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On a Class of Multi-symplectic Structure

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abstract

By developing a work of Geoffrey Martin, we study a class of multi-symplectic structures, called symplectic structures of order k, in analogy with the well-known classical symplectic geometry. Also, we introduce the Liouville form of degree k and the notion of Hamiltonian systems and Hamiltonian $p-$ system on a manifold equipped with a symplectic structure of order k.

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I. INTRODUCTION

The $(k+1)-$form symplectic, introduced by Geoffrey Martin[1−12], have allowed to describe a class of dynamical structures that give a natural extension of electrodynamics to sections of a fibered manifolds pair (N, S, π) satisfying $\dim S = n$ and $\dim N = c(n,k) = n + C_n^k$.

A $(k+1)-$form symplectic on vector space V in the sense of Geoffrey Martin[12], is $(k+1)-$form ω, such that (1) ω is a non degenerate, and there exist a subspace $W \subset V$, such that (2) for $u, v \in W$ we have $i\,(u \wedge v) = 0$, and (3) $\dim W = \dim \Lambda^k\,(V/W)$ and $\dim W \geq \dim\,(V/W)$.

The approach proposed by Geoffrey Martin[12] concerns a multi-symplectic model for electrodynamics based on the notion of symplectic $(k+1)-$vector fields which generalizes the concept of cosymplectic structure on a Poisson manifold and he gives an extension of the Darboux-Moser -Weinstein theorem and presents a new class of dynamical structures[11 ,12].

The study proposed by geoffrey Martin have led us to introduce the notion of a symplectic structure of order k on a smooth manifold M of dimension $c(n,k) = n + C_n^k$ by a pair $(\theta, \mathfrak{F})$ wher $\mathfrak{F}$ is an $n-$codimensional foliation and θ is a non degenerate differential closed form of degree $k+1$ on M satisfying $i\,(X)\,i\,(Y)\,\theta = 0$ for all vector fields X, Y tangent to $\mathfrak{F}$.

The particular case $k = 1$ corresponds to a classical symplectic structure endowed with a Lagrangian foliation. This last structure is usually called a real polarization of a symplectic manifold or a quasi-cotangent structure in the sense of Molino, Clark and Goel[13].

The fundamental example of the symplectic structure of order k is the existence of a $(k+1)-$form on the space $\Lambda^k(T^*B)$, of $k-$differential forms over an $n-$dimensional manifold B, generalizing the Liouville form on the tangent bundle and having for local model

$$\sum_{1\leq i_1<i_2<\ldots<i_k\leq n} dx^{i_1 i_2\ldots i_k}\wedge dx^{i_1}\wedge\ldots\wedge dx^{i_k}.$$

The contact structures of superior order in the sense of[3], developed in the Mulhouse Mathematical laboratory whose idea has been suggested for a long time by G. Reeb, are connected to the symplectic structure of order k in analogy with classic bonds between symplectic structures and contact, or well, with bonds between the $k-$symplectic geometry and that defined by a contact $k-$systems[2,1].

A multi-symplectic geometry in a classical field theory was initiated by Dedecker in 1953 and was developed by Tulczyjew around 1968, Gawkedzki in 1972, Kijowski in 1973, Krupka since 1975, Szczyrba and Kondracki in 1979 (see for instance the works[8,12].

In this paper, we study the basic algebraic notions of the symplectic geometry of order k in analogy with the well-known classical symplectic and $k-$symplectic structures : the classification theorem of the symplectic structure of order k, the symplectic group $Sp_{(k,n)}(\mathbb{K})$ and its Lie algebra of order 2 in dimension 3 and of order 3 in dimension 6, the symplectic orthogonality of order k, the existence of totally isotropic subspaces, etc…

For the local differential case, we introduce the Liouville form of degree k and notion of Hamiltonian systems, Hamiltonian forms and Poisson brackets on a manifold of dimension $c(n,k)$, endowed with a symplectic structure of order k, we give also some properties of the Hamiltonian $p-$systems and the Hamiltonian of Riemannian metric.

II. SYMPLECTIC GROUP OF ORDER K

Let E be a vector space of dimension $c(n,k)=n+C_n^k$ over a commutative field $\mathbb{K}$ of characteristic different from 2, let Ω be an alternating $(k+1)-$linear form on E and let F be an $n-$codimensional subspace of E.

We say that (Ω,F) is a symplectic structure of order k on E, if the following conditions are satisfied (1) Ω is non degenerate, that is the map $X\longmapsto i(X)\Omega$, from E into $\Lambda^k(E)$ is injective and (2) $i(X)\,i(Y)\,\Omega=0$ for all $X,Y\in F$.

Considering for example the real space $E=\mathbb{R}^{c(n,k)}$ equipped with its canonical basis $(e_{i_1\ldots i_k},e_1,\ldots,e_n)_{1\leq i_1<\cdots<i_k\leq n}$ with dual basis $(\omega^{i_1\cdots i_k},\omega^1,\ldots,\omega^n)$, let F be the subspace of E spanned by the vectors $(e_{i_1\ldots i_k})_{1\leq i_1<\cdots<i_k\leq n}$ and let Ω be the alternating $(k+1)-$linear form given by $\Omega=\sum_{1\leq i_1<\cdots<i_k\leq n}\omega^{i_1\cdots i_k}\wedge\omega^{i_1}\wedge\ldots\wedge\omega^{i_k}$. Then, the pair (Ω,F) defines a symplectic structure of order k on this space; this structure is called a canonical symplectic of order k on $\mathbb{R}^{c(n,k)}$.

Théorème II.1 *Each symplectic structure* (Ω,F) *of order* k *on a* $\mathbb{K}-$*vector space of dimension* $c(n,k)$ *is isomorphic to the canonical symplectic of order* k *on* $\mathbb{K}^{c(n,k)}$*, that is, there exists a basis* $(\omega^{i_1\cdots i_k},\omega^1,\ldots,\omega^n)_{1\leq i_1<\cdots<i_k\leq n}$*, such that* $\Omega=\sum_{1\leq i_1<\cdots<i_k\leq n}\omega^{i_1\cdots i_k}\wedge\omega^{i_1}\wedge\ldots\wedge\omega^{i_k}$ *and* F *is defined by equations* $\omega^1=0,\ldots,\omega^n=0$.

For all $C=\left(C_j^i\right)$ be an $n\times n$ matrix where $C_j^i\in\mathbb{K}$ and for all integer k such that $1\leq k\leq n$, we associate a $C_n^k\times C_n^k$ square matrix denoted by ${}^{\wedge k}C$, indexed by the set $I=\{i_1\ldots i_k\mid 1\leq i_1<\ldots<i_k\leq n\}$ whose the entries ${}^{\wedge k}C_{j_1\cdots j_k}^{i_1\ldots i_k}$ are given by :

$${}^{\wedge k}C_{j_1\cdots j_k}^{i_1\ldots i_k}=\varepsilon_{s_1\ldots s_k}^{i_1\ldots i_k}C_{j_1}^{s_1}C_{j_2}^{s_2}\ldots C_{j_k}^{s_k}$$

where $\varepsilon^{i_1\ldots i_k}_{s_1\ldots s_k}$ is the Kronecker's tensor :

$$\varepsilon^{i_1\ldots i_k}_{s_1\ldots s_k} = \begin{cases} 0 & \text{if } \binom{i_1\ldots i_k}{s_1\ldots s_k} \text{ is not a permutation} \\ -1 & \text{if } \binom{i_1\ldots i_k}{s_1\ldots s_k} \text{ is an odd permutation} \\ +1 & \text{if } \binom{i_1\ldots i_k}{s_1\ldots s_k} \text{ is an even permutation,} \end{cases}$$

We have the following properties :

1. For $k = 1$, the matrix $^{\wedge 1}C$ coincides with C and for $k = n$ we have
$$^{\wedge n}C = \varepsilon^{1\ldots n}_{s_1\ldots s_n} C_1^{s_1} C_2^{s_2} \ldots C_n^{s_n} = \det C.$$
2. If C is an $n \times n$ symmetric matrix, then $^{\wedge k}C$ is also symmetric for all $k = 1, \ldots, n$.
3. If C is an $n \times n$ upper triangular matrix, then the matrix $^{\wedge k}C$ is also a upper triangular matrix for all $k = 1, \ldots, n$.
4. If C is an $n \times n$ diagonal matrix, then the matrix $^{\wedge k}C$ is also a diagonal matrix for all $k = 1, \ldots, n$.
5. If $C = \lambda I_n$ ($\lambda \in \mathbb{K}$) is an $n \times n$ a scalar matrix, then the matrix $^{\wedge k}C = \lambda^k I_{C_n^k}$ is also a scalar matrix for all $k = 1, \ldots, n$.
6. If C is an $n \times n$ skew symmetric matrix, then $^{\wedge k}C$ is a skew symmetric matrix if k is odd and it is symmetric if k is even.

Let E be a $c(n, k)-$dimensional vector space endowed with a symplectic structure of order k defined by the pair (Ω, F).

Let $(e_{i_1\ldots i_k}, e_1, \ldots, e_n)_{1\leq i_1<\cdots<i_k\leq n}$ be a symplectic basis of E of order k, that is, with respect to its dual basis $(\omega^{i_1\ldots i_k}, \omega^1, \ldots, \omega^n)_{1\leq i_1<\cdots<i_k\leq n}$, such that $\Omega = \sum_{1\leq i_1<\cdots<i_k\leq n} \omega^{i_1\ldots i_k} \wedge \omega^{i_1} \wedge \ldots \wedge \omega^{i_k}$and the subspace F is defined by equations $\omega^1 = 0, \ldots, \omega^n = 0$.

Let u be an endomorphism of E. We say that it preserves the symplectic structure (Ω, F) of order k if it leaves invariant both the differential form Ω and the subspace F, in other word, if (1) $u^*\Omega = \Omega$ and (2) $u(F) \subseteq F$.

We denote by $Sp_{(k,n)}(E)$ the group of all automorphisms of E preserving the symplectic structure (Ω, F) of order k. Then, we have $u \in Sp_{(k,n)}(E)$ if and only if, (1) $\Omega(u(X_0), \ldots, u(X_k)) = \Omega(X_0, \ldots, X_k)$, for all $X_0, \ldots X_k$ belonging to E and (2) $u(F) \subseteq F$. The group $Sp_{(k,n)}(E)$ is termed *symplectic group of order k.*

Let $Sp_{(k,n)}(\mathbb{K})$ be the group of matrices of symplectic automorphisms of order k of E with respect to the symplectic basis of order k. Then we have

Proposition II.1 *The group $Sp_{(k,n)}(\mathbb{K})$ is formed by the matrices of the type*

$$\begin{pmatrix} (^{\wedge k}C^T)^{-1} & B \\ 0 & C \end{pmatrix}$$

such as

$$\sum_{s=1}^{k+1} (-1)^{s-1} \left(^{\wedge k}C^T B\right)_{i_s}^{i_1\ldots i_{s-1} i_{s+1}\ldots i_{k+1}} = 0, \tag{2.1}$$

where $C = \left(C_i^j\right)$ is an $n \times n$ matrix whose the associated matrix $^{\wedge k}C^T$ is invertible and $B = \left(B_i^{i_1\ldots i_k}\right)$ is a $C_n^k \times n$ matrix.

Proof. Let $u \in Sp_{(k,n)}(\mathbb{K})$. Since $u(F) \subseteq F$, one can write

$$u(e_{i_1 \dots i_k}) = A^{j_1 \dots j_k}_{i_1 \dots i_k} e_{j_1 \dots j_k} \quad , \quad u(e_i) = B_i^{i_1 \dots i_k} e_{i_1 \dots i_k} + C_i^j e_j \ ,$$

where $A^{j_1 \dots j_k}_{i_1 \dots i_k}, B_i^{i_1 \dots i_k}$ and $C_i^j \in \mathbb{K}$.

For $X_1 = e_{r_1}, \dots, X_k = e_{r_k}$ with $1 \leq r_1 < \dots < r_k \leq n$ and $X_0 = e_{j_1 \dots j_k}$, the relationships

$$\Omega(u(X_0), u(X_1), \dots, u(X_k)\ ) = \Omega(X_0, X_1, \dots, X_k), \tag{2.2}$$

and

$$\Omega(e_{i_1 \dots i_k}, e_{s_1}, e_{s_2}, \dots, e_{s_k}) = \sum_{1 \leq l_1 < \cdots < l_k \leq n} \delta^{l_1 \dots l_k}_{i_1 \dots i_k} \varepsilon^{l_1 \dots l_k}_{s_1 \dots s_k} \ = \varepsilon^{s_1 \dots s_k}_{i_1 \dots i_k},$$

imply,

$$A^{i_1 \dots i_k}_{j_1 \dots j_k} C^{s_1}_{r_1} C^{s_2}_{r_2} \dots C^{s_k}_{r_k} \Omega(e_{i_1 \dots i_k}, e_{s_1}, e_{s_2}, \dots, e_{s_k}) = \Omega(e_{j_1 \dots j_k}, e_{r_1}, e_{r_2}, \dots, e_{r_k}),$$

and therefore,

$$\begin{aligned} \textstyle\sum_{s_1 \dots s_k\,;\, i_1 \dots i_k} A^{i_1 \dots i_k}_{j_1 \dots j_k} C^{s_1}_{r_1} C^{s_2}_{r_2} \dots C^{s_k}_{r_k} \varepsilon^{s_1 \dots s_k}_{i_1 \dots i_k} \ &= \sum_{i_1 \dots i_k} A^{i_1 \dots i_k}_{j_1 \dots j_k} \sum_{s_1 \dots s_k} C^{s_1}_{r_1} C^{s_2}_{r_2} \dots C^{s_k}_{r_k} \varepsilon^{s_1 \dots s_k}_{i_1 \dots i_k} \\ &= \varepsilon^{r_1 \dots r_k}_{j_1 \dots j_k}. \end{aligned}$$

Consequently,

$$\sum_{i_1 \dots i_k} A^{i_1 \dots i_k}_{j_1 \dots j_k} \ {}^{\wedge k}C^{i_1 \dots i_k}_{r_1 \dots r_k} = \varepsilon^{r_1 \dots r_k}_{j_1 \dots j_k} = \delta^{r_1 \dots r_k}_{j_1 \dots j_k}$$

but $1 \leq r_1 < \dots < r_k \leq n$, then $A \left({}^{\wedge k}C \right)^T = I_{C_n^k}$, this proves that $A = \left({}^{\wedge k}C^T \right)^{-1}$.
The relationship

$$\Omega(u(e_{i_1}), \dots, u(e_{i_k}), u(e_{i_{k+1}})) = 0,$$

for $1 \leq i_1 < i_2 < \dots < i_k < i_{k+1} \leq n$, implies

$$\sum_{s=1}^{k+1} (-1)^{s-1} B^{j_1^s \dots j_k^s}_{i_s} \prod_{l=1\,;\, l \neq s} C^{j_l}_{i_l} \Omega(e_{j_1^s \dots j_k^s}, e_{j_1}, \dots, \overset{\vee}{e}_{j_s}, \dots, e_{j_{k+1}}) = 0,$$

here the hat over an index means that this index is omitted; then,

$$\sum_{s=1}^{k+1} (-1)^{s-1} B^{j_1^s \dots j_k^s}_{i_s} \prod_{l=1\,;\, l \neq s} C^{j_l}_{i_l} \varepsilon^{j_1^s \dots j_{s-1}^s, j_s^s \dots j_k^s}_{j_1 \dots j_{s-1}, j_{s+1} \dots j_{k+1}} = 0$$

and thus,

$$\sum_{s=1}^{k+1} (-1)^{s-1} \sum_{1 \leq j_1 < \dots \overset{\vee}{j}_s \dots < j_{k+1} \leq n\,;\, 1 \leq j_1^s < j_2^s < \dots < j_k^s \leq n} B^{j_1^s \dots j_k^s}_{i_s} \prod_{l=1\,;\, l \neq s} C^{j_l}_{i_l} \ \varepsilon^{j_1^s \dots j_{s-1}^s, j_s^s \dots j_k^s}_{j_1 \dots j_{s-1}, j_{s+1} \dots j_{k+1}} = 0$$

it follows,

$$\sum_{s=1}^{k+1}(-1)^{s-1}\sum_{1\leq j_1^s<\ldots<j_k^s\leq n} B_{i_s}^{j_1^s\cdots j_k^s}\sum_{1\leq j_1<\ldots\overset{\vee}{j_s}\ldots<j_{k+1}\leq n}\ \prod_{l=1\,;\,l\neq s} C_{i_l}^{j_l}\ \varepsilon_{j_1\cdots j_{s-1},j_{s+1}\cdots j_{k+1}}^{j_1^s\cdots j_{s-1}^s,j_s^s\cdots j_k^s}=0$$

hence,

$$\sum_{s=1}^{k+1}(-1)^{s-1}\sum_{1\leq j_1^s<j_2^s<\ldots<j_k^s\leq n} B_{i_s}^{j_1^s\cdots j_k^s}\ \wedge^k C_{i_1\ldots i_{s-1},i_{s+1}\ldots i_{k+1}}^{j_1^s\cdots j_k^s}=0$$

consequently

$$\sum_{s=1}^{k+1}(-1)^{s-1}\left(\wedge^k C^T B\right)_{i_s}^{i_1\ldots i_{s-1}i_{s+1}\ldots i_{k+1}}=0.$$

The proposition is then proven.
The relationship 2.1 becomes

$$\left(C^T B\right)_{i_1}^{i_2}=\left(C^T B\right)_{i_2}^{i_1}$$

this proves that the matrix $C^T B$ is symmetric and we find the classic case of symplectic structures.

We suppose there, that $\mathbb{K}=\mathbb{R}$ or $\mathbb{K}=\mathbb{C}$. In this case, $Sp_{(k,n)}(E)$ is a Lie group. The Lie algebra of this group, which is denoted by $\mathfrak{sp}_{(k,n)}(E)$ and identified below with the tangent space of the symplectic group $Sp_{(k,n)}(E)$ of order k in the unity mapping of E, is formed by the endomorphisms u of E according to :

$$\Omega(u(X_0),X_1,\ldots,X_k)+\Omega(X_0,u(X_1),\ldots,X_k)+\ldots+\Omega(X_0,X_1,\ldots,u(X_k))=0,$$

for all $X_0,X_1,\ldots,X_k\in E$.

The determination of these Lie algebras is equivalent to one of an algebraic equations system with $(k+1)\left(n+C_n^k\right)$ variables. We restrict ourselves to cases $(n=2,\ k=2)$ and $(n=3,\ k=2)$ only.

We denote by $\mathfrak{sp}_{(k,n)}(\mathbb{K})$ the Lie algebra of all matrices of endomorphisms of E belonging to the symplectic Lie algebra $\mathfrak{sp}_{(k,n)}(E)$, with respect to the symplectic basis of order k of E.

For $n=2$ and $k=2$, the Lie algebra $\mathfrak{sp}_{(2,2)}(E)$ is formed by the matrices of type

$$\begin{pmatrix} -Tr\ C & B \\ 0 & C \end{pmatrix}$$

where C and B are respectively 2×2 and 1×2 matrices whose entries belong to $\mathbb{K}$.

For $n=3$ and $k=2$, the Lie algebra $\mathfrak{sp}_{(2,3)}(E)$ is formed by the matrices of type

$$\begin{pmatrix} A & B \\ 0 & -({}^tA) \end{pmatrix}$$

where A and B are two 3×3 matrices whose entries belong to $\mathbb{K}$.

III. SYMPLECTIC GEOMETRY OF ORDER K

Let E be a vector space of dimension $c(n,k)$ over a commutative field $\mathbb{K}$ with characteristic different from 2, equipped with a symplectic structure (Ω, F) of order k.

1. Let x and y be two vectors of E. We say that x is orthogonal to y, with respect to (Ω, F), or symbolically, $x \perp y$, if $i(x)i(y)\Omega = 0$.
2. Let L and M be two vector subspaces. We say that L is orthogonal to M, with respect to (Ω, F), or symbolically, $L \perp M$, if $x \perp y$ for all $x \in L$ and $y \in M$.

The symplectic orthogonal of order k, of a non empty set A, is defined by :

$$A^{\perp} = \{x \in E \mid i(x)i(y)\Omega = 0, \text{ for all } y \in A\}.$$

We say that E is an orthogonal sum of two vector subspaces L and M (of E) if we have : $E = L \oplus M$ and $L = M^{\perp}$.

Remarque III.1 *1. The non degeneracy of Ω is equivalent to $E^{\perp} = \{0\}$.*

2. The symplectic orthogonal $X^{\perp}$ of order 2, of an element $X \in E$, is not necessarily a hyperplane. Considering, for instance, the real space $\mathbb{R}^6$ endowed with its canonical symplectic structure of order 2 defined by :

(a) the three form

$$\Omega = \omega^4 \wedge \omega^1 \wedge \omega^2 + \omega^5 \wedge \omega^1 \wedge \omega^3 + \omega^6 \wedge \omega^2 \wedge \omega^3$$

(b) the subspace F defined by equations $\omega^1 = 0$, $\omega^2 = 0$ and $\omega^3 = 0$,

where $\{\omega^1, \ldots, \omega^6\}$ is the dual basis of the canonical basis $\{e_1, \ldots, e_6\}$ of $\mathbb{R}^6$. The symplectic orthogonal of order 2 of e_4 is the vector subspace $\{e_3, e_4, e_5, e_6\}$, then $\dim\{e_4^{\perp}\} = 4$, consequently, $e_4^{\perp}$ is not a hyperplane of E.

For all non empty subsets A and B of E, we have :

Proposition III.1 *1. $A \subseteq B \Longrightarrow B^{\perp} \subseteq A^{\perp}$*

2. $A \subseteq A^{\perp\perp}$.

A vector subspace L of E is called totally isotropic if $L \subseteq L^{\perp}$.

Définition III.1 *A vector subspace L of E is totally isotropic subspace, if and only if, $i(x)i(y)\Omega = 0$, for all $x, y \in L$.*

Every totally isotropic subspace is contained in a maximal totally isotropic subspace.

A maximal totally isotropic subspace will be called a Lagrangian subspace of E.

Proposition III.2 *Every element of the symplectic $Sp_{(k,n)}(E)$ of order k transforms the totally isotropic subspaces (resp. Lagrangian subspaces) of E in totally isotropic subspaces (resp. Lagrangian subspaces).*

Proposition III.3 *Let L be a Lagrangian subspace of E, and let M_0 be a totally isotropic subspace of E supplementary to L. Then there exists a Lagrangian subspace M of E supplementary to L and containing M_0.*

Proposition III.4 *For all vector subspace L of E, the following properties are equivalent :*

1. L is a Lagrangian subspace of E,

2. $L = L^{\perp}$.

Proposition III.5 *Let L be a vector subspace of E. If L is totally isotropic subspace of E, then* $\dim L \leq c(n,k)$.

Proof. Let L be a totally isotropic subspace of E, $\{f_1, \ldots, f_p\}$ a basis of L, which extends to a basis of E denoted by $\{f_1, \ldots, f_p, u_1, \ldots, u_{m-p}\}$ and let $\{f_1^*, \ldots, f_p^*, u_1^*, \ldots, u_{m-p}^*\}$ be its dual basis. We can write the form Ω under the form :

$$\Omega = \sum a_{ji_1\ldots i_k} f_j^* \wedge u_{i_1}^* \wedge \ldots \wedge u_{i_k}^* + \sum b_{i_1\ldots i_{k+1}} u_{i_1}^* \wedge \ldots \wedge u_{i_{k+1}}^* .$$

Let C be the $C_{m-p}^k \times p$ matrix where the entries are $c_{(i_1\ldots i_k,j)} = a_{ji_1\ldots i_k}$, with $1 \leq i_1 < \ldots < i_k \leq m-p, 1 \leq j \leq p$.
The matrix C is of rank p. In fact, for all $X = (x_1, \ldots, x_p) \in \mathbb{K}'$, the relationship $C^T X = 0$ implies that the vector $x_1 f_1 + \ldots + x_p f_p$ belongs to $A(\Omega)$; the non-degeneracy of Ω implies that $X = 0$, thus the matrix C is of rank p. It follows that we have, in particular, $p \leq C_{m-p}^k$, consequently $p \leq C_n^k$.
Let E be a vector space of dimension $c(n,k) = C_n^k + n$, and let (Ω, F) be a symplectic structure of order k on E. Let $\widetilde{\eta}$ be the mapping from E into the $\mathbb{K}-$vector space $\hom(E, \Lambda^{k-1}(E))$ defined by $\widetilde{\eta}(x)(y) = i(x)i(y)\Omega$ for every $x, y \in E$. The non-degeneracy of Ω implies that the mapping $\widetilde{\eta}$ is injective.
For all endomorphism u of E, we associate a map from $\hom(E, \Lambda^{k-1}(E))$ into itself, denoted by ${}^t u$ and called *transpose* of u, defined by ${}^t u(\xi) = \xi \circ u$, for each $\xi \in \hom(E, \Lambda^{k-1}(E))$.
For all $u, v \in End(E)$ and $\lambda \in \mathbb{K}$ we have :

1. ${}^t(u+v) = {}^t u + {}^t v \quad , \qquad {}^t(\lambda u) = \lambda\, {}^t u$,
2. ${}^t(u \circ v) = {}^t v \circ {}^t u$,
3. ${}^t\,(id_E)$ is the identity mapping of $\hom(E, \Lambda^{k-1}(E))$,
4. if $u \in GL(E)$ then ${}^t u \in GL(\hom(E, \Lambda^{k-1}(E))$
   and $({}^t u)^{-1} = {}^t\left(u^{-1}\right)$.

Let u be an endomorphism of E verifying the hypothesis :

$$Im\left({}^t u \circ \widetilde{\eta}\right) \subseteq Im\widetilde{\eta}. \tag{3.1}$$

For each $t \in E$, there exists a unique $t^{'} \in E$ such that $\left({}^t u \circ \widetilde{\eta}\right)(t) = \widetilde{\eta}(t^{'})$. It is clear that the association $t \longmapsto t^{'}$ defines a mapping from E into itself, denoted by u^*, such that for all t and x belonging to E, we have :

$$i\,(t)\, i\,(u(x))\, \Omega \ = i\,(u^*(t))\, i\,(x)\, \Omega\ . \tag{3.2}$$

Consequently, we deduce that for all $u, v \in End(E)$ satisfying the relationship (3.1) and $\lambda \in \mathbb{K}$, we have the following properties :

1. the mapping u^* is linear,
2. $(u+v)^* = u^* + v^*, (\lambda u)^* = \lambda u^*$
3. $(u \circ v)^* = v^* \circ u^*, u^{**} = u, (id_E)^* = id_E$
4. if $u \in GL(E)$ then $u^* \in GL(E)$, and we have $(u^*)^{-1} = (u^{-1})^*$.

Conversely, let u be an element of $End(E)$ in such a way that there exists an endomorphism u^* of E satisfying :

$$i(t)i(u(x))\Omega = i(u^*(t))i(x)\Omega,$$

for all $t, x \in E$; then we have ${}^t u(\widetilde{\eta}(t)) = \widetilde{\eta}(u^*(t))$. Consequently u satisfies the relationship 3.1. Furthermore :
For all endomorphism u of E, the following properties are equivalent :

Proposition III.6 *1. There exists an endomorphism* u^* *of* E *such that*

$$i(t)i(u(x))\Omega = i(u^*(t))i(x)\Omega,$$

for all $t, x \in E$.

2. $Im\,({}^t u \circ \widetilde{\eta}) \subseteq Im\widetilde{\eta}$.

If an endomorphism u of E satisfies one of the equivalent conditions of the previous proposition, the endomorphism u^* defined above is called an adjoint endomorphism of u.

Remarque III.2 *For* $k = 1$, *the mapping* $\widetilde{\eta}$ *is an isomorphism of vector spaces from* E *into* E^*, *and the relationship* $Im\,({}^t u \circ \widetilde{\eta}) \subseteq Im\widetilde{\eta} = E^*$ *is satisfied for any endomorphism* u *of* E*; hence, each endomorphism of* E *possesses an adjoint endomorphism. This situation is not automatic when* $k \geq 2$, *because* $Im\widetilde{\eta}$ *is of dimension* $C_n^k + n$ *(*$\widetilde{\eta}$ *is injective); then it is strictly contained in* $\hom(E, \Lambda^{k-1}(E))$.

IV. DIFFERENTIAL SYMPLECTIC STRUCTURE OF ORDER K

Let B be a differential manifold, (T^*B, p, B) its cotangent bundle, let $M = \Lambda^k(T^*B)$ be the space of differential $k-$forms on B and let $(\Lambda^k(T^*B), \pi, B)$ be the bundle over B of differential $k-$forms on B. For each local coordinates system $(U, \varphi) = (U, x^1, \ldots, x^n)$ of B, we associate the pair $\left(\widetilde{U}, \tilde{\varphi}\right)$, where $\widetilde{U} = \pi^{-1}(U)$ and $\tilde{\varphi}$ is the mapping from $\tilde{U}$ into $U \times \mathbb{R}^{C_n^k}$ defined by :

$$\tilde{\varphi}(\omega_x) = \left(\omega_x\left(\frac{\partial}{\partial x^{i_1}}, \ldots, \frac{\partial}{\partial x^{i_k}}\right), \varphi(\pi(\omega_x))\right)_{1 \leq i_1 < \ldots < i_k \leq n}.$$

For $\omega_x = \sum x^{i_1 i_2 \ldots i_k} dx^{i_1} \wedge \ldots \wedge dx^{i_k}$, we have $\tilde{\varphi}(\omega_x) = (x^{i_1 i_2 \ldots i_k}, \varphi(x))_{1 \leq i_1 < \ldots < i_k \leq n}$, then $(\tilde{U}, \tilde{\varphi}) = (\tilde{U}, x^{i_1 i_2 \ldots i_k},\ x^1, \ldots, x^n)_{1 \leq i_1 < \ldots < i_k \leq n}$ is a local coordinates system of $M = \Lambda^k(T^*B)$.

Définition IV.1 *Let* α *be a differential* $k-$*form on* B *and let* $\lambda\,(\alpha)$ *be the differential* $k-$*form on* M, *defined by :*

$$\lambda(\alpha)\,(X_1(\alpha), \ldots, X_k(\alpha)) = \alpha(x)(p^T(X_1(\alpha), \ldots p^T(X_k(\alpha)))$$

for all $X_1, \ldots, X_k \in \mathfrak{X}\,(M)$.

The differential $k-$form λ is called the *Liouville form of degree* k.

Proposition IV.1 *With respect to the local coordinates system*

$$(x^{i_1 i_2 \ldots i_k}, x^1, \ldots, x^n)_{1 \leq i_1 < i_2 < \ldots < i_k \leq n},$$

the exterior derivative of the Liouville form of degree k *is given by :*

$$\Omega = \sum_{1 \leq i_1 < i_2 < \ldots < i_k \leq n} dx^{i_1 i_2 \ldots i_k} \wedge dx^{i_1} \wedge \ldots \wedge dx^{i_k}.$$

Let M be a smooth manifold of dimension $c(n,k)$ equipped with a foliation $\mathfrak{F}$ of codimension n, let Ω be a closed differential form on M of degree $k+1$ and let E be the sub-bundle of TM defined by the foliation $\mathfrak{F}$.

Définition IV.2 *We say that* (Ω, E) *is a symplectic structure of order* k *on* M *if the following properties are satisfied :*

1. *the differential form* Ω *is non degenerate, that is, the mapping*

$$X \longmapsto i(X)\Omega,$$

from $\mathfrak{X}\,(M)$ *into* $\Lambda^k\,(M)$, *is injective,* *2. $i(X)i(Y)\Omega = 0$, for all $X, Y \in \Gamma(E)$,*

where $\Gamma(E)$ is the set of the cross sections of the sub-bundle E.

Exemples IV.1 *1.* ***Canonical symplectic structure of order*** k.

Let $M = \mathbb{R}^{c(n,k)}$ be the $c(n,k)-$dimensional real space equipped with its Cartesian coordinates $(x^{i_1 i_2 \ldots i_k}, x^1, \ldots, x^n)_{1 \le i_1 < i_2 < \ldots < i_k \le n}$. Let E be the sub-bundle of $T\mathbb{R}^{c(n,k)}$ defined by equations $dx^1 = 0, \ldots, dx^n = 0$, and let Ω be the $(k+1)-$ differential form on M defined by :

$$\Omega = \sum_{1 \le i_1 < i_2 < \ldots < i_k \le n} dx^{i_1 i_2 \ldots i_k} \wedge dx^{i_1} \wedge \ldots \wedge dx^{i_k}.$$

The pair (Ω, E) defines a symplectic structure of order k on M, called the canonical symplectic structure of order k.

2. ***Symplectic structure of order k on the bundle over B of differential $k-$forms.*** *Let B be a differential manifold, (T^*B, p, B) its cotangent bundle, $M = \Lambda^k(T^*B)$ the space of differential $k-$forms on B and $(\Lambda^k(T^*B), \pi, B)$ the bundle over B of differential $k-$forms on B, and let $\Omega = d\lambda$ be the differential derivative of the Liouville form of degree k. Then (Ω, E) determines a symplectic structure of order k on $M = \Lambda^k(T^*B)$, where E is defined by the fibration $\pi : \Lambda^k(T^*B) \longrightarrow B$.*

3. ***Symplectic structure of order k on spheres.*** *Let S^n be the unit sphere of $\mathbb{R}^{n+1}$, it is an orientable connected compact manifold. If the integer n is odd, there exists a vector field without singularity on S^n, for example, the vector field*

$$X(x_1, \ldots, x_{n+1}) = (x_2, -x_1, \ldots, x_{n+1}, -x_n),$$

is without singularity on S^n (if $n = 3$, the vector fields $X_1(x) = ix$, $X_2(x) = jx$ and $X_3(x) = kx$ are without singularity on S^3, where $1, i, j, k$ is the basis of the non commutative field of quaternions).

Let E be the $1-$dimensional subbundle defined by X and Ω be a volume form S^n. The pair (Ω, E) is a symplectic structure of order $n-1$ on S^n.

By the Lefschetz theorem, if n is even, the sphere S^n does not admit a symplectic structure of order $n-1$ subordinate to a foliation defined by a vector field without singularity.

Let M be a $c(n,k)-$ dimensional smooth manifold. A *symplectic Darboux atlas* of order k is an atlas $\mathfrak{A}$ whose the changes of local coordinates belong to the pseudo-group Γ of local diffeomorphisms of $\mathbb{R}^{c(n,k)}$ leaving the canonical symplectic structure of order k invariant, in other word, if the atlas $\mathfrak{A}$ defines a $\Gamma-$structure on M.
Let M be a $c(n,k)-$dimensional manifold equipped with a symplectic Darboux atlas of order k. Then M is equipped with a symplectic structure (Ω, E) of order k such that :

$$\Omega = \sum_{1 \le i_1 < i_2 < \ldots < i_k \le n} dx^{i_1 i_2 \ldots i_k} \wedge dx^{i_1} \wedge \ldots \wedge dx^{i_k}$$

for all $(U, \varphi = (x^{i_1 i_2 \ldots i_k}, x^1, \ldots, x^n)_{1 \le i_1 < i_2 < \ldots < i_k \le n}) \in \mathfrak{A}$, and the subbundle E is defined through equations $dx^1 = 0, \ldots, dx^n = 0$.

Let M be a smooth manifold of dimension $c(n,k) = n + C_n^k$ equipped with a symplectic structure (Ω, E) of order k, and let j be the mapping from $\mathfrak{X}(M)$ into $\Lambda^k(M)$ defined by $j(X) = i(X)\Omega$, for all $X \in \mathfrak{X}(M)$.
A vector field X on M is called an infinitesimal automorphism for E, if, the local one parameter group associated with X, on a neighborhood of an arbitrary point of M, leaves the sub-bundle E invariant. We denote by $\mathfrak{I}(M, E)$ the Lie algebra of infinitesimal automorphisms of $\mathfrak{F}$.
A vector field X on M is called a Hamiltonian system if it is an infinitesimal automorphism for the symplectic structure (Ω, E) of order k, that is, the following conditions are satisfied : (1) X is an infinitesimal automorphism for E and (2) $L_X\Omega = 0$, where L_X is the Lie derivative with respect to X.
The relationship $L_X\Omega = i(X)d\Omega + di(X)\Omega$ shows that the second condition of the previous definition

is equivalent to requiring that the $k-$differential form $i(X)\Omega$ is closed.

Let X be a Hamiltonian system, it results from the Poincaré lemma that for every $x \in M$, there exists an open neighborhood U of M containing x and a differential form H of degree $k-1$ on U, such that $j(X) = -dH$. Conversely, $H \in \Lambda^{k-1}(M)$ such that $dH \in j(\Im(M,E))$, there exists a unique vector field on M, denoted by X_H , and called the Hamiltonian system associated to H, such that $j(X) = -dH$.
The $(k-1)-$differential form H is called a Hamiltonian form of the symplectic structure (Ω, E) of order k. Let H and K be two Hamiltonian forms and X_H, X_K the associated Hamiltonian systems. The Lie bracket $[\ X_H, X_K]$ is a Hamiltonian system; more precisely, the mapping denoted by $\{H, K\}$ of $\Lambda^{k-1}(M)\times\ \Lambda^{k-1}(M)$ into $\Lambda^{k-1}(M)$ defined by :

$$\{H, K\} = -i(X_H)i(X_K)\Omega$$

Satisfies

$$[X_H, X_K] = X_{\{H,K\}}.$$

The mapping $\{H, K\}$ will be called the Lie bracket of Hamiltonian forms H and K. In fact, we have

$$\begin{aligned} i\left([X_H, X_K]\right)\Omega &= [L_{X_H}, i(X_H)]\,\Omega = d(i\left(X_H\right)i\left(X_K\right)\Omega) \\ &= -d\left\{H, K\right\}. \end{aligned}$$

Let B be a differential manifold equipped with a Riemannian metric g, let $M = \Lambda^k(T^*B)$ be the space of differential $k-$forms on B endowed with the symplectic structure (Ω, E) of order k, where $\Omega = d\lambda$ be the differential derivative of the Liouville form of degree k and E is the subbundle of TM defined by the fibration $\pi : \Lambda^k(T^*B) \longrightarrow B$.
A Hamiltonian mapping of a metric g^3, is the function

$$H = \sum g^{i_1 \ldots i_k j_1 \ldots j_k}(x) x^{i_1 \ldots i_k} x^{j_1 \ldots j_k},$$

where

$$g^{i_1 \ldots i_k j_1 \ldots j_k} = \det \begin{pmatrix} g^{i_1 j_1} & \ldots & g^{i_1 j_k} \\ \vdots & \vdots & \vdots \\ g^{i_k j_1} & \ldots & g^{i_k j_k} \end{pmatrix},$$

and where $g^{ij}g_{jl} = \delta^i_l$, g_{ij} being the components of the metric g with respect to the coordinates system $\left(x^i\right)_{1 \le i \le n}$ of B. Let

$$\begin{aligned} X &= \sum a_{i^1_1 \ldots i^1_k \ldots i^k_1 \ldots i^k_k} \frac{\partial}{\partial x^{i^1_1 \cdots i^1_k}} \wedge \ldots \wedge \frac{\partial}{\partial x^{i^k_1 \cdots i^k_k}} + \sum c_{j_1 \cdots j_k} \frac{\partial}{\partial x^{j_1}} \wedge \ldots \wedge \frac{\partial}{\partial x^{j_k}} \\ &+ \sum b_{i^1_1 \ldots i^1_k \ldots i^p_1 \ldots i^p_k j_1 \ldots j_{k-p}} \frac{\partial}{\partial x^{j_{k-p}}} \wedge \ldots \wedge \frac{\partial}{\partial x^{j_1}} \wedge \frac{\partial}{\partial x^{i^p_1 \cdots i^p_k}} \wedge \ldots \wedge \frac{\partial}{\partial x^{i^1_1 \cdots i^1_k}}. \end{aligned}$$

be a $k-$vector field on M. If $i(X)\Omega = -dH$, then we have,

$$i(X)\Omega = \sum_{p=1}^{n} \left(\sum_{s=1}^{k} (-1)^{k-s} \sum_{1 \le i_1 < \ldots < i_{s-1} < i_s = p < i_{s+1} < \ldots < i_k \le n} b_{i_1 \ldots i_k, i_1, \ldots, \overset{\vee}{i_s}, \ldots, i_k} \right) dx^p$$

$$+ \sum_{1 \le i_1 < \ldots < i_k \le n} c_{i_1 \ldots i_k} dx^{i_1 \ldots i_k},$$

consequently,

$$c_{i_1 \ldots i_k} = -\frac{\partial H}{\partial x^{i_1 \ldots i_k}}$$

and

$$\sum_{s=1}^{k}(-1)^{k-s}\left(\sum_{1\leq i_1<\ldots<i_{s-1}<i_s=p<i_{s+1}<\ldots<i_k\leq n} b_{i_1\ldots i_k,i_1,\ldots,i_{s-1},i_{s+1},\ldots,i_k}\right)=-\frac{\partial H}{\partial x^p}.$$

Let M be a smooth manifold of dimension $c(n,k)$ equipped with a symplectic structure (Ω, E) of order k and let $A(M)$ be the exterior algebra of multivector fields on M.
A $p-$vector field $(p = 1, \ldots, k)$, is called a Hamiltonian $p-$system if $i(X)\Omega$ is closed. We recall ([9],[10]) that there exist a unique $\mathbb{R}-$linear mapping, defined on $A(M) \times A(M)$ with values in $A(M)$, called the Schouten-Nijenhuis bracket and denoted by $(X,Y) \longmapsto [X,Y]$, which satisfies the following properties :

1. $[f, g] = 0$, for all $f,\ g \in A^0(M) = C^\infty(M)$,
2. $[X,Y] = L_XY$, for $X \in A^1(M)$ and $Y \in A(M)$, L_XY is the Lie derivative of Y with respect to X,
3. $[X,Y] = -(-1)^{(p-1)(q-1)}[Y,X]$, for all $X \in A^p(M)$ and $Y \in A^q(M)$
4. $[X, Y \wedge Z] = [X,Y] \wedge Z + (-1)^{(p-1)q} Y \wedge [X,Z]$, for all $X \in A^p(M)$, $Y \in A^q(M)$ and $Z \in A(M)$.

By[9] and[10], we deduct that we have :

$$i([X,Y])\Omega = (-1)^{pq+q} i(X)di(Y)\Omega + (-1)^p i(Y)di(X)\Omega$$

for $p + q - 1 = k + 1$, for all $X \in A^p(M)$ and $Y \in A^q(M)$. And it results from the relation $i([X,Y]) = -[[i(Y), d], i(X)]$, that we have. If X is a Hamiltonian $p-$system and Y is a Hamiltonian $q-$system then the Schouten-Nijenhuis bracket $[X,Y]$ is also a Hamiltonian $(p+q-1)-$system, more precisely, $i([P,Q])\Omega$ is exact, in particular if $X \in A^p(E)$ and $Y \in A^q(E)$, then $[P,Q] = 0$.

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