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The Hilbert Transform on Marcinkiewicz Spaces

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abstract

In this work we extend the classical definition of Hilbert transform to the Marcinkiewicz space $\mathfrak{M}^p(\mathbf{R})$, $1 < p < +\infty$.

Let $\mathfrak{M}^p(\mathbf{R})$ be the set of functions $f \in L^p_{loc}(\mathbf{R}), 1 \le p < \infty$ such that

$$\limsup_{T \to +\infty} \frac{1}{2T} \int_{-T}^{T} |f(s)|^p \, ds < +\infty.$$

$\mathfrak{M}^p(\mathbf{R})$, $1 \le p < +\infty$ is a vector space and

$$\|f\|_p^p = \limsup_{T \to +\infty} \frac{1}{2T} \int_{-T}^{T} |f(s)|^p \, ds$$

defines a seminorm on $\mathfrak{M}^p(\mathbf{R})$ .It is possible to prove that $\mathfrak{M}^p(\mathbf{R}), 1 \le p < +\infty$ is complete with respect to such seminorm.

We have then the following

Proposition 1

Let $f \in \mathfrak{M}^p(\mathbf{R}), 1 \le p < +\infty$ and $\varphi \in L^1(\mathbf{R})$ be such that the function

$$\psi(x) = supess_{|t| \ge |x|} |\varphi(t)|$$

belongs to $L^1(\mathbf{R})$. Then the integral

$$\int_{-\infty}^{+\infty} f(x-t)\varphi(t)\, dt$$

exists for $x \in \mathbf{R}$ a.e.

Proof:

Since

$$\frac{1}{2T}\int_{-T}^{T}\int_{-\infty}^{+\infty}|f(x-t)\varphi(t)|\,dtdx=\int_{-\infty}^{+\infty}\frac{1}{2T}\left(\int_{-T}^{T}|f(x-t)|\,dx\right)|\varphi(t)|\,dt<+\infty,$$

we get that

$$\int_{-\infty}^{+\infty}f\left(x-t\right)\varphi\left(t\right)dt$$

exists for almost all x.■

Proposition 2 (cfr. [4] , Prop.1.25, page 13)

Let $f\in\mathfrak{M}^{p}\left(\mathbf{R}\right)$, $1\leq p<+\infty$, and let $\varphi\in L^{1}\left(\mathbf{R}\right)$ be such that the function

$$\psi\left(x\right)=supess_{|t|\geq|x|}\left|\varphi\left(t\right)\right|$$

belongs to $L^{1}\left(\mathbf{R}\right)$. Then, if we set $\varphi_{\varepsilon}(t)=\frac{1}{\varepsilon}\varphi(\frac{t}{\varepsilon})$, we get that

$$\lim_{\varepsilon\to0^{+}}\int_{-\infty}^{+\infty}f\left(x-t\right)\varphi_{\varepsilon}\left(t\right)dt=f\left(x\right)\int_{-\infty}^{+\infty}\varphi\left(t\right)dt,$$

for $x\in\mathbf{R}$ almost everywhere.

Proof:

Let $g\in L^{1}\left(a,b\right)$. Then for $t_{o}\in[a,b]$ almost everywhere, the limit

$$\lim_{h\to0}\frac{1}{h}\int_{t_{o}}^{t_{o}+h}\left|g\left(s\right)-g\left(t_{o}\right)\right|ds,\quad t_{o}+h\in[a,b]$$

exists and is equal to 0.

The set of Lebesgue points of g is the set of points $t_{o}\in[a,b]$ that satisfy such property.

Let x_{o} be a Lebesgue point of f, such that $f\left(x_{o}\right)$ is finite.

For any $\widetilde{\varepsilon}>0$ there exists $\delta_{\widetilde{\varepsilon}}>0$, such that

$$\int_{o}^{s}\left|f\left(x_{o}+t\right)-f\left(x_{o}\right)\right|dt\leq\widetilde{\varepsilon}\left|s\right|$$

for all s such that $|s|<\delta_{\widetilde{\varepsilon}}$.

We set $\delta_{\widetilde{\varepsilon}}=\delta$, for simplicity of notation.

One has then

$$\begin{aligned}&\left|\int_{-\infty}^{+\infty}f\left(x_{o}-t\right)\varphi_{\varepsilon}\left(t\right)dt-f\left(x_{o}\right)\int_{-\infty}^{+\infty}\varphi\left(t\right)dt\right|\\
&=\left|\int_{-\infty}^{+\infty}\left(f\left(x_{o}-t\right)-f\left(x_{o}\right)\right)\varphi_{\varepsilon}\left(t\right)dt\right|\\
&\leq\int_{-\infty}^{+\infty}\left|f\left(x_{o}-t\right)-f\left(x_{o}\right)\right|\left|\varphi_{\varepsilon}\left(t\right)\right|dt\\
&\leq\int_{-\infty}^{-\delta}\left|f\left(x_{o}-t\right)-f\left(x_{o}\right)\right|\psi_{\varepsilon}\left(t\right)dt\\
&\quad+\int_{\delta}^{+\infty}\left|f\left(x_{o}-t\right)-f\left(x_{o}\right)\right|\psi_{\varepsilon}\left(t\right)dt+\int_{-\delta}^{\delta}\left|f\left(x_{o}-t\right)-f\left(x_{o}\right)\right|\psi_{\varepsilon}\left(t\right)dt.\end{aligned}$$

Let us estimate first the integral

$$\int_{-\delta}^{\delta} |f(x_o - t) - f(x_o)| \psi_\varepsilon(t) dt.$$

One has

$$\begin{aligned}
&\int_{-\delta}^{\delta} |f(x_o - t) - f(x_o)| \psi_\varepsilon(t) dt \\
&= \int_{-\delta}^{\delta} \left(\frac{d}{dt} \int_o^t |f(x_o - s) - f(x_o)| ds \right) \psi_\varepsilon(t) dt \\
&= \int_o^t |f(x_o - s) - f(x_o)| ds \cdot \psi_\varepsilon(t) |_{-\delta}^{\delta} - \int_{-\delta}^{\delta} \left(\int_o^t |f(x_o - s) - f(x_o)| ds \right) d\psi_\varepsilon(t) \\
&= \int_o^{\delta} |f(x_o - s) - f(x_o)| ds \cdot \psi_\varepsilon(\delta) - \int_o^{-\delta} |f(x_o - s) - f(x_o)| ds \cdot \psi_\varepsilon(-\delta) \\
&\quad - \int_{-\delta}^{\delta} \left(\int_o^t |f(x_o - s) - f(x_o)| ds \right) d\psi_\varepsilon(t) \\
&\leq \widetilde{\varepsilon}\delta\psi_\varepsilon(\delta) + \widetilde{\varepsilon}\delta\psi_\varepsilon(-\delta) - \int_{-\delta}^{\delta} \left(\int_o^t |f(x_o - s) - f(x_o)| ds \right) d\psi_\varepsilon(t) \\
&= 2\widetilde{\varepsilon}\frac{\delta}{\varepsilon}\psi\left(\frac{\delta}{\varepsilon}\right) - \int_{-\delta}^{\delta} \left(\int_o^t |f(x_o - s) - f(x_o)| ds \right) d\psi_\varepsilon(t).
\end{aligned}$$

Let us observe that

$$\begin{aligned}
&- \int_{-\delta}^{\delta} \left(\int_o^t |f(x_o - s) - f(x_o)| ds \right) d\psi_\varepsilon(t) \\
&\leq - \int_{-\delta}^{\delta} \widetilde{\varepsilon} |t| d\psi_\varepsilon(t) = - \int_o^{\delta} \widetilde{\varepsilon} t d\psi_\varepsilon(t) - \int_{-\delta}^{o} \widetilde{\varepsilon} t d\psi_\varepsilon(t) \\
&= -\widetilde{\varepsilon} t \psi_\varepsilon(t) |_o^{\delta} + \widetilde{\varepsilon} \int_o^{\delta} \psi_\varepsilon(t) dt - \widetilde{\varepsilon} t \psi_\varepsilon(t) |_{-\delta}^{o} + \widetilde{\varepsilon} \int_{-\delta}^{o} \psi_\varepsilon(t) dt \\
&= -\widetilde{\varepsilon} t \psi_\varepsilon(t) |_o^{\delta} - \widetilde{\varepsilon} t \psi_\varepsilon(t) |_{-\delta}^{o} + 2\widetilde{\varepsilon} \int_o^{\delta} \psi_\varepsilon(t) dt \\
&\leq -\widetilde{\varepsilon} t \psi_\varepsilon(t) |_o^{\delta} - \widetilde{\varepsilon} t \psi_\varepsilon(t) |_{-\delta}^{o} + 2\widetilde{\varepsilon} \int_o^{+\infty} \psi_\varepsilon(t) dt \\
&\leq \widetilde{\varepsilon}\delta\psi_\varepsilon(\delta) + \widetilde{\varepsilon}\delta\psi_\varepsilon(-\delta) + 2\widetilde{\varepsilon} \int_o^{+\infty} \psi_\varepsilon(t) dt \\
&= \widetilde{\varepsilon} \left\{ \delta\psi_\varepsilon(\delta) + \delta\psi_\varepsilon(\delta) + 2 \int_o^{+\infty} \psi_\varepsilon(t) dt \right\} \leq c_o \widetilde{\varepsilon}.
\end{aligned}$$

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Let us observe that the function

$$\psi(x) = supess_{|t| \geq |x|} |\varphi(t)|$$

is even. In fact

$$\psi(-x) = supess_{|t| \geq |-x|} |\varphi(t)| = supess_{|t| \geq |x|} |\varphi(t)| = \psi(x).$$

Furthermore, if we set $r = |x|$, we get that $\psi(r) = \psi(|x|)$ is a decreasing function.
In fact if $r' > r$ one has

$$supess_{|t|\geq r'} |\varphi(t)| \leq supess_{|t|\geq r} |\varphi(t)|$$

since

$$\{t : |t| \geq r'\} \subset \{t : |t| \geq r\}.$$

For each $r > 0$ we have then

$$\int_{r<|x|<2r} \psi(x)\, dx \geq \int_{r<|x|<2r} \psi(2r)\, dx = 2r\psi(2r).$$

Let us observe that

$$\int r < |x| < 2r\psi(x)\, dx \longrightarrow 0$$

for $r \to 0^+$ and for $r \to +\infty$,
and hence

$$\psi(r)\, r \longrightarrow 0$$

for $r \to 0^+$ and for $r \to +\infty$.
Furthermore

$$\int r < |x| < 2r\psi(x)\, dx \leq \int \mathbf{R}\psi(x)\, dx$$

and so we get that $\psi(r)\, r$ is bounded in $\mathbf{R}_+$ (and hence in $\mathbf{R}$).
Let us give an estimate of the other two integrals:

$$\int_{\delta}^{+\infty} |f(x_o - t) - f(x_o)|\, \psi_\varepsilon(t)\, dt$$

and

$$\int_{-\infty}^{-\delta} |f(x_o - t) - f(x_o)|\, \psi_\varepsilon(t)\, dt.$$

It is sufficient to estimate the first one, i.e.

$$\int_{\delta}^{+\infty} |f(x_o - t) - f(x_o)|\, \psi_\varepsilon(t)\, dt.$$

Since $f \in \mathfrak{M}^p(\mathbf{R})$, for $|t| \geq \delta$ we get that

$$\int_o^t |f(x_o - s) - f(x_o)|\, ds \leq \int_o^t |f(x_o - s)|\, ds + \int_o^t |f(x_o)|\, ds$$
$$\leq \int_o^t |f(x_o - s)|\, ds + |f(x_o)|\,|t| \leq c_\delta\,|t|$$

and hence

$$\begin{aligned}
&\int_\delta^{+\infty} |f(x_o - t) - f(x_o)|\, \psi_\varepsilon(t)\, dt \\
&= \int_\delta^{+\infty} \frac{d}{dt}\left(\int_o^t |f(x_o - s) - f(x_o)|\, ds\right)\psi_\varepsilon(t)\, dt \\
&= \int_o^t |f(x_o - s) - f(x_o)|\, ds \cdot \psi_\varepsilon(t)\,|_\delta^{+\infty} - \int_\delta^{+\infty}\left(\int_o^t |f(x_o - s) - f(x_o)|\, ds\right) d\psi_\varepsilon(t) \\
&\leq \int_o^\delta |f(x_o - s) - f(x_o)|\, ds \cdot \psi_\varepsilon(\delta) - \int_\delta^{+\infty}\left(\int_o^t |f(x_o - s) - f(x_o)|\, ds\right) d\psi_\varepsilon(t) \\
&\leq \widetilde{\varepsilon}\delta\psi_\varepsilon(\delta) - c_\delta \int_\delta^{+\infty} |t|\, d\psi_\varepsilon(t) \\
&= \widetilde{\varepsilon}\delta\psi_\varepsilon(\delta) - c_\delta t\psi_\varepsilon(t)\,|_\delta^{+\infty} + c_\delta \int_\delta^{+\infty} \psi_\varepsilon(t)\, dt \\
&\leq \widetilde{\varepsilon}c + c_\delta \int_\delta^{+\infty} \psi_\varepsilon(t)\, dt = \widetilde{\varepsilon}c + c_\delta \int_{\frac{\delta}{\varepsilon}}^{+\infty} \psi(t)\, dt.
\end{aligned}$$

One may choose then $\eta > 0$ such that for every $\varepsilon > 0$ such that $\varepsilon < \eta$ one gets

$$\widetilde{\varepsilon}c + c_\delta \int_{\frac{\delta}{2}}^{+\infty} \psi(t)\, dt < \widetilde{\varepsilon}$$

and this ends the proof.■

Proposition 3 (see [3] ,page 386)

Let $f \in \mathfrak{M}^p(\mathbb{R})$, $1 \leq p < \infty$ be such that

$$\forall x \in \mathbb{R}, \quad \left|\int_0^t f\left(x + \widetilde{t}\right) d\widetilde{t}\right| \leq C\,|t|^\alpha, \quad 0 < \alpha < 1.$$

Then the limit

$$\lim_{N\to\infty} \frac{1}{\pi i}\int_{-N}^{N} \frac{f(t)}{t - z} dt, \quad z \in \mathbb{C}_+,$$

exists uniformly on compact subsets of $\mathbb{C}_+$.

Proof:

Let us set

$$F(z, N) = \frac{1}{\pi i}\int_{-N}^{N} \frac{f(t)}{t - z} dt$$

For $N'' > N'$

$$F(z, N'') - F(z, n') = \frac{1}{\pi i}\int_{-N''}^{N''} \frac{f(t)}{t - z} dt - \frac{1}{\pi i}\int_{-N'}^{N'} \frac{f(t)}{t - z} dt$$
$$= \frac{1}{\pi i}\int_{-N''}^{-N'} \frac{f(t)}{t - z} dt + \frac{1}{\pi i}\int_{N'}^{N''} \frac{f(t)}{t - z} dt.$$

Let us give an estimate of the integral

$$\frac{1}{\pi i}\int_{N'}^{N''}\frac{f(t)}{t-z}dt.$$

We have that

$$\begin{aligned}\int_{N'}^{N''}\frac{f(t)}{t-z}dt &= \int_{N'}^{N''}\frac{f(t)}{(t-x)-iy}dt\\ &= \int_{N'-x}^{N''-x}\frac{f(x+s)}{s-iy}ds\\ &= \int_{N'-x}^{N''-x}\frac{\frac{d}{dt}\int_0^t f(x+s)ds}{t-iy}dt\\ &= \left.\frac{\int_0^t f(x+s)ds}{t-iy}\right|_{N'-x}^{N''-x}+\int_{N'-x}^{N''-x}\frac{\int_0^t f(x+s)ds}{(t-iy)^2}dt.\end{aligned}$$

Hence

$$\begin{aligned}&\left|\frac{1}{\pi i}\int_{N'}^{N''}\frac{f(t)}{t-z}dt\right|\\ &\leq \frac{1}{\pi}\left(\frac{\left|\int_0^{N''-x}f(x+s)ds\right|}{|(N''-x)-iy|}+\frac{\left|\int_0^{N'-x}f(x+s)ds\right|}{|(N'-x)-iy|}+\int_{N'-x}^{N''-x}\frac{\left|\int_0^t f(x+s)ds\right|}{|t-iy|^2}dt\right)\\ &\leq \frac{1}{\pi}\left(\frac{C\,|N''-x|^{\alpha}}{\sqrt{(N''-x)^2+y^2}}+\frac{C\,|N'-x|^{\alpha}}{\sqrt{(N'-x)^2+y^2}}+\int_{N'-x}^{N''-x}\frac{C\,|t|^{\alpha}}{t^2+y^2}dt\right).\end{aligned}$$

Since $|x|\leq T,\quad y\geq y_o>0$ one may choose N', N'' such that the last quantity becomes less than $\frac{\varepsilon}{2}$. The second integral may be estimated in an analogous way.

Then the function

$$F(z)=\frac{1}{\pi i}\int_{-\infty}^{+\infty}\frac{f(t)}{t-z}dt=\lim_{N\to\infty}\frac{1}{\pi i}\int_{-N}^{N}\frac{f(t)}{t-z}dt,\quad z\in\mathbb{C}_+$$

is holomorphic in $\mathbb{C}_+=\{z=x+iy : y>0\}$.

Proposition 4

In the same hypotheses of above the limit

$$\forall x\in\mathbb{R},\varepsilon>0\quad \lim_{N\to\infty}\frac{1}{\pi}\int_{N\geq|t|\geq\varepsilon}\frac{f(x+t)}{t}dt.$$

exists.

Proof:

$$\int_{N\geq|t|\geq\varepsilon}\frac{f(x+t)}{t}dt=\int_{\varepsilon}^{N}\frac{f(x+t)}{t}dt+\int_{-N}^{-\varepsilon}\frac{f(x+t)}{t}dt.$$

Let us prove that

$$\lim_{N\to\infty}\int_{\varepsilon}^{N}\frac{f(x+t)}{t}dt.$$

exists.
We have that

$$\int_{\varepsilon}^{N''} \frac{f(x+s)}{s} ds - \int_{\varepsilon}^{N'} \frac{f(x+s)}{s} ds$$
$$= \int_{N'}^{N''} \frac{f(x+s)}{s} ds = \int_{N'}^{N''} \frac{\frac{d}{dt}\int_0^t f(x+s)ds}{t} dt$$
$$= \left.\frac{\int_0^t f(x+s)ds}{t}\right|_{N'}^{N''} + \int_{N'}^{N''} \frac{\int_0^t f(x+s)ds}{t^2} dt.$$

It follows that

$$\left| \cdot \frac{1}{\pi i} \int_{\varepsilon}^{N''} \frac{f(x+s)}{s} ds - \int_{\varepsilon}^{N'} \frac{f(x+s)}{s} ds \right|$$
$$\leq \left(\frac{C\,|N''|^{\alpha}}{|N''|} + \frac{C\,|N'|^{\alpha}}{|N'|} + \int_{N'}^{N''} \frac{C\,|t|^{\alpha}}{t^2} dt \right)$$

and we may conclude as usual.
Let us set

$$\lim_{N\to\infty} \frac{1}{\pi} \int_{N\geq|t|\geq\varepsilon} \frac{f(x+t)}{t} dt = \frac{1}{\pi} \int_{|t|\geq\varepsilon} \frac{f(x+t)}{t} dt = H_{\varepsilon} f(x), \quad x \in \mathbb{R}, \quad a.e.$$

By the theorem of Fatou-Privalov (cfr. [1] and [2], page 196) we get that the limit

$$\lim_{y\to 0^+} \frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{t f(x+s)}{t^2+y^2} dt = \lim_{y\to 0^+} v(x,y),$$

exists for $x \in \mathbf{R}$ **a.e**.
Let us show that

$$\lim_{y\to 0^+} |v(x,y) - H_y f(x)| = 0, \quad x \in \mathbb{R}, \quad a.e.$$

and hence

$$\lim_{y\to 0^+} \frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{t f(x+s)}{t^2+y^2} dt = \lim_{y\to 0^+} \frac{1}{\pi} \int_{|t|\geq y} \frac{f(x+t)}{t} dt$$
$$= \lim_{y\to 0^+} H_y f(x) = H f(x)$$

the Hilbert transform of f.
Let us define the function

$$\Phi(t) = \begin{cases} \frac{t}{t^2+1} - \frac{1}{t}, & |t| \geq 1 \\ \frac{t}{t^2+1}, & |t| < 1 \end{cases}$$

Let us observe that $\Phi \in L^1(\mathbb{R})$,

$$\int_{-\infty}^{+\infty} \Phi(t) dt = 0$$

and that $\Psi(x) = supess\, lim_{|t|\geq|x|} |\Phi(t)| \in L^1(\mathbb{R})$ since

$$\Psi(x) = supess\, lim_{|t|\geq|x|} |\Phi(t)| = \begin{cases} \frac{1}{|x|(x^2+1)}, & |x| \geq 1 \\ \frac{1}{2}, & |x| < 1. \end{cases}$$

We have then

$$v(x,y) - H_y f(x) = \frac{1}{\pi}\int_{-\infty}^{+\infty} \frac{tf(x+s)}{t^2+y^2}dt - \frac{1}{\pi}\int_{|t|\geq y} \frac{f(x+t)}{t}dt$$
$$= \frac{1}{\pi}\int_{-\infty}^{+\infty} f(x+s)\Phi_y(t)dt.$$

We may apply hence Proposition 1 and conclude that

$$\lim_{y\to 0^+} \frac{1}{\pi}\int_{-\infty}^{+\infty} \frac{tf(x+s)}{t^2+y^2}dt = Hf(x), \quad x\in\mathbb{R}, \quad a.e. \blacksquare$$

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