Y is simple.

Proof. Let ϕ be a function of $l^2(X)$,show that if $R\phi = 0$ then $\phi = 0$. Suppose that $R\phi = 0$, from (2.4) we have

$$R\phi = \sum_{x \in X} \phi(x) \chi_{G_x}. \tag{2.5}$$

Multiplying the two members of this equality by $\chi_{G_{x_1}}$, we obtain

$$\begin{aligned} R\phi.\chi_{G_{x_1}} &= \sum_{x \in X} \phi(x) \chi_{G_{x_1}} \chi_{Gx} \qquad (2.6)\\ &= \sum_{x \in X} \phi(x) \chi_{G_{x_1} \cap G_x} \\ &= \phi(x_1) \chi_{G_{x_1}} + \phi(x_2) \beta_{l(x_1,x_2)} + \cdots + \phi(x_N) \beta_{l(x_1,x_N)}, \end{aligned}$$

where $N = card(X)$ and $X = \{x_1, x_2, ..., x_N\}$(see introduction). The last equality is justified by the proposition 4. Let $y = \{x_1, x_2\}$ be a line of Y , the equality
(2.6) implies
$(R\phi.\chi_{G_{x_1}})(y) = (\phi(x_1) \chi_{G_{x_1}} + \phi(x_2) \beta_{l(x_1,x_2)} + \cdots + \phi(x_N) \beta_{l(x_2,x_N)})(y) = 0,$
since $R\phi = 0$, we have $\phi(x_1) + \phi(x_2) = 0$. Multiplying also the equality (2.5) by $\chi_{G_{x_2}}$,we have

$$\begin{aligned} \chi_{G_{x_2}}.R\phi &= \sum_{x \in X} \phi(x) \chi_{G_x \cap G_{x_2}} \\ &= \phi(x_2) \chi_{G_{x_2}} + \phi(x_1) \beta_{l(x_1,x_2)} + \cdots + \phi(x_N) \beta_{l(x_2,x_N)}. \end{aligned}$$

As before, at the line $y = \{x_2, x_3\} = l(x_2, x_3)$ the above equality gives

$$\begin{aligned} (\chi_{G_{x_2}}.R\phi)(y) &= (\phi(x_2) \chi_{G_{x_2}} + \phi(x_1) \beta_{l(x_1,x_2)} + \cdots + \phi(x_N) \beta_{l(x_2,x_N)})(y) \\ &= \phi(x_2) + \phi(x_3), \end{aligned}$$

since $R\phi = 0$,we have $\phi(x_2) + \phi(x_3) = 0$.
Gradually we obtain the following system

$$(S_1) \begin{cases} \phi(x_1) + \phi(x_2) = 0 \\ \phi(x_2) + \phi(x_3) = 0 \\ \phi(x_3) + \phi(x_4) = 0 \\ \vdots \\ \phi(x_{N-2}) + \phi(x_{N-1}) = 0 \\ \phi(x_{N-1}) + \phi(x_N) = 0 \end{cases}$$

Now, we return to the equality (2.6). At the line $y = \{x_1, x_3\} = l(x_1, x_3)$,
this equality implies that $\phi(x_1) + \phi(x_3) = 0$. For $y = \{x_1, x_4\}$, we obtain also in the same way $\phi(x_1) + \phi(x_4) = 0$. As precedentley, we get the following system

$$(S_2) \begin{cases} \phi(x_1) + \phi(x_3) = 0 \\ \phi(x_1) + \phi(x_4) = 0 \\ \phi(x_1) + \phi(x_5) = 0 \\ \vdots \\ \phi(x_1) + \phi(x_{N-1}) = 0 \\ \phi(x_1) + \phi(x_N) = 0 \end{cases}$$

The systems (S_1) and (S_2) imply $\phi(x_1) = \phi(x_2) = \phi(x_3) = \cdots = \phi(x_{N-1}) = \phi(x_N) = 0$,then $\phi = 0$. Consequently R is injective.And this completes the proof of theorem 7. ■

Remark 8 *Rang* $(R) = dim(l^2(X)) = N$.

Proposition 9 *Let* K_0 *be a subset of* X *and* f *a complex-valued function defined on* X. *Then*
$(*)$ *supp*$f \subset K_0 \Rightarrow Rf(y) = 0$ *for all* $y \in Y$ *such that* $y \cap K_0 = \emptyset$.
If Y *is simple and* $card(K_0) = 1$, *with* $card(X) \geq 4$, *then the above implication becomes an equivalence.*

Proof. Let f be an element of $l^2(X)$ such that supp$f \subset K_0$. Let $y \in Y$ for which $y \cap K_0 = \emptyset$ .We will prove that $Rf(y) = 0$. Since supp$(f) \subset K_0$,we have

$$|Rf(y)| \leq \sum_{x \in y \cap K_0} |f(x)|,$$

as $y \cap K_0 = \emptyset$,we have $\sum_{x \in y \cap K_0} |f(x)| = 0$. Hence $Rf(y) = 0$.
We suppose now that Y is simple and $card(K_0) = 1$, with $card(X) \geq 4$,and showing that if $Rf(y) = 0$ for all $y \in Y$ such that $y \cap K_0 = \emptyset$, then supp$f \subset K_0$. Suppose that $K_0 = \{x_1\}$ and let A be the set of all $y \in Y$ for which $y \cap K_0 = \emptyset$. Since Y is simple, A is constitued by the elements $\{x_2, x_3\}, \{x_2, x_4\}, \ldots, \{x_2, x_N\}, \{x_3, x_4\}, \ldots, \{x_3, x_N\}, \ldots, \{x_{N-1}, x_N\}$. It is clear that $card(A) = \dfrac{(card(X) - 1)(card(X) - 2)}{2}$. In vertue of the condition $Rf(y) = 0$ for all $y \in Y$ such that $y \cap K_0 = \emptyset$, we obtain the following systems

$$(S_1)\begin{cases} f(x_2) + f(x_3) = 0 \\ f(x_2) + f(x_4) = 0 \\ \vdots \\ f(x_2) + f(x_N) = 0 \end{cases} \quad \text{and} \quad (S_2)\begin{cases} f(x_3) + f(x_4) = 0 \\ f(x_3) + f(x_5) = 0 \\ \vdots \\ f(x_3) + f(x_N) = 0 \end{cases}$$

It follows that the systems (S_1) and (S_2) imply supp$f \subset K_0$. This finish the proof of proposition. ■

Let us now turn our attention to the converse of the implication $(*)$. The answer is given by the following remark.

Remark 10 *The converse of the implication* $(*)$ *is false, indeed if* $X = \{a, b, c\}$ *and* $K_0 = \{a\}$ *,with* Y *is simple, then the lines* $y \in Y$ *such that* $y \cap \{a\} = \emptyset$ *are exactly the line* $y = \{b, c\}$. *Taking now* ϕ *a function of* $l^2(X)$ *such that* $R\phi(y) = \phi(b) + \phi(c) = 0$ *and* $\phi(b) \neq 0$, $\phi(c) \neq 0$. *It follows that supp*$\phi \nsubseteq \{a\}$.

III. INVERSION FORMULAS FOR THE OPERATORS R AND R^*

In this section,we seek a kernel $G(x, y)$ which is solution of this equation

$$RG(y_0, .) = \beta_{y_0} \quad (y_0 \in Y). \tag{3.1}$$

We shall now calculate explicitly the solution $G(y, x)$ of the functional equation (3.1), via which we establish the inversion formulas for R and its dual R^*.
We state first the theorem which gives the expression of the kernel $G(y, x)$.

Theorem 11 *The solution of the equation (*3.1*) has for expression:*

$$G(y_0, x) = \frac{1}{(\eta(x) - 1)} \left[\chi_{y_0}(x) - \frac{(card(y_0) - 1)}{(card(X) - 1)}\right]. \tag{3.2}$$

Before proving this theorem, we shall need several technical lemmas.

Lemma 12 *Let x be an element of X, then*

$$\sum_{y\ni x}(card\,(y)-1)=card\,(X)-1 \tag{3.3}$$

Proof. If $X=\{a,b,c\}$ and $x=a$,then the elements $y\in\mathcal{P}(X)$ ($\mathcal{P}(X)$ is the set of parts of X) which contained x are $l(a,b)$, $l(a,c)$ and in this case we have

$$\begin{aligned}\sum_{y\ni x}(card\,(y)-1)&=\sum_{y\in Y}\chi_{G_x}(y)\,(card\,(y)-1)\\&=card\,(X)-1\\&=2\end{aligned}$$

Suppose that the formula (3.3) is true for all sets of cardinal inferior or equal to $N-1$ and show that (3.3) holds for a set X of cardinal equal to N. Let x be an element of $X=\{a_1,a_2,\cdots,a_N\}$ and let $G_x^{N-1}(X-\{a_N\})$ be the set of the lines $y\in\mathcal{P}(X-\{a_N\})$ such that $y\ni x$. It is clear that

$$G_x^N(X)=G_x^{N-1}(X-\{a_N\})\cup\{\{x,a_N\}\}, \tag{3.4}$$

because if $y\in G_x^N(X)$ and $a_N\in y$ then there exists a unique line y_1 such that $y_1\ni a_N$ and $y_1\ni x$, hence $y_1=y=l(x,a_N)=\{x,a_N\}$; if $y\in G_x^N(X)$ and $a_N\notin y$ then $y\in G_x^{N-1}(X-\{a_N\})$. Consequently

$$\sum_{y\ni x}(card\,(y)-1)=\sum_{y\in G_x^N(X)}(card\,(y)-1),$$

from (3.4), the above equality becomes

$$\sum_{y\ni x}(card\,(y)-1)=1+\sum_{y\in G_x^{N-1}(X-\{a_N\})}(card\,(X)-1),$$

by recurrence hypothesis, we obtain

$$\begin{aligned}\sum_{y\ni x}(card\,(y)-1)&=(N-2)+1\\&=N-1\\&=card\,(X)-1.\end{aligned}$$

This proves the lemma. ■

Lemma 13 *Let ψ be a function of $l^2(Y)$ defined by*

$$\psi(y)=\frac{1}{(card\,(X)-1)}(card\,(y)-1), \tag{3.5}$$

then $R^*\psi(x)=1$ for all $x\in X$.

Proof. From definition of R^*,we have for all $x\in X$

$$\begin{aligned}R^*\psi(x)&=\sum_{y\ni x}\psi(y)\\&=\frac{1}{(card\,(X)-1)}\sum_{y\ni x}(card\,(y)-1),\end{aligned}$$

by lemma 12 and the last equality, we obtain $R^*\psi(x)=1$ for all $x\in X$. This proves lemma 13.

Lemma 14 *Let $f \in l^2(X)$, then*

$$\sum_{x \in X} f(x) = \frac{1}{(card(X) - 1)} \sum_{y \in Y} (card(y) - 1) Rf(y) \tag{3.6}$$

■

Proof. Since $R^* \psi(x) = 1$ for all $x \in X$ (see lemma 13), we have the following equality

$$\sum_{x \in X} f(x) = \sum_{x \in X} f(x) R^* \psi(x),$$

from the duality formula, the above equality becomes

$$\sum_{x \in X} f(x) = \sum_{y \in Y} Rf(y) \psi(y).$$

Replacing $\psi(y)$ (see lemma 13) by its expression, we obtain

$$\sum_{x \in X} f(x) = \frac{1}{(card(X) - 1)} \sum_{y \in Y} (card(y) - 1) Rf(y).$$

This completes the proof of lemma 14. ■

we shall now give the proof of theorem 11.

We seek a kernel $G(y, x)$ such that

$$RG(y_0, .) = \beta_{y_0}, \tag{3.7}$$

where $RG(y_0, .)(y) = \sum_{x \in y} G(y_0, x)$. By lemma 2, the kernel $G(y_0, x)$ has for expression

$$G(y_0, x) = \frac{1}{(\eta(x) - 1)} \left[R^* RG(y_0, x) - \sum_{x \in X} G(y_0, x) \right].$$

From (3.7), we can rewrite the above equality as follows

$$G(y_0, x) = \frac{1}{(\eta(x) - 1)} \left[R^* \beta_{y_0}(x) - \sum_{x \in X} G(y_0, x) \right], \tag{3.8}$$

but $R^* \beta_{y_0}(x) = \sum_{y \ni x} \beta_{y_0}(y) = \begin{cases} 1 \text{ if } x \in y_0 \\ 0 \text{ if } x \notin y_0 \end{cases}$,

hence

$$R^* \beta_{y_0} = \chi_{y_0}. \tag{3.9}$$

In addition, lemma 14 gives

$$\sum_{x \in X} G(y_0, x) = \frac{1}{(card(X) - 1)} \sum_{y \in Y} (card(y) - 1) RG(y_0, .)(y).$$

Replacing $RG(y_0, .)$ by β_{y_0} in the above formula, we have

$$\sum_{x \in X} G(y_0, x) = \frac{1}{(card(X) - 1)} \sum_{y \in Y} (card(y) - 1) \beta_{y_0}(y).$$

It follows that

$$\sum_{x\in X} G(y_0, x) = \frac{(card(y_0) - 1)}{(card(X) - 1)}. \tag{3.10}$$

Combining (3.8), (3.9) and (3.10), we obtain the equality (3.2), and this finsh the proof of theorem 11.

As a consequence of theorem 11, we prove some properties of the kernel $G(y, x)$. More precisely, we have the following corollary.

Corollary 15 *The kernel* $G(y, x)$ *verify the following property*

$$\sum_{y\in Y} G(y, x) = \frac{1}{(\eta(x) - 1)} \left[\eta(x) - \frac{\alpha(Y)}{(card(X) - 1)}\right], \tag{3.11}$$

where $\alpha(Y) = \sum_{y\in Y} (card(y) - 1)$.

We remark that $\alpha(Y) = card(Y)$ if Y is simple and $\alpha(Y) \leq card(Y)$ if Y is not simple.

Proof. Using the equality (20) and summing $G(y, x)$ over all $y \in Y$, it follows

$$\sum_{y\in Y} G(y, x) = \frac{1}{(\eta(x) - 1)} \left[\sum_{y\in Y} \chi_y(x) - \frac{\alpha(Y)}{(card(X) - 1)}\right],$$

since $\sum_{y\in Y} \chi_y(x) = \eta(x)$,the above equality can be written in the form

$$\sum_{y\in Y} G(y, x) = \frac{1}{(\eta(x) - 1)} \left[\eta(x) - \frac{\alpha(Y)}{(card(X) - 1)}\right].$$

This proves the corollary. ■

In this subsection, we shall give the explicit inversion formulas for the finite Radon transform and its dual, using the theorem 11.

a) **Inversion formula for $\mathbf{R}^*$**

Theorem 16 *The finite dual Radon transform* R^* *is inverted as follows*

$$F(y_0) = \sum_{x\in X} G(y_0, x) R^*F(x), \text{ for all } F \in l^2(Y). \tag{3.12}$$

Proof. The duality formula gives

$$\sum_{y\in Y} RG(y_0, .)(y) F(y) = \sum_{x\in X} G(y_0, x) R^*F(x), \text{ for all } F \in l^2(Y).$$

Replacing $RG(y_0, .)$ by β_{y_0}(see formula (19)), we can write

$$\sum_{x\in X} G(y_0, x) R^*F(x) = \sum_{y\in Y} \beta_{y_0}(y) F(y),$$

since $\beta_{y_0}(y) = 0$ if $y_0 \neq y$ and $\beta_{y_0}(y_0) = 1$,the above equality becomes

$$F(y_0) = \sum_{x\in X} G(y_0, x) R^*F(x).$$

This completes the proof. ■

Remark 17 *From formula(3.12) we have for all $F \in l^2(Y)$*

$$F(y_0) = \sum_{x\in X} G(y_0, x) R^* F(x).$$

Replacing $R^*F(x)$ by $\sum_{y\in Y} F(y)\chi_{G_x}(y)$ in the above equality, we obtain

$$F(y_0) = \sum_{x\in X} G(y_0, x) F(y)\chi_{G_x}(y).$$

Thus

$$F(y_0) = \left\langle F, \sum_{x\in X} G(y_0, x)\chi_{G_x} \right\rangle,$$

but

$$F(y_0) = \langle F, \beta_{y_0} \rangle.$$

Whence

$$\beta_{y_0} = \sum_{x\in X} G(y_0, x)\chi_{G_x}, \text{ for all } y_0 \in Y.$$

This formula is analogous to the Calderon's identity (see[4],[5],[15] and[16]).

Remark 18 *In[1] the author gives the Radon inversion formulas for the Damek-Ricci space analogous to those established in theorem 16.That is the transforms R and R^* are inverted by a remarkable kernel.*

b) **Inversion formula for the finite Radon transform**

Let x_0 be a fixed element of X. We shall find a kernel $\phi(.,.)$ defined on $Y \times X$ such that

$$R^*\phi(., x_0) = \chi_{x_0}, \tag{3.13}$$

recall that $R^*\phi(., x_0)(x) = \sum_{y\ni x} \phi(y, x_0)$,for all $x \in X$.
We will show that the equation (31) admits a unique solution.We begin by proving the existence of the solution of this equation.We need the following results.

Lemma 19 *Let x and x_0 be two elements of X, then*

$$\sum_{y\ni x} \chi_y(x_0) = \begin{cases} 1 & \text{if } x \neq x_0 \\ \eta(x_0) & \text{if } x = x_0 \end{cases}$$
$$= \eta(x, x_0).$$

Proof. Suppose that $x \neq x_0$ and consider the line $y_0 = l(x, x_0)$,then $\sum_{y\ni x} \chi_y(x_0) = \chi_{y_0}(x_0) = 1$ if $x \neq x_0$,and if $x = x_0$ it is clear that $\sum_{y\ni x} \chi_y(x_0) = \eta(x_0)$ .This proves the lemma. ■

Lemma 20 *The kernel $G(.,.) \in l^2(Y \times X)$ is a solution of the equation (31).*

Proof. Let x be an element of X. Using the expression of the kernel $G(.,.)$ (see formula (20))and summing it over all $y \in Y$ which contained the point x, we obtain

$$\sum_{y \ni x} G(y, x_0) = \frac{1}{(\eta(x_0) - 1)} \left[\sum_{y \ni x} \chi_y(x_0) - \frac{\sum_{y \ni x} (card(y) - 1)}{(card(X) - 1)} \right] \tag{3.14}$$

From lemmas (12) and (19), one has

$$\sum_{y \ni x} G(y, x_0) = \begin{cases} 0 \text{ if } x \neq x_0 \\ 1 \text{ if } \ x = x_0 \end{cases}$$

Thus

$$\sum_{y \ni x} G(y, x_0) = \chi_{x_0}(x),$$

but, the first member of the last equality is exactly $R^* G(., x_0)$.
Consequently $R^* G(., x_0) = \chi_{x_0}$, and the lemma is proved. ■

Lemma 21 *The equation (31) admits a unique solution.*

Proof. In lemma (20), we have showed the existence of a solution of the equation (31). We will now pove the uniquness of this solution. Let $\phi(.,.)$ a kernel for which $R^*\phi(., x_0) = \chi_{x_0}$, with $x_0 \in X$. By applying the dual Radon inversion formula (see theorem (16)) to $\phi(., x_0) \in l^2(Y)$, we obtain

$$\phi(y, x_0) = \sum_{x \in X} G(y, x) R^* \phi(., x_0)(x),$$

since $R^*\phi(., x_0) = \chi_{x_0}$, the above equality becomes $\phi(y, x_0) = G(y, x_0)$. This proves the lemma. ■

We establish in the following theorem the inversion formula for the finite Radon transform.

Theorem 22 *The finite Radon transform R is inverted as follows*

$$f(x) = \sum_{y \in Y} G(y, x) Rf(y), \textit{ for all } f \in l^2(X) \textit{ and } x \in X. \tag{3.15}$$

Proof. Applying the duality formula, we obtain for each $x_0 \in X$

$$\sum_{y \in Y} G(y, x_0) Rf(y) = \sum_{x \in X} f(x) R^* G(., x_0)(x), \text{ for all } f \in l^2(X). \tag{3.16}$$

Since $R^* G(., x_0) = \chi_{x_0}$ (see lemma (20)), the formula (34) can be written in the form

$$f(x_0) = \sum_{y \in Y} G(y, x_0) Rf(y),$$

and this completes the proof of theorem (22). ■

Remark 23 *If we replace f by χ_{x0} ($x_0 \in X$) in the formula (33), we obtain*

$$\chi_{x_0}(x) = \sum_{y \in Y} R(\chi_{x_0})(y) G(y, x), \textit{ for all } x \in X, \tag{3.17}$$

but $R(\chi_{x_0}) = \chi_{G_{x_0}}$ *(see*[3]*,page 29), then (35) can be transformed as follows*

$$\chi_{x_0}(x) = \sum_{y \in Y} (\chi_{G_{x_0}})(y)\, G(y,x).$$

Thus

$$\chi_{x_0}(.) = \sum_{y \ni x} G(y,.). \tag{3.18}$$

The formula (36) shows that χ_{x_0} *"Dirac's measure" can be developed into plane waves (see Gelfand's works*[5] *and also*[16]*).*

c) **Inversion formulas for the operators R^*R and RR^***

Now, we establish the inversion formulas analogous to Helgason's inversion formulas.
From theorem (16) we have

$$\sum_{x \in X} R^*Rf(x)\, G(y_0,x) = Rf(y_0), \text{ for all } f \in l^2(X) \text{ and } y_0 \in Y,$$

and by theorem (22), the above formula becomes

$$\begin{aligned} \sum_{y \in Y} Rf(y)\, G(y,x_0) &= \sum_{y \in Y}(\sum_{x \in X} R^*Rf(x)\, G(y,x))G(y,x_0) \\ &= \sum_{x \in X}(\sum_{y \in Y} G(y,x)\, G(y,x_0))R^*Rf(x) \\ &= f(x_0). \end{aligned} \tag{3.19}$$

Let $A(x,x_0) = \sum_{y \in Y} G(y,x)\, G(y,x_0)$ and $\mathbb{B}$ the operator defined on $l^2(X)$ by

$$\mathbb{B}f(x_0) = \sum_{x \in X} A(x,x_0)\, f(x), \text{for all } f \in l^2(X).$$

The equality (37) becomes

$$\mathbb{B}R^*Rf = f, \text{for all } f \in l^2(X). \tag{3.20}$$

This can be formulated as follows.

Theorem 24 *The operator R^*R can be inverted as follows*

$$\mathbb{B}R^*Rf = f, \tag{3.21}$$

where $\mathbb{B}f(x_0) = \sum_{x \in X}\left(\sum_{y \in Y} G(y,x_0)\, G(y,x)\right) f(x)$ *, with* $f \in l^2(X)$ *and* $x_0 \in X$.

Using the same techniques as the previous demonstration, we have for all $F \in l^2(Y)$ the following formula

$$F(y_0) = \sum_{y \in Y}\left(\sum_{x \in X} G(y,x)\, G(y_0,x)\right) RR^*F(y), \text{ for all } F \in l^2(Y). \tag{3.22}$$

Let $\mathbb{D}$ be the kernel defined by $\mathbb{D}(y_0,y) = \sum_{x \in X} G(y,x)\, G(y_0,x)$,for all $(y_0,y) \in Y \times Y$,and putting $\Lambda F(y_0) = \sum_{y \in Y} \mathbb{D}(y_0,y)\, F(y)$. Replacing, in the formula (40), $\mathbb{D}$ and Λ by their expressions, one has

$$F = \Lambda RR^*F, \text{ for all } F \in l^2(Y). \tag{3.23}$$

So, we obtain the following theorem

Theorem 25 *The operator RR^* can be inverted as follows*

$$I = \Lambda RR^*,$$

where $\Lambda F(y_0) = \sum_{y \in Y}\left(\sum_{x \in X} G(y,x)\, G(y_0,x)\right) F(y)$*, for all* $F \in l^2(Y)$ *and* $y_0 \in Y$.

The operators Λ and $\mathbb{B}$ are the Calderon-Zygmund's operators type (see[11],[10])

IV. RANGE THEOREMS FOR THE FINITE RADON TRANSFORM

In his paper[12] John was giving a characterization of the image for the Radon transform via a remarkable differential operator.The John's works are generalized by several authors particularly[6],[7],[9],[13].

In the finite case[3] has establish a characterization of the range for the finite Radon transform.We adopt an algebraic method.

We begin by the example $X = \{x_1, x_2, x_3, x_4\}$,the general case deal with by the same manner.We suppose that Y is simple,for $j \neq k$ let $y_{ij} = \{x_i, x_j\}$be the line of Y and let $y_{12}, y_{13}, y_{14}, y_{23}, y_{24}, y_{34}$ the elements of Y. In the preceding sections, we have seen that the system $\chi_{x_{1,}} \chi_{x_{2,}} \chi_{x_3}, \chi_{x_{4,}}$ is a base of $l^2(X)$ and $\chi_{y_{12}}, \chi_{y_{13}}, \chi_{y_{14}}, \chi_{y_{23}}, \chi_{y_{24}}, \chi_{y_{34}}$ is also a base of $l^2(Y)$,in addition we know that $R : l^2(X) \longrightarrow l^2(Y)$ is a matrix R in its bases,we have

$$R = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix},$$

because

$$\begin{aligned} R(\chi_{x_1}) &= \chi_{G_{x_1}} = \chi_{y_{12}} + \chi_{y_{13}} + \chi_{y_{14}} \\ R(\chi_{x_2}) &= \chi_{G_{x_1}} = \chi_{y_{12}} + \chi_{y_{23}} + \chi_{y_{24}} \\ R(\chi_{x_3}) &= \chi_{G_{x_3}} = \chi_{y_{23}} + \chi_{y_{13}} + \chi_{y_{34}} \\ R(\chi_{x_4}) &= \chi_{G_{x_4}} = \chi_{y_{14}} + \chi_{y_{24}} + \chi_{y_{34}} \end{aligned}$$

(with A= $y_{12} = \{x_1, x_2\}$, B= $y_{23} = \{x_2, x_3\}$,C= $y_{34} = \{x_3, x_4\}$, D= $y_{14} = \{x_1, x_4\}$,E= $y_{13} = \{x_1, x_3\}$, F= $y_{24} = \{x_2, x_4\}$).

For all $\phi \in l^2(X)$,we have $\phi = \phi\,(x_1)\,\chi_{x_1} + \phi(x_2)\,\chi_{x_2} + \phi(x_3)\,\chi_{x_3} + \phi(x_4)\,\chi_{x_4}$.

Thus

$$R\phi = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} \phi(x_1) \\ \phi(x_2) \\ \phi(x_3) \\ \phi(x_4) \end{pmatrix} = \begin{pmatrix} \phi(x_1) + \phi(x_2) \\ \phi(x_1) + \phi(x_3) \\ \phi(x_1) + \phi(x_4) \\ \phi(x_3) + \phi(x_2) \\ \phi(x_4) + \phi(x_2) \\ \phi(x_3) + \phi(x_4) \end{pmatrix},$$

it follows that

$$R\phi = (u_1, u_2, u_3, u_4, u_5, u_6) = \psi,$$

with

$$\begin{cases} u_1(y_{12}, y_{13}, y_{14}, y_{23}, y_{24}, y_{34}) = \phi(x_1) + \phi(x_2) \\ u_2(y_{12}, y_{13}, y_{14}, y_{23}, y_{24}, y_{34}) = \phi(x_1) + \phi(x_3) \\ \vdots \\ u_6(y_{12}, y_{13}, y_{14}, y_{23}, y_{24}, y_{34}) = \phi(x_4) + \phi(x_3) \end{cases}$$

We construct an operator P (here P is a matrix) such that $Im\,R = \ker P$. Show at first that $ImR \subset \ker P$, it suffices for this to seek a operator P for which $PR = 0$. take the following matrix

$$P = \begin{pmatrix} -1 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -1 \\ -1 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -1 \\ -1 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -1 \\ 1 & -1 & 0 & 0 & -1 & 1 \end{pmatrix}$$

It is clear that $PR = 0$,then $Im\, R \subset \ker P$.
To show the other inclusion, it suffices to prove that $ImR) = \dim(\ker P)$.
It is easy to see that vectors space $\ker P$ admits a base formed by the vectors $(1,1,1,0,0,0)$, $(0,-1,1,0,1,0)$,$(0,0,-1,1,0,,0)$, $(0,1,1,0,0,1)$.We observe also that ImR have for base the system $(1,0,0,1,1,0)\,,(1,1,0,0,0,1)\,,.$
$(0,0,1,0,1,0)\,,(0,1,1,0,1,0)$ .Then $ImR = \ker P$.
In the general case, where $\mathrm{card}(X) = N$, we use exactly the same techniques as precedently. Recall that the system $\chi_{x_1},...,\chi_{x_N}$ is a base of the N-dimentional vector space $l^2(X)$ and $\chi_{y_{12}},\chi_{y_{13}},...,\chi_{y_{1N}},\chi_{y_{23}},\chi_{y_{24}},...,\chi_{y_{2N}},...,$
$\chi_{y_{j_0,j_0+1}},...,\chi_{y_{j_0,N}},...,\chi_{y_{N-2,N-1}},\chi_{y_{N-2,N}},\chi_{y_{N-1,N}}$ is a base of the $\frac{N(N-1)}{2}$-dimentional vector space $l^2(Y)$ if Y simple.
The Radon transform is then given by the matrix

$$\begin{pmatrix}
1 & & & & & & \\
1 & & & & & & \\
\vdots & & & I_{N-1} & & & \\
1 & & & & & & \\
0 & 1 & & & & & \\
0 & 1 & & & I_{N-2} & & \\
\vdots & \vdots & & & & & \\
0 & 1 & & & & & \\
0 & 0 & 1 & & & & \\
0 & 0 & 1 & & I_{N-3} & & \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 1 & \cdots & \cdots & \cdots & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & \cdots & \cdots & 0 & 1 & 1 & 0 \\
0 & \cdots & \cdots & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & \cdots & 0 & 1 & 1
\end{pmatrix},$$

where the I_j $(2 \le j \le N-1)$ are the unity matrix .
We calculate P as in the above example.

Theorem 26 *Let X be a finite set and R the finite Radon transform of $l^2(X)$ into $l^2(Y)$, then there exists a operator P (matrix)such that $ImR = \ker P$.*

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