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Maximal Operators on Semisimple Lie Groups

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abstract

In this article I give an overview of the so-called real variable method in harmonic analysis, especially, the role of maximal operators in harmonic analysis on the Euclidean spaces and on semi simple Lie groups. The most of the classical results on the Euclidean spaces can be extended to the spaces of homogeneous type, where the doubling condition holds. However, analogous approach to the spaces of non-homogeneous type (cf. [FS]), such as semi simple Lie groups, encounters much difficulties and sometimes a new method is required. The aim of this article is to give a rapid introduction to the recent topics on maximal operators on Lie groups; L^p boundedness, real Hardy spaces, BMO, singular integrals, etc, obtained by various people, and to propose some problems in the further study.

I. AVERAGES OF FUNCTIONS

The most basic tool in harmonic analysis is the average of functions. I brief this subject on the one-dimensional Euclidean spcae, of course, any results can be easily extended to higer dimensional cases and to the spaces of homogeneous type. The family of the averages of a function f is defined by for $t > 0$

$$\begin{aligned} &\frac{1}{2t}\int_{-t}^{t} f(x-y)dy \quad \text{(average around } x\text{)} \\ &= \frac{1}{|B(t)|} f * \chi_{B(t)}(x) \\ &= f * (\chi_{B(1)})_t(x), \end{aligned}$$

where $\chi_{B(t)}$ is the characteristic function on the interval $B(t) = [-t, t]$ and $(\chi_{B(1)})_t$ is the dilation given by

$$(\chi_{B(1)})_t(x) = \frac{1}{2t}\chi_{B(1)}\left(\frac{x}{t}\right).$$

By using these averages the Hardy-Littlewood maximal function $M_{HL}f$ of f is defined by

$$M_{HL}f(x) = \sup_{t>0} |f| * \chi_t(x).$$

Our natural questions are, (1) in what sense does $f * \chi_t$ approache to f as $t \to 0$? and (2) how about L^p inequality for the maximal operator? Then the answers are the following.

(1) For $f \in L^p$ $(1 \le p < \infty)$ $f * \chi_t$ approaches to f in L^p sense, and for $f \in L^1_{loc}$, $f * \chi_t(x)$ approaches $f(x)$ almost every x.

(2) Maximal theorem holds; $M_{HL} \in (L^p, L^p)$ $(1 < p \le \infty)$ and $(L^1, weak L^1)$.

We can prove these facts by showing that they hold for a larger non-centered maximal function $M_{NC}f$ defined by

$$M_{NC}f(x) = \sup_{0 \in B} |f| * (\chi_B)_t(x),$$

where the supremum is taken over all balls B containing 0. Actually, there exist constants c_1, c_2 such that for all x

$$c_1 M_{HL}f(x) \le M_{NC}f(x) \le c_2 M_{HL}f(x).$$

Then the fact that $M_{NC} \in (L^1, weak L^1)$ is essential, from which (1) and $M_{NC} \in (L^p, L^p)$ $(1 < p \le \infty)$ easily follow. The key to obtain this fact is the following Vitali and Whitney's covering lemma: there exist constants $c_1 > 1, c_2 > 1$ such that for all x, y and $\delta > 0$

$$B(x,\delta) \cap B(y,\delta) \ne \emptyset \Rightarrow B(y,\delta) \subset B(x, c_1\delta), \text{ and } |B(x, c\delta)| \le c_2 |B(x,\delta)|.$$

Now we review the corresponding results on semisimple Lie groups G. By using the ball on G, the Hardy-Littlewood maximal operator M_{HL} on G is defined by

$$M_{HL}f(x) = \sup_{t>0} \frac{1}{|B(t)|} f * \chi_{B(t)}(x)$$

where $B(t)$ is the ball with redius t centered at the origin and $|B(t)|$ the volume of the ball. Since the volume has exponential growth when the redius t goes to ∞, Vitali and Whitney's covering lemma does not hold on G. Hence the analogous proof of the maximal theorem on the Euclidean spaces could not apply in this case. However, we have

Maximal theorem: $M_{HL} \in (L^p, L^p)$ $(1 < p \le \infty)$ and $M_{HL} \in (L^1, weak\ L^1)$.

This theorem was proved by Clerc and Stein [CS] when $p > 1$, and by Strömberg [Str] when $p = 1$. The idea of the proof is the following. First we divide the operator M_{HL} into two parts:

$$Mf(x) \le M_{local}f(x) + M_{global}f(x),$$

where the local part $M_{local}f$ is defined by restricting the supremum over $0 < t < 1$, and the global part M_{global} is defined by doing over $t \ge 1$. Since the Vitali and Whitney's covering lemma locally holds, so we can apply the Euclidean argument for M_{local}. As for $M_{global}f$ we note that

$$M_{global}f(x) \le |f| * \tau(x), \quad \tau(x) = \frac{1}{1 + |B(\sigma(x))|},$$

where $\sigma(x)$ is the Riemannian distance of x from the origin. When $p > 1$, we can essentially use the Kunze-Stein phenomenon (cf. [CS]) to obtain the strong L^p boundedness. However, when $p = 1$, we need a deep estimates of $|f| * \tau$ and the arguments are very complicated. Anyway, without using the covering lemma and the non-centered maximal operator we can deduce the desired result for the global part.
We here introduce the recent topics on the Hardy-Littlewood maximal operator on semisimple Lie groups G. To estimate the global part $|f| * \tau$ Strömberg obtains a pointwise estimate such as

$$|\tau(kak'^{-2\rho(a)}\omega(a), \quad x, x' \in K, \ a \in A_+$$

and deduces a condition on ω under which $|f| * \tau$ satisfies the weak L^1 estimate. However, this process requires really deep estimates, so we would like to simplify his proof.

Let $G = KAN = KCL(A_+)K$ denote the Iwasawa and Cartan decompositions of G and ρ is half the sum of positive restricted roots. In the case of K-biinvariant functions on real rank one semisimple Lie groups, more generally, in the case of the Fourier-Jacobi transfrom, J. Liu [L] gives a quick proof of the maximal theorem. Actually, he shows that for all $x, y \in A_+$, the inequality

$$\int_K |\tau(xky)|dk \le ce^{-2\rho(x)}$$

holds and therefy, all K-biinvariant functions $f \in L^1$ satisfy

$$|f| * \tau(x) \le c\|f\|_1 e^{2\rho(x)}, \quad x \in A_+.$$

This pointwise estimate easily yields the maximal theorem. For higher rank case, Kawazoe and Liu [KL] notice the kernel form of the the product of two spherical functions (see [FK]);

$$\phi_\lambda(x)\phi_\lambda(y) = \int_{A_+} \phi_\lambda(z)K(x,y,z)\Delta(z)dz, \quad x, y \in A_+,$$

where ϕ_λ is the zonal spherical function on G and Δ is the weight on A_+ corresponding to the Haar measure on G. By using this kernel we can deduce that

$$\int_K \tau(xky)dk = \int_{A_+} \tau(z)K(x,y,z)\Delta(z)dz$$

and thereby, if the kernel satisfies

$$K(x,y,z) \le ce^{-(\rho(x)+\rho(y)+\rho(z))}(1+\sigma(z))^{n-1} \tag{1.1}$$

for $|x-y| > z, x, y, z \in A_+$, then we have

$$\int_K |\tau(xky)|dk \le ce^{-2\rho(x)}e^{-(\|\rho\|\|x-y\| - \langle\rho, x-y\rangle)}\|x-y\|^{n-1}, \tag{1.2}$$

where n is the real rank of G and A is identified with the n dimensional Euclidean space. Form (2) we can easily deduce the maximal theorem. Fortunately, except three simple Lie groups; $SL(3, \mathbf{R})$, $SL(4, \mathbf{R})$, and $SO(3, 2)$, since the Harish-Chandra C-function (see below) is integrable, all simple Lie groups satisfy (1) and thus (2). Roughly speaking, in his original proof Strömberg obtains a pointwise estimate of τ and use the Iwasawa decomposition of G, however, in our case, since f is K-biinvariant, we can replace the pointwise estimate with the one of the integral (2) over K and use the Cartan decomposition of G. Recently, Ionescue [I1] obtained a partial maximal theorem on the non-centered Hardy-Littlewood maximal operator M_{NC} on the real rank one semisimple Lie groups. He introduces the Lorentz space $L^{p,q}$ on G with the norm

$$\|f\|_{p,q} = p\left(\int_0^\infty |\{x;\ |f(x)| > \lambda\}|^{q/p}\lambda^{q-1}d\lambda\right)^{1/q}.$$

Clearly, if $p \ge 1$, then

$$L^{p,1} \subset L^{p,p}(= L^p) \subset L^{p,\infty}(= weak\ L^p).$$

Weak maximal theorem: M_{NC} is not boundend on L^2 but $M_{NC} \in (L^{2,1}, L^{2,1})$. Moreover, $M_{NC} \in (L^p, L^p)$ $(2 < p \le \infty)$.

II. TINY EXTENTION

Now we replace $(\chi_{B(1)})_t$ in the definition of the Hardy-Littlewood maximal operator M_{HL} on $\mathbf{R}$ by Φ_t, where Φ is a rapidly decreasing function on $\mathbf{R}$ with

$$\int_{-\infty}^{\infty} \Phi(x)dx \ne 0.$$

Then the tangential maximal function on **R** is defined by

$$M_\Phi f(x) = \sup_{t>0} f * \Phi_t(x),$$

where

$$\Phi_t(x) = \frac{1}{t}\Phi\left(\frac{x}{t}\right).$$

Since $M_\Phi f$ is pointwisely dominated by the Hardy-Littlewood maximal function

$$M_\Phi f(x) \leq cM_{HL} f(x),$$

it also satisfies

(1) For $f \in L^p$ $(1 \leq p < \infty)$ $f * \Phi_t$ approaches to f in L^p sense, and for $f \in L^1_{loc}$, $f * \Phi_t(x)$ approaches $f(x)$ almost every x.

(2) Maximal theorem holds; $M_\Phi \in (L^p, L^p)$ $(1 < p \leq \infty)$ and $(L^1, weakL^1)$.

Now we shall consider the corresponding extension on semisimple Lie groups G. We suppose that the real rank of G eqauls one. In this process we must define a "dilation" Φ_t of Φ on G. We here note that one of essential properties of the dilation Φ_t on **R** is to preserve the L^1 norm of Φ. We suppose that Φ is K-biinvariant on G and we identify it with an even function on A. In the sense of preserving the L^1-norm of Φ we shall define the dilation Φ_t on G as

$$\Phi_t(x) = \frac{1}{t}\frac{1}{\Delta(t)}\Delta\left(\frac{x}{t}\right)\Phi\left(\frac{x}{t}\right).$$

Then the approximation of the identity; $f * \Phi_t \to f$ $(t \to 0)$ in L^p $(1 \leq p < \infty)$ is proved by Flensted-Jensen [F], more generally, for the Fourier-Jacobi transform. As for the L^p boundedness of the maximal operator M_Φ defined by

$$M_\Phi f(x) = \sup_{t>0} |f * \Phi_t(x)|,$$

we have:

Maximal theorem: $M_\Phi \in (L^p, L^p)$ $(1 < p \leq \infty)$ and $M_\Phi \in (L^1, weakL^1)$.
For the proof, as before we devide the maximal function into the local part and the global part;

$$M_\Phi f(x) \leq M_{\Phi,local} f(x) + M_{\Phi,global} f(x).$$

Then each part is dominated as

$$M_{\Phi,local} \leq cM_{HL} f(x) \quad \text{and} \quad M_{\Phi,global} f(x) \leq c|f| * \tau(x).$$

Hence the maximal theorem follows from the arguments used in the case of the Hardy-Littlewood maximal operator on G (see [K1]).

III. OTHER TOPICS ON $F * \Phi_T$

We summarize other topics on $f * \Phi_t$. Actually, this convolution is really important and plays an essential role in harmonic analysis. On the Euclidean space we have the following (see [St2], [FS] for general references).
(a) Nontangential maximal functions: We define

$$M^*_\Phi f(x) = \sup_{|x-y|<t} |f * \Phi_t(y)|.$$

Then Maximal theorem; $M_\Phi^* \in (L^p, L^p)$ $(1 < p \le \infty)$ and $M_\Phi^* in (L^1, weak L^1)$, holds.

(b) Square-functions: We define

$$S_\Phi f(x) = \left(\int_0^\infty |f * \Phi_t(x)|^2 \frac{dt}{t} \right)^{1/2}.$$

Then Maximal theorem; $S_\Phi \in (L^p, L^p)$ $(1 < p \le \infty)$ and $S_\Phi \in (L^1, weak L^1)$, holds.

(c) Hardy spaces: The Hardy space H^p $(0 < p \le \infty)$ is defined as

$$f \in H^p \;\text{ if }\; M_\Phi f \in L^p \;\; (\text{or } M_\Phi^* f \in L^p).$$

(d) Carleson measure: A Borel measure $d\mu$ on $\mathbf{R}_+^{\mathbf{n+1}}$ is called a Carleson measure if

$$\sup_{x \in \mathbf{R^n}} \left(\sup_{x \in B} \frac{1}{|B|} \int_{T(B)} |d\mu| \right) = \|d\mu\|_{\mathcal{C}} < \infty,$$

where

$$T(B) = \{(x,t) \; ; \; |x - x_0| \le r - t\}, \;\; \text{if } B = B(x_0, r).$$

Let $d\mu(x,t)$ be a Carleson measure on $\mathbf{R}_+^{\mathbf{n+1}}$. Then

$$\left| \int_{\mathbf{R}_+^{\mathbf{n+1}}} f * \Phi_t(x) d\mu(x,t) \right| \le \|M_\Phi^* f\|_1 \|d\mu\|_{\mathcal{C}}.$$

(e) If f is BMO, then

$$|f * \Phi_t(x)|^2 \frac{dxdt}{t}$$

is a Carleson measure.

On semisimple Lie groups we don't have much results on these topics. In further study we should challenge these topics. In the next two sections we study Hardy sapces and BMO on G.

IV. HARDY SPACES

The (real) Hardy space H^p on $\mathbf{R}$ is defined as follows. $f \in H^p$ if there exists a rapidly decreasing function Φ with $\int_{-\infty}^\infty \Phi(x)\, dx = 1$ such that the tangential maximal function $M_\Phi f(x) = \sup_{t>0} |f * \Phi_t(x)|$ belongs to L^p. Then it is easy to see that $H^p = L^p$ for $1 < p < \infty$ and $H^1 \subset L^1$. For $0 < p \le 1$ a quasi-norm is given as

$$\|f\|_{H^p} = \|M_\Phi f\|_{L^p}^p.$$

The natural question is how to characterize H^p when $0 < p \le 1$. There are several ways to characterize these real Hardy spaces. As I said, the nontangential maximal operator also characterizes H^p: $f \in H^p$ if and only if the nontangential maximal function $M_\Phi^* f(x) = \sup_{|x-y| \le t} |f * \Phi_t(y)|$ belongs to $\in L^p$, that is,

$$\|M_\Phi f\|_{L^p} \sim \|M_\Phi^* f\|_{L^p}.$$

Another way to characterize H^p is to define atomic Hardy spaces. We say a function a on $\mathbf{R}$ is a (p,q,r)-atom if $r \ge [1/p - 1]$ and a satisfies

(i) supp a $\subset$ B,

(ii) $\|a\|_q \le |B|^{1/q - 1/p}$,

(iii) $\int_{-\infty}^\infty a(x) x^k dx = 0, \;\; 0 \le k \le r,$

where B is a ball (interval) on $\mathbf{R}$. By using these atoms the atomic Hardy space $H^p_{q,r}$ is defined by

$$H^p_{q,r} = \{\ f = \sum \lambda_i a_i\ ;\ \text{each } a_i \text{ is a } (p,q,r)\text{-atom and } \sum |\lambda_i|^p < \infty\ \}$$

and its quasi-norm is given by

$$\|f\|_{H^p_{q,r}} = \inf \sum |\lambda_i|^p,$$

where the infimum is taken over all expression $f = \sum \lambda_i a_i$. Then we have

$$H^p = H^p_{q,r}$$

and

$$\|f\|_{H^p} \ \sim\ \|f\|_{H^p_{q,r}}.$$

Now we shall consider the following problem on semisimple Lie groups G. On the Euclidean space M_Φ is not strong L^1 bounded and $M_\Phi \in (L^1, weakL^1)$, however, if we replace L^1 by H^1, a subspace of L^1, then $M_\Phi \in (H^1, L^1)$ by the definition. Our question is
Problem. On G is there a subspace of L^1 from which M_Φ is bounded into L^1?
We suppose that the real rank of G is 1 and we treat K-biinvariant functions on G. In what follows we shall construct two subspaces of L^1 from which M_Φ is bounded into L^1.
(a) Pull back of $H^1(\mathbf{R})$: In order to construct a desired subspace of $L^1(G)$ we shall pull back the real Hardy space H^1 on $\mathbf{R}$ by using Fourier transform on $\mathbf{R}$ and the spherical Fourier transfrom on G. We prepare several operators:

$\mathcal{F}_{\mathbf{R}}$: Fourier transfrom on $\mathbf{R}$,
$\mathcal{M}_C$: Fourier multiplier with $C(\lambda + i\rho)$,
$\mathcal{S}_\rho$: Shift operator such that $h(\lambda) \to h(\lambda - i\rho)$,
$\mathcal{F}_G$: Spherical Fourier transform on G,

where C is Harish-Chandra's C-function. Then we define a space $H^1(G)$ as

$$H^1(G) = \mathcal{F}_G^{-1} \circ \mathcal{S}_\rho \circ \mathcal{M}_C \circ \mathcal{F}_{\mathbf{R}} \left(H^1(\mathbf{R})\right)$$

and the norm is given by

$$\|f\|_{H^1(G)} = \|F(f)\|_{H^1(\mathbf{R})}.$$

where

$$F(f) = \mathcal{F}_{\mathbf{R}}^{-1} \circ \mathcal{M}_C^{-1} \circ \mathcal{S}_\rho^{-1} \circ \mathcal{F}_G(f).$$

Theorem. $H^1(G) \subset L^1(G)$ and $M_\Phi \in (H^1(G), L^1(G))$.

The idea behind this definition of $H^1(G)$ and the proof of Theorem is the following. For a suitable K-biinvariant function f on G the Fourier inversion formula on G, the Harish-Chandra expansion of ϕ_λ, and the Gangolli expansion (cf. [GV]) yield that, if we denote $\mathcal{F}_G(f) = \hat{f}$, then

$$\begin{aligned} f(x) &= \int_{-\infty}^{\infty} \hat{f}(\lambda)\phi_\lambda(x)|C(\lambda)|^{-2} d\lambda \\ &= e^{-\rho x} \int_{-\infty}^{\infty} \hat{f}(\lambda)\Phi(\lambda, x)C(-\lambda)^{-1} e^{-i\lambda x} d\lambda \\ &= e^{-\rho x} \sum_{m=0}^{\infty} e^{-2mx} \int_{-\infty}^{\infty} \hat{f}(\lambda)\Gamma_m(\lambda)C(-\lambda)^{-1} e^{-i\lambda x} d\lambda \\ &= e^{-2\rho x} \sum_{m=0}^{\infty} e^{-2mx} \int_{-\infty}^{\infty} \Gamma_m(\lambda + i\rho)C(-\lambda - i\rho)^{-1} \hat{f}(\lambda + i\rho) e^{-i\lambda x} d\lambda. \end{aligned}$$

Since $\Gamma_0 \equiv 1$, the leading term in the last series is the term corresponding $m = 0$;

$$e^{-2\rho x} \int_{-\infty}^{\infty} C(-\lambda - i\rho)^{-1} \hat{f}(\lambda + i\rho) e^{-i\lambda x} d\lambda.$$

We recall that $\Delta(x) \sim e^{2\rho x}$ $(x > 1)$. Thereby, it is really natural to think that

$$\int_{-\infty}^{\infty} C(-\lambda - i\rho)^{-1} \hat{f}(\lambda + i\rho) e^{-i\lambda x} d\lambda$$

should be related with harmonic analysis on the Euclidean space. This term is noting but $F(f) = \mathcal{F}_{\mathbf{R}}^{-1} \circ \mathcal{M}_C^{-1} \circ \mathcal{S}_\rho^{-1} \circ \mathcal{F}_G(f)$ in the definition. To prove that $H^1(G) \subset L^1(G)$ we note that the Fourier multiplies $C(-\lambda - i\rho)^{-1}$ and $\Gamma_m(\lambda + i\rho)$ satisfy the Hörmander condition, which guarantees the (H^1, L^1) boundedness of these Fourier multipliers. Since $F(f)$ belongs to $H^1(\mathbf{R})$, we see that each term

$$\int_{-\infty}^{\infty} \Gamma_m(\lambda + i\rho) C(-\lambda - i\rho)^{-1} \hat{f}(\lambda + i\rho) e^{-i\lambda x} d\lambda$$

belongs to $L^1(\mathbf{R})$. By estimating the L^1 norm carefully, especially on m, and summing up we see that f belongs to $L^1(G)$ (see [K2]). As for the proof of $M_\Phi \in (H^1(G), L^1(G))$, we first note that the spherical Fourier transform $\mathcal{F}_G(\Phi_t)$ satisfies

$$\left| \left(\frac{d}{d\lambda} \right)^l \mathcal{F}_G(\Phi_t)(\lambda + i\rho) \right| \leq ct^l (1+t)^k (1 + |t\lambda|)^{-2k}$$

for $l \in \mathbf{N}$, where we assume that $\Phi(x) = O(x^k)$ as $x \to 0$. Now we suppose that f belongs to $H^1(G)$, that is, $F(f)$ belongs to $H^1(\mathbf{R})$. Then, by using the above estimate and the fact that $F(f)$ has an atomic decomposition, we can deduce that $f * \Phi_t$ is a function with a good shape. Then we can prove that $M_\Phi(f)$ belongs to $L^1(G)$ (see [K2] and [K4]).

(b) Atomic Hardy space: The second way to define a subspace of $L^1(G)$ from which the maximal operator M_Φ is bounded into $L^1(G)$ is to construct an atomic Hardy space. We say that a K-bi invariant function a on G is $(1, \infty, 0, \delta)$-atom, $\delta > 0$, if

$$\begin{aligned} &(i) \quad \text{supp a} \subset \text{B(r)}, \\ &(ii) \quad \|a\|_\infty \leq \frac{1}{|B(r)|} \left(\frac{1}{1+r} \right)^\delta, \\ &(iii) \quad \int_G a(g) dg = 0. \end{aligned}$$

Compared with the Euclidean atoms, we restrict the support of a to the ball $B(r)$ centered at the origin and we require an extra decay of a when the radius $r > 1$. We shall translate them around x as

$$a_x(g) = \int_K a(xkg) dk.$$

By using these translations we define

$$H^{1,\delta}_{\infty,0}(G) = \{ \sum_i \lambda_i a_{i,x_i} \text{ ;each } a_i \text{ is } (1, \infty, 0, \delta)\text{-atom and } \sum |\lambda_i| < \infty,\ x_i \in G \}$$

and the norm is given by

$$\|f\|_{H^{1,\delta}_{\infty,0}} = \inf \sum |\lambda_i|,$$

where the infimum is taken over all expression $f = \sum \lambda_i a_{i,x_i}$.

Theorem. $H^{1,\delta}_{\infty,0}(G) \subset L^1(G)$ and $M_\Phi \in (H^1_{\infty,0}(G), L^1(G))$.

Since $\|a_x\|_1 \le 1$ for all $x \in G$ and all $(1,\infty,0,\delta)$-atoms on G, it is clear that $H^{1,\delta}_{\infty,0}(G) \subset L^1(G)$. For the proof of $M_\Phi \in (H^1_{\infty,0}(G), L^1(G))$, we must estimate $a * \Phi_t$ for each $(1,\infty,0,\ \delta)$-atom a on G. As before, locally we can apply the arguments used in the Euclidean space. On the other hand, to estimate far from the origin, we prove that

$$|a * \Phi_t(x)| \le \frac{1}{\Delta(x)} \begin{cases} \left(\dfrac{1}{|x| - r}\right)^{1+\delta} & |x| > r + 1, \\ \dfrac{r}{|x|^2} & |x| \le r. \end{cases}$$

Then, using the first line when $r \ge 1$ and the second one when $r < 1$, we can deduce that $\|M_\Phi a\|_1 \le c$ for all $(1,\infty,0,\ \delta)$-atoms a on G, where c does not depend on a.

Can we take $\delta = 0$ in this process? Maybe "No".
In the Euclidean case, we see that $M_\Phi \chi_{B(1)}(x) \le c(1+|x|)^{-1}$ and it is not integrable. However, if a is supported on $B(1)$ and satisfies the moment condition $\int_{B(1)} a(x)dx = 0$, then $M_\Phi a(x)$ has an extra decay of $|x|^{-1}$ at infinity as $M_\Phi a(x) \le cr(1+|x|)^{-2}$ and it is integrable:

$$M_\Phi \chi_{B(1)}(x) \sim \frac{1}{|x|} \quad \text{and} \quad M_\Phi a(x) \sim \frac{1}{|x|^2}, \quad |x| \to \infty.$$

This situation is exactly same on semi simple Lie groups G. The moment condition yields the extra decay of $|x|^{-1}$ at infinity:

$$M_\Phi \chi_{B(1)}(x) \sim \frac{1}{\Delta(x)} \quad \text{and} \quad M_\Phi a(x) \sim \frac{1}{\Delta(x)|x|}, \quad |x| \to \infty.$$

We notice that, different from the Euclidean case, this $M_\Phi a(x)$ is not integrable on $\mathbf{R}$ with the weight Δ, so $M_\Phi a$ does not belong to $L^1(G)$. Therefore, to obtain the integrability of $M_\Phi a$ on G, we need an extra condition; a modification as *(ii)* with $\delta > 0$ (see [K3] and [K4]).

How about a relation between $H^1(G)$ and $H^{1,\delta}_{\infty,0}(G)$?
At present I have no information on this question.

V. BMO

We shall consider the space BMO. On the Euclidean space $\mathbf{R}$ it is defined by using the sharp maximal operator

$$f^\sharp(x) = \sup_{x \in B} \frac{1}{|B|} \int_B |f(t) - f_B| dt,$$

where f_B is the average over B given by

$$f_B = \frac{1}{|B|} \int_B |f(x)| dx.$$

Then f belongs to BMO if $f^\sharp$ belongs to L^∞ and the norm is given by

$$\|f\|_{\text{BMO}} = \|f^\sharp\|_{L^\infty}.$$

We see that $L^\infty \subset$ BMO and BMO is identified with the dual space of H^1. Moreover, for $f \in$BMO, $|f * \Phi_t(x)|^2 \dfrac{dxdt}{t}$ is a Carleson measure.

On semi simple Lie groups G we can define the space BMO as an analogous way. However, since G is not homogeneous type, so we could not apply the arguments used in the Euclidean space directly. Thereby, BMO on G is not studied much. Recently, Ionescu [I2] fond an very interesting phenomenon. He defines BMO on the real rank one semi simple Lie groups G by using the local sharp function

$$f^{\sharp}_{local}(x) = \sup_{x\in B, r(B)\leq 1} \frac{1}{|B|}\int_B |f(t) - f_B|dt,$$

where $r(B)$ is the redius of the ball B. Then, f belongs to BMO(G) if $f^{\sharp}_{local} \in L^\infty$ and the norm is given by

$$\|f\|_{\mathrm{BMO}} = \|f^{\sharp}_{local}\|_{L^\infty}.$$

We easily see that for $p > 1$

$$\|f^{\sharp}_{local}\|_p \leq C_p\|f\|_p,$$

because $f^{\sharp}_{local}$ is pointwisely dominated by the local noncentered maximal operator. Surprisingly, he obtains the converse.

Theorem. $\|f\|_p \leq A_p\|f^{\sharp}_{local}\|_p,\ 1 \leq p < \infty.$

This means that the global part of the sharp function is not essential and it can be ignored. Of course, this converse inequality of the local sharp function does not hold on the Euclidean case, and it is a unique phenomenon on semisimple Lie groups. He also obtains (L^∞, BMO) boundedness of some Fourier multipliers.

VI. OPERATORS ASSOCIATED WITH LAPLACIAN

Here we remark some operators on Lie groups associated with the Laplacian, which are introduced by Stein [St1] in a general setting.
In the semi simple case these operators are the following. Let Δ denote the Laplace-Beltrami operator on G and T_t, P_t the symmetric diffusion semi-groups symbolically denoted by

$$T^t = e^{t\Delta} \text{ and } P^t = e^{t(-\Delta)^{1/2}}.$$

Let $T^t f$ and $P^t f$ denote the solutions of the heat and Poisson equations respectively.

(a) For $1 < p < \infty$

$$\|\left(\int t|\frac{\partial P^t f}{\partial t}(x,t)|^2 dt\right)^{1/2}\|_p \sim \|f\|_p,$$

$$\|\left(\int t|\frac{\partial T^t f}{\partial t}(x,t)|^2 dt\right)^{1/2}\|_p \sim \|f\|_p.$$

(b) The Littlewood-Paley g-function: We define

$$\mathrm{g(f)(x)} = \left(\int \mathrm{t}|\mathbf{\nabla P^t f}|^{\mathbf{2}}\mathbf{(x,t)dt}\right)^{1/2},$$

where

$$|\mathbf{\nabla u}|^{\mathbf{2}} = \left(\frac{\partial \mathbf{u}}{\partial \mathbf{t}}\right)^{\mathbf{2}} + \sum \mathbf{a_{ij}X_i f X_j f}, \quad \mathbf{\Delta} = \sum \mathbf{a_{ij}X_i X_j}.$$

Then

$$\|\mathrm{g(f)}\|_{\mathrm{p}} \sim \|\mathrm{f}\|_{\mathrm{p}}, \quad 1 < \mathrm{p} < \infty.$$

(c) The Riesz transforms: We define

$$R_i(f) = X_i(-\Delta)^{-1/2} f, \quad \Delta = \sum a_{ij} X_i X_j.$$

Then

$$\|R_i(f)\|_p \sim \|f\|_p, \quad 1 < p < \infty.$$

(d) The heat and Poisson maximal operators: We define

$$M_H f(x) = \sup_{t>0} |T^t f(x)| \text{ and } M_P f(x) = \sup_{t>0} |P^t f(x)|.$$

Then

$$\|M_H f\|_p \sim \|f\|_p \text{ and } \|M_P\|_p \sim \|f\|_p, \quad 1 < p \le \infty.$$

These facts show that the operators g, R_i, M_H, M_P are L^p bounded when $p > 1$. As for $p = 1$, Anker [A] shows that they satisfy the weak type L^1 estimate.

Theorem. g, R_i, M_H, $M_P \in (L^p, L^p)$ $(1 < p \le \infty)$ and $(L^1, weak L^1)$.

How about (H^1, L^1) boundedness? Here H^1 is the real Hardy space defined in **V**. To obatin a partial answer we shall define some modified maximal operators as

$$M_H^\epsilon f(x) = \sup_{t>0} \left(\frac{1}{1+t}\right)^\epsilon T^t f(x),$$

$$M_P^\epsilon f(x) = \sup_{t>0} \left(\frac{1}{1+t}\right)^\epsilon P^t f(x).$$

Then we have the following (see [K4]).

Theorem. (1) $M_H^\epsilon \in (H^1, L^1)$ if $\epsilon > 1/2$, and (2) $M_P^\epsilon \in (H^1, L^1)$ if $\epsilon > 0$.

As for $H^{1,\delta}_{\infty,0}$ we have the same results. Can we take $\epsilon = 1/2$ and 0 respectively? At present I have no information on this question.

VII. SINGULAR INTEGRALS

Last we give some well-known examples of the family of singular integrals on the Euclidean space **R**, which should be generalized to semi simple Lie groups in the further study.

The class of operators called singular integrals is the family of the operators being of the form

$$Tf(x) = \int_{-\infty}^{\infty} K(x,y) f(y) dy,$$

where $K(x,y)$ has singularity at $x = y$. Well-known examples are the following.

a) Fourier multipliers:

$$T_m f(x) = \int_{-\infty}^{\infty} m(\lambda) \hat{f}(\lambda) e^{-i\lambda x} d\lambda.$$

b) Hilbert transform:

$$H(f)(x) = T_m(f)(x) = p.v.\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{f(x-y)}{y}dy, \quad m(\lambda) = -i\frac{\lambda}{|\lambda|}.$$

c) Riesz transforms:

$$R_j(f) = T_{m_j}(f)(x), \quad m_j(\lambda) = -i\frac{\lambda_j}{|\lambda_j|}$$

d) Calderón-Zygmund operators (CZO): The kernel satisfies

$$|\partial_x^\alpha \partial_y^\beta K(x,y)| \le C|x-y|^{-n-|\alpha|-|\beta|}.$$

e) Fractional integrals:

$$I_\alpha(f)(x) = \frac{1}{\Gamma(\alpha)}\int_{-\infty}^{\infty} \frac{f(x-y)}{|y|^{1-\alpha}}dy.$$

f) Fourier integral operators:

$$T(f)(x) = \int_{-\infty}^{\infty} e^{2\pi i\Phi(x,\lambda)}a(x,\lambda)\hat{f}(\lambda)d\lambda$$

g) Oscillatory integrals:

$$T_\lambda(f)(x) = \int_{-\infty}^{\infty} e^{i\lambda\Phi(y,x)}a(y,x)f(y)dy$$

Our natural questionas are (L^p, L^p), (H^1, L^1), (L^∞, BMO) boundedness of the singular integrals. For example, one of criteria for which the sigular integral T satisfies the maximal theorem is that, if T is (L^q, L^q) for a $q > 1$ and there exists $c > 0$ such that for all $t > 0$

$$\int_{B(y,ct)^c} |K(x,y) - K(x,y')|dx \le C \text{ if } y' \in B(y,t),$$

then T is in (L^p, L^p), $1 < p \le q$ and $(L^1, weakL^1)$. If we can prove that $T \in (L^1, weakL^1)$, then an interpolation between 1 and q yields that $T \in (L^p, L^p)$, $1 < p \le q$. So, the weak L^1 boundedness is essential in this criteria. In order to obatin the weak type L^1 estimate we use the so called the Calderón-Zygmund decomposition of f such as

$$f = g + b, \;\; |f| \le \alpha, \;\; b = \sum b_i, \;\; \mathrm{supp b_i} \subset \mathrm{B_i}, \;\; \int \mathrm{b_i} = 0, \;\; \sum |\mathrm{B_i}| \le \frac{1}{\alpha}\|\mathrm{f}\|_1.$$

and estimate Tg and Tb separately.
Our last question is how to generalize this type of criteria to the semi simple case G. As before, if we divide Tf into a local part and a global part, then we can apply the Euclidean arguments to the local part. How to treat the global part. We have two strategies. One way is to reconstruct the above process to the global part. In this case we need to generalize the Calderón-Zygmund decomposition based on the family of the averages of f and the covering lemma. Another one is to find a quit new argument. Actually, we have high possibility to find unique properties, which hold on the semi simple case, but not on the Euclidean case. The Kunze-Stein phenomenon and the converse inequality for the local sharp function are typical examples. These phenomena will help the analysis on the global part.

VIII. REFERENCES

[A] J-Ph. Anker, *Sharp estimates for some functions of the Laplacian on noncompact symmetric spaces,* Duke Math. J., Vol. 65, 1992, pp. 257-297

[CS] J. C. Clerc and E. M. Stein, *L^p-multipliers for non-compact symmetric spaces,* Proc. Nat. Acad. Sci. USA, Vol. 71, 1974, pp. 3911-3912

[I1] A. Ionescu, *An endpoint estimate for the Kunze-Stein phenomenon and related maximal operators,* Ann. of Math., Vol. 152, 2000, pp. 259-275

[I2] A. Ionescue, *Fourier integral operators on noncompact symmetric spaces of real rank one,* J. Funct. Analysis, Vol. 174, 2000, pp. 274-300

[F] M. Flensted-Jensen *Paley-Wiener type theorems for a differential operator connected with symmetric spaces,* Ark. Mat., Vol. 10, 1972, pp. 143-162

[FK] M. Flensted-Jensen and T. Koornwinder *The convolution structure for Jacobi function expansions,* Ark. Mat., Vol. 11, 1973, pp. 245-262

[FS] G. B. Folland and E. M. Stein, Hardy Spaces on Homogeneous Groups, Mathematical Notes, No. 28, Princeton University Press, 1982

[GV] R. Gangolli and V. S. Varadarajan, *Harmonic Analysis of Spherical Functions on Real Reductive Groups,* A Series of Mordern Surveys in Math, Vol. 101, Springer, 1988

[K1] T. Kawazoe, *Atomic Hardy spaces on semisimple Lie groups,* Japan. J. Math., Vol. 11, 1985, pp. 293-343

[K2] T. Kawazoe, *L^1 estimates for maximal functions and Riesz transform on real rank 1 semisimple Lie groups,* J. Funct. Analysis, Vol. 157, 1998, pp. 327-357

[K3] T. Kawazoe, *Hardy spaces and maximal operators on real rank one semisimple Lie groups I,* Tohoku Math. J., Vol. 52, 2000, pp. 1-18

[K4] T. Kawazoe, *Hardy spaces and maximal operators on real rank one semisimple Lie groups II,* preprint

[L] J. Liu, *Maximal functions associated with the Jacobi transform,* Bull. London Math., vol. 32, 2000, pp. 1-7

[St1] E. M. Stein *Topics in Harmonic Analysis related to the Littlewood-Paley theory,* Ann. of Mathematics Studies, Num. 63, Princeton University Press, 1970

[St2] E. M. Stein, *Harmonic Analysis: real-variable methods, orthogonality, and oscillatory integrals,* Princeton Mathematical Series., Num. 43, Princeton University Press, 1993

[Str] J-O, Strömberg *Weak type L^1 estimates for maximal functions on non-compact symmetric spaces,* Ann. of Math., Vol. 114, 1981, pp. 115-126