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Reproducing Kernels of Eigenspaces of a Family of Magnetic Laplacians on Complex Projective Spaces $P^n(\mathbb{C})$ and their Heat Kernels.

A. Hafoud and A. Intissar

Université Mohammed V-Agdal,
Faculté des Sciences, Rabat , Morocco

abstract

In this paper, we give the explicit formulae of Reproducing Kernels of eigenspaces of a family of Magnetic Laplacians H_ν on the complex projective space $P^n(\mathbb{C})$ and those of their Heat Kernels.
Keywords: Magnetic Laplacian; hypergeometric function; reproducing kernels; heat kernel

I. INTRODUCTION

Let S^2 be the unit sphere of $\mathbb{R}^3$ endowed with its canonical metric ds^2 written into geodesic polar coordinates as:

$$ds^2 = d\theta^2 + \sin^2(\theta)d\varphi^2, \quad 0 < \theta < \pi, \quad 0 < \varphi < 2\pi.$$

Then, in their investigation of reproducing arbitrary multiplicity of the first eigenvalue for Schrödinger operators on the sphere S^2, the authors of [1] had considered on S^2 the following family of Magnetic Laplacians H_a given by:

$$H_a = \frac{\partial^2}{\partial^2\theta} + \cot(\theta)\frac{\partial}{\partial\theta} + \frac{1}{\sin^2(\theta)}\frac{\partial^2}{\partial^2\varphi} + \frac{2i}{\sin(\theta)}a(\theta)\frac{\partial}{\partial\varphi} - a^2(\theta),$$

where $a(\theta)$ is a real smooth function of θ.

The above operators H_a are of Magnetic Laplacians type. Indeed, by considering the connection operator:

$$\nabla_a = d + \sqrt{-1}\alpha,$$

where α is the real differential 1-form given by:

$$\alpha(\theta,\varphi) = a(\theta)\sin(\theta)d\varphi,$$

we can rewrite the operator H_a as Magnetic Laplacians given by:

$$H_a = -\nabla_a^*\nabla_a,$$

where ∇_a^* is the formal adjoint of ∇_a with respect to the natural hermitian scalar product on differential p-forms of S^2 induced by the metric ds^2.

Note in passing that this family of operators H_a forms a large class of Magnetic Laplacians on the sphere S^2 when varying $a(\theta)$. However, the case of $a(\theta) = \nu \cot(\frac{\theta}{2})$ i.e. $\alpha(\theta, \varphi) = 2\nu \cos^2(\frac{\theta}{2})d\varphi$, where ν is a non negative integer, is of particular interest since after using stereographic projection $z = \cot(\frac{\theta}{2})e^{i\varphi}$ from the sphere $S^2 \backslash \{\text{south pole}\}$ onto the z-plane $\mathbb{C}$, the differential 1-form $\alpha(\theta, \varphi)$ reads in the z-complex plane as:

$$\alpha(z) = -\sqrt{-1}\nu(\frac{\overline{z}dz - zd\overline{z}}{1 + |z|^2}) = -\sqrt{-1}\nu(\partial - \overline{\partial})log(1 + |z|^2),$$

and the associated Magnetic Laplacian H_ν can nicely be written in the local complex coordinate z as follows:

$$H_\nu = (1 + z\overline{z})\{(1 + z\overline{z})\frac{\partial^2}{\partial z \partial \overline{z}} - \nu(z\frac{\partial}{\partial z} - \overline{z}\frac{\partial}{\partial \overline{z}})\} - \nu^2|z|^2, \tag{1.1}$$

or in the following form:

$$H_\nu = (1 + z\overline{z})^2(\frac{\partial}{\partial z} + \nu\frac{\overline{z}}{1 + z\overline{z}})(\frac{\partial}{\partial \overline{z}} - \nu\frac{z}{1 + z\overline{z}}) - \nu. \tag{1.2}$$

The latter expression (1.2) for H_ν had been recognized in [5] and [6] to be the z-representation expression of the Hamiltonian of a spinless particle constrained to move on the surface of the sphere S^2 in the presence of a magnetic monopole placed at its center and of magnitude ν. We mention also that the operators H_ν given in (1.1) can be intertwined to give rise to the operators

$$\overline{\Delta}_{2\nu} = (1 + |z|^2)\{(1 + |z|^2)\frac{\partial^2}{\partial z \partial \overline{z}} - 2\nu z\frac{\partial}{\partial z}\}$$

that were considered in [8] on the quantized sphere $S^2 = P^1(\mathbb{C})$.

Now our objective in this paper is, firstly, to provide in higher complex dimension $n \geq 2$ the analogous of the Magnetic Laplacian H_ν given for $n = 1$ by the formula (1.1). This will be possible by considering the Hopf fibration $S^1 \hookrightarrow S^{2n+1} \to P^n(\mathbb{C})$, where $P^n(\mathbb{C})$ is the complex projective space endowed with the Fubini-Study metric and where the Magnetic Laplacians H_ν are in the form $H_\nu = -\nabla_\nu^*\nabla_\nu$, where ∇_ν is a hermitian connection on some complex line bundles $\mathcal{L}_{-\nu,\nu}$ over the complex projective space $P^n(\mathbb{C})$ that are associated canonically to the above Hopf fibration (see section 2).

Secondly, the main aim of this paper is to discuss the eigenfunction problem of the above defined Magnetic Laplacian $H_\nu = -\nabla_\nu^*\nabla_\nu$ when acting on sections of $\mathcal{L}_{-\nu,\nu}$ over $P^n(\mathbb{C})$. More precisely, we will give explicit formulas for the reproducing kernels of the eigenspaces of sections for the Magnetic Laplacian H_ν, and we derive a series expansion of the corresponding Heat kernels (see section 3). But the problem of producing arbitrary multiplicity of the first eingenvalue of Schrödinger operators with a "Magnetic field " on $P^n(\mathbb{C})$, $n \geq 2$ is more complicated and will not be discussed in this paper (see section 4 for finals remarks).

II. PRELIMINARIES: GEOMETRIC ASPECTS OF THE GENERALIZED COMPLEX HOPF FIBRATION AND DEFINITION OF THE ASSOCIATED MAGNETIC LAPLACIANS H_ν.

Let $P^n(\mathbb{C})$ be the projective complex space of dimension n whose elements are the complex lines of $\mathbb{C}^{n+1}$. Let $S^1 = \{\lambda \in \mathbb{C}, |\lambda| = 1\}$ be the unit circle of the complex plane $\mathbb{C}$ acting by complex multiplication $\lambda.\omega = (\lambda\omega_1, ..., \lambda\omega_{n+1})$ on the unit sphere $S^{2n+1} = \{\omega = (\omega_1, ..., \omega_{n+1}) \in \mathbb{C}^{n+1} : |\omega| = 1\}$ of the usual Hermitian complex space $\mathbb{C}^{n+1}$. Then, it is well known that the set of orbits of the above action of S^1 on the sphere S^{2n+1} is exactly the complex projective space i.e. $P^n(\mathbb{C}) = S^1 \setminus S^{2n+1}$.

Let π be the canonical projection from S^{2n+1} onto the quotient space $S^1 \setminus S^{2n+1} = P^n(\mathbb{C})$, and let $P^n(\mathbb{C})$ be the complex projective space endowed with the Fubini-Study metric ds^2_{FS}. Then the above projection mapping π gives rise to a Riemannian submersion from (S^{2n+1}, ds^2_{can}) onto $(P^n(\mathbb{C}), \mathrm{ds}^2_{\mathrm{FS}})$ and whose fibers (S^1, ds^2_{can}) are totally geodesic see [3]. The above fibration $S^1 \hookrightarrow S^{2n+1} \to P^n(\mathbb{C})$ is called the generalized complex Hopf fibration and it defines a principal circle bundle on $P^n(\mathbb{C})$ whose associated complex line bundle L is, modulo a possible sign convention, the Hopf line bundle

$$L = \{(l, v) \in P^n(\mathbb{C}) \times \mathbb{C}^{\mathrm{n}+1} / \mathrm{v} \in \mathrm{l}\}.$$

Now let ∇_L denote the unique hermetian connection (up to normalization) on the above Hopf line bundle over the compact Kähler manifold $(P^n(\mathbb{C}), \mathrm{ds}^2_{\mathrm{FS}})$ and whose curvature Ω is the (1,1)-Kähler differential form on $(P^n(\mathbb{C}), \mathrm{ds}^2_{\mathrm{FS}})$ associated to the Hermitian Fubini-Study metric ds^2_{FS} on $P^n(\mathbb{C})$ see in bellow the local expression of the metric ds^2_{FS} as well as of the connection 1-form associated to the connection operator ∇_L on the Hopf line bundle L.

Now for fixed non negative integer ν let $\mathcal{L}_{-\nu,\nu}=\overline{L}^{*\otimes\nu} \otimes L^{\otimes\nu}$ denote the complex line bundle over $P^n(\mathbb{C})$ where $\overline{L}^*$ is the conjugate dual of the complex line bundle L and let ∇_ν be the natural associated Hermitian connection of $\mathcal{L}_{-\nu,\nu}$.
Therefore, using the above notations we can define the family H_ν of Magnetic Laplacians on the complex projective space $P^n(\mathbb{C})$ $(n \geq 2)$that generalizes those given in the introduction by (1.1) in the case of $S^2 = P^1(\mathbb{C})$ that corresponds to $n = 1$.

Definition II.1 *Let $\nu \in \mathbb{Z}_+$ and ∇_ν be the connection defined on $\mathcal{L}_{-\nu,\nu}$. Then the operators $H_\nu = -\nabla^*_\nu \nabla_\nu$ acting on $C^\infty(P^n(\mathbb{C}), \mathcal{L}_{-\nu,\nu})$ are called Magnetic Laplacians on $P^n(\mathbb{C})$. Here ∇^*_ν is the formal adjoint of ∇_ν with respect to the natural Hermitian scalar product on $\mathcal{L}_{-\nu,\nu}$-valued p-forms of $P^n(\mathbb{C})$ induced by the metric ds^2_{FS}.*

In below, we give explicit formulae for the above operator H_ν in local complex coordinates of $P^n(\mathbb{C})$ and we state some of their invariance properties with respect to the group $G = SU(n+1)$. For this let fix an arbitrary standard affine chart of $P^n(\mathbb{C})$, say $\mathbb{C}^{\mathrm{n}} = \{\mathrm{z} = (\mathrm{z}_1, ..., \mathrm{z}_\mathrm{n}), \mathrm{z}_\mathrm{i} \in \mathbb{C}\}$. Then in these local complex coordinates $z = (z_1, ..., z_n)$, the Fubini-Study metric ds^2_{FS} can be written as [4]:

$$ds^2_{FS} = (1 + |z|^2)^{-2} \sum_{ij} [(1 + |z|^2)\delta_{ij} - \overline{z_i} z_j] dz_i \otimes \overline{dz_j}, \quad (z_1, ..., z_n) \in \mathbb{C}^{\mathrm{n}}$$

where δ_{ij} is the usual Kronecker symbol, normalized here so that the holomorphic sectionnal curvature of $P^n(\mathbb{C})$ is equal to +4. The (1,1)-form Ω and the canonical connection ∇_L of the Hopf complex line bundle L can be written respectively as follows:

$$\Omega = \frac{\sqrt{-1}}{2} (1 + |z|^2)^{-2} \sum_{ij} [(1 + |z|^2)\delta_{ij} - \overline{z_i} z_j] dz_i \wedge \overline{dz_j}.$$

$$\nabla_L = d + \frac{\sum_{j=1}^n \overline{z_j} dz_j}{1 + |z|^2} = d + \partial log(1 + |z|^2).$$

Also in these coordinates, the volume form $d\mu(z)$ and the Laplace-Beltrami $\triangle_0$ are respectively given by:

$$d\mu(z) = \frac{dm(z)}{(1+|z|^2)^{n+1}}, \quad dm(z) \text{ is the Lebesgue measure on } \mathbb{C}^{\mathrm{n}}$$
$$\Delta_0 = 4(1 + |z|^2) \sum_{ij} [\delta_{ij} + z_i \overline{z_j}] \frac{\partial^2}{\partial z_i \partial \overline{z_j}}.$$

Let $G = SU(n+1)$ be the group of unimodular complex matrices on $\mathbb{C}^{\mathrm{n}+1}$ preserving the Hermitian form $Q(z) = |z_1|^2 + ... + |z_{n+1}|^2$. Then for $g \in SU(n+1)$ written into bloc matrices $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, A, B, C, D are $n \times n, n \times 1, 1 \times n, 1 \times 1$ matrices respectively, we consider the almost every where defined mapping g given by $g.z = (Az + B)(Cz + D)^{-1}$. This is a biholomorphic mapping from $\mathbb{C}^{\mathrm{n}} \setminus \mathrm{H}_\mathrm{g}$ onto $\mathbb{C}^{\mathrm{n}} \setminus \mathrm{H}_{\mathrm{g}^{-1}}$ where H_g, $H_{g^{-1}}$ are the following affine hyperplane of $\mathbb{C}^{\mathrm{n}}$ given respectively by:

$$H_g = \{z \in \mathbb{C}^{\mathrm{n}}, \quad \mathrm{Cz} + \mathrm{D} = 0\}$$

$$H_{g^{-1}} = \{z \in \mathbb{C}^n, \quad B^*z + D^* = 0\}.$$

Now for $\nu \in \mathbb{Z}_+$, we consider the following action $T^\nu(g)$ on C^∞-function $F(z)$ on $\mathbb{C}^n$ given by:

$$(T^\nu(g)F)(z) = (\overline{Cz+D})^\nu (Cz+D)^{-\nu} F(g.z) \quad g \in SU(n+1), z \notin H_g. \tag{2.1}$$

Using these coordinates $z = (z_1, ..., z_n)$, we will list the following local expressions of ∇_ν and H_ν together with some of their properties that we will be using in the sequel of this letter.

Proposition II.1 *i)* $T^\nu(g)$ *are unitary operators in the Hilbert space* $L^2(\mathbb{C}^n, d\mu(z))$.

ii) Let $\theta = -\sqrt{-1}(\partial - \bar{\partial})Log(1+|z|^2)$. *Then the pull-back* $g^*(\theta)$ *of* θ *satisfies* $g^*(\theta) = \theta - \sqrt{-1}d[log(\frac{\overline{Cz+D}}{Cz+D})]$ *for all* $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in SU(n+1)$.

iii) Let ∇_ν *be the canonical connection on the Hermitian line bundle* $\mathcal{L}_{-\nu,\nu}$. *Then* $\nabla_\nu = d + \sqrt{-1}\nu\theta$ *and it satisfies* $T^\nu(g)\nabla_\nu = \nabla_\nu T^\nu(g)$ *for all* $g \in SU(n+1)$.

iv) Let $H_\nu = -\nabla_\nu^* \nabla_\nu$ *as defined in above. Then in the local complex coordinates* $z = (z_1, ..., z_n)$, *the Magnetic Laplacian* H_ν *can be written as follows:*

$$H_\nu = 4(1+|z|^2)\{\sum_{ij}[\delta_{ij} + z_i\overline{z_j}]\frac{\partial^2}{\partial z_i \partial \overline{z_j}} - \nu(E - \overline{E})\} - 4\nu^2|z|^2$$

and it satisfies the following invariance properties: $H_\nu T^\nu(g) = T^\nu(g) H_\nu$, *for all* $g \in SU(n+1)$. *Here* E *is the Euler operator* $E = \sum_i z_i \frac{\partial}{\partial z_i}$ *and* $\overline{E}$ *is its complex conjugate.*

III. EIGENSPACES OF H_ν AND EXPLICIT FORMULAE FOR THEIR REPRODUCING KERNELS.

Let H_ν be the family of Magnetic Laplacians on $P^n(\mathbb{C})$ as defined in section (2) which, in local complex coordinates $z = (z_1, ..., z_n)$ are given on $\mathbb{C}^n$ by:

$$H_\nu = 4(1+|z|^2)\{\sum_{ij}[\delta_{ij} + z_i\overline{z_j}]\frac{\partial^2}{\partial z_i \partial \overline{z_j}} - \nu(E - \overline{E})\} - 4\nu^2|z|^2 \tag{3.1}$$

In these coordinates, we will discuss the following eigenfunction problem (3.2) associated to H_ν. Namely, for λ a complex number we consider the equation:

$$H_\nu F(z) = (\lambda^2 - n^2 + 4\nu^2)F(z), \tag{3.2}$$

where F is a bounded function on $\mathbb{C}^n$, and we denote by $E_b^\nu(\lambda)$ the associated eigenspace, i.e:

$$E_b^\nu(\lambda) = \{F : \mathbb{C}^n \to \mathbb{C}, \quad F \quad \text{is} \quad \text{bounded} \quad \text{and} \quad H_\nu F(z) = (\lambda^2 - n^2 + 4\nu^2)F(z)\}.$$

Now, in order to state the main results of this paper, we need to fix some further notations. Namely, for $p, q \in \mathbb{Z}_+$, let $H(p,q)$ denote the finite dimension space of all Euclidean harmonic polynomials $h_{p,q}(z,\overline{z})$ on $\mathbb{C}^n$, which are homogeneous of degree p in z and degree q in $\overline{z}$ and let $\mathcal{H}(p,q)$ the space of the restrictions of $h_{p,q}(z,\overline{z})$ to the sphere S^{2n-1}. The spaces $\mathcal{H}(p,q)$ are pairwise orthogonal in $L^2(S^{2n-1}, d\omega)$ and $L^2(S^{2n-1}, d\omega) = \oplus_{p,q\geq 0}\mathcal{H}(p,q)$. Let $h_{p,q}^1, ... h_{p,q}^{d(n,p,q)}$, where $d(n,p,q) = dim(\mathcal{H}(p,q))$, be any orthonormal basis for $\mathcal{H}(p,q)$, then the reproducing kernel $H_n^{p,q}$ of $\mathcal{H}(p,q)$ is given by Koorwinder formulae in terms of Jacobi polynomials (see [2]):

$$\sum_{j=1}^{d(n,p,q)} h_{p,q}^j(\xi)\overline{h_{p,q}^j(\zeta)} = H_n^{p,q}(<\xi,\zeta>), \quad \xi = (\xi_i)_{1\leq i\leq n}, \quad \zeta = (\zeta_i)_{1\leq i\leq n} \in S^{2n-1}$$

where $<\xi,\zeta> = \sum_{i=1}^n \xi_i\overline{\zeta_i}$. Then, we set $G_{p,q}^n(z,w) = (|z||w|)^{p+q}H_n^{p,q}(<\frac{z}{|z|}, \frac{w}{|w|}>)$.
Also, let D_ν denote the discrete set of $\mathbb{C}$ given by:

$$D_\nu = \{\lambda \in \mathbb{C}, \frac{n \pm \lambda}{2} + \nu \in \mathbb{Z}_-\}\bigcup\{\lambda \in \mathbb{C}, \frac{n \pm \lambda}{2} - \nu \in \mathbb{Z}_-\}$$

Hencefore, using the above notations, the first result of this paper is given by the following theorem:

Theorem III.1 *i) Let $\lambda \notin D_\nu$. Then the corresponding eigenspace $E_b^\nu(\lambda)$ is trivial i.e. $E_b^\nu(\lambda) = \{0\}$*

ii) Let $\lambda \in D_\nu$. Then the corresponding eigenspace $E_b^\nu(\lambda)$ is not trivial if and only if such λ has the form $\lambda_l = \pm[2(l+\nu)+n]$ for some $l \in \mathbb{Z}_+$. In this case $E_b^\nu(\lambda_l)$ is of finite dimension and any function $F(z)$ in $E_b^\nu(\lambda_l)$ can be written in the form:

$$F(z) = (1+|z|^2)^{-(l+\nu)} \sum_{0\le p\le l, 0\le q\le l+2\nu} {}_2F_1(p-l, q-l-2\nu, n+p+q; -|z|^2) h_{pq}(z,\overline{z}) \tag{3.3}$$

for some $h_{pq}(z,\overline{z})$ in $H(p,q)$ where ${}_2F_1(a,b,c;x)$ is the Gauss hypergeometric function.

iii) Let $F(z)$ be as in ii) and let $z = r\omega$, $r > 0$, $\omega \in S^{2n-1}$. Then $\lim_{r\to\infty} F(r\omega)$ exists and it is given by:

$$F_\infty(\omega) := \lim_{r\to\infty} F(r\omega) = \sum_{0\le p\le l} (-1)^{l-p} \frac{\Gamma(l-p+1)\Gamma(n+2p+2\nu)}{\Gamma(l+n+p+2\nu)} h_{p,p+2\nu}(\omega,\overline{\omega})$$

Furthermore, the above function F_∞ satisfies the following invariance property: $F_\infty(\lambda\omega) = (\frac{\overline{\lambda}}{\lambda})^\nu F_\infty(\omega)$ for every $\lambda \in S^1 = \{\lambda \in \mathbb{C}, |\lambda| = 1\}$, and $\omega \in S^{2n-1}$.

Remark III.1 *Note that the statement iii) of theorem 1) permits to view the bounded eigenfunction $F(z)$ of H_ν as sections of the complex line bundle $\mathcal{L}_{-\nu,\nu}$ over $P^n(\mathbb{C}) = \mathbb{C}^n \bigcup P^{n-1}(\mathbb{C}) = \mathbb{C}^n \bigcup (S^1 \setminus S^{2n-1})$.*

To state the next result of this paper, let $E_b^\nu(\lambda_l)$ be the above finite dimensional eigenspace equipped with the Hermitian scalar product induced from $L^2(\mathbb{C}^n, d\mu(z))$:

$$< f, g > = \int_{\mathbb{C}^n} f(z)\overline{g(z)} d\mu(z)$$

Since the space $(E_b^\nu(\lambda_l), <,>)$ is finite dimensional space, then it admits a reproducing kernel $K_l^\nu(z,w)$. That is $f(z) = \int_{\mathbb{C}^n} K_l^\nu(z,w) f(w) d\mu(w)$ for every $f \in E_b^\nu(\lambda_l)$. In bellow, we give explicit formula for $K_l^\nu(z,w)$.

Theorem III.2 *Let $K_l^\nu(z,w)$ be the reproducing kernel of $(E_b^\nu(\lambda_l), <,>)$ then we have:*

i) $K_l^\nu(z,w) = [(1+|z|^2)(1+|w|^2)]^{-(l+\nu)} \sum_{0\le p\le l, 0\le q\le l+2\nu} c_{n,l,\nu}(p,q) \times$
${}_2F_1(p-l, q-l-2\nu, n+p+q; -|z|^2)\, {}_2F_1(p-l, q-l-2\nu, n+p+q;$
$-|w|^2) G_{pq}^n(z,w)$

where the constants $c_{n,l,\nu}(p,q)$ are given by:

$$c_{n,l,\nu}(p,q) = \frac{2(2l+2\nu+n)\Gamma(l+q+n)\Gamma(l+2\nu+p+n)}{\Gamma(l-p+1)\Gamma^2(n+p+q))\Gamma(l+2\nu-q+1)}$$

In particular:

$$K_l^\nu(z,0) = c(n,\nu,l)(1+|z|^2)^{-(l+\nu)} {}_2F_1(-l, -l-2\nu, n; -|z|^2),$$

where $c(n,\nu,l)$ is given by:

$$c(n,\nu,l) = \frac{2(2l+2\nu+n)\Gamma(l+n)\Gamma(l+2\nu+n)}{vol(S^{2n-1})\Gamma^2(n)\Gamma(l+1)\Gamma(l+2\nu+1)}$$

ii) The reproducing kernel K_l^ν of the space $(E_b^\nu(\lambda_l), <,>)$ is given by the following closed formula:

$$K_l^\nu(z,w) = c(n,\nu,l) \frac{(\overline{1+ < z,w >})^{2\nu}}{(1+|z|^2)^\nu (1+|w|^2)^\nu} {}_2F_1(-l, l+2\nu+n, n; 1 - \frac{|1+ < w,z > |^2}{(1+|z|^2)(1+|w|^2)})$$

and in terms of Jacobi polynomials $P_l^{(\alpha,\beta)}(x)$, *it can be rewritten as :*

$$K_l^\nu(z,w) = \frac{2(2l+2\nu+n)\Gamma(l+n+2\nu)}{vol(S^{2n-1})\Gamma(n)\Gamma(l+2\nu+1)}\left[\frac{(1+<w,z>)^2}{(1+|z|^2)(1+|w|^2)}\right]^\nu P_l^{(n-1,2\nu)}(cos(2d(z,w)))$$

where $d(z,w)$ *is the Fubini-Study distance given by* $cos^2(d(z,w)) = \frac{|1+<z,w>|^2}{(1+|z|^2)(1+|w|^2)}$.

Corollary III.1 *Let* $l = 0,1,....,\mu_l(\nu) = -4(l+\nu)(l+\nu+n)+4\nu^2$ *and let* $d_{n,\nu}(l) = dim(E_b^\nu(\mu_l(\nu)))$ *then:*

i) $d_{n,\nu}(l) = (2l+n+2\nu)\frac{\Gamma(l+n)\Gamma(l+n+2\nu)}{n\Gamma^2(n)\Gamma(l+1)\Gamma(l+2\nu+1)}$

ii) The kernel $Q_\nu(t,z,w)$ *given by:*

$$Q_\nu(t,z,w) = \frac{\Gamma(n)}{\pi^n}\left[\frac{(1+<w,z>)^2}{(1+|z|^2)(1+|w|^2)}\right]^\nu e^{4\nu^2 t} \times$$
$$\sum_{l=0}^{\infty} d_{n,\nu}(l)e^{-4(l+\nu)(l+\nu+n)t}{}_2F_1(-l,l+2\nu+n,n;sin^2(d(z,w)))$$

solve the following Heat equation:

$$\begin{cases} \frac{d}{dt}u(t,z) = H_\nu u(t,z) \\ u(0,z) = F(z);\ F \in \oplus_{l=0}^{\infty} E_b^\nu(\lambda_l) \end{cases}$$

Precisely we have:

$$u(t,z) = \int_{\mathbb{C}^n} Q_\nu(t,z,w)F(w)\frac{dm(w)}{(1+|w|^2)^{n+1}}.$$

Proof of theorem 3.1. This will be essentially based on the following two lemmas:

Lemma III.1 *i) The Laplacian* H_ν *on* $\mathbb{C}^n$ *has the following polar decomposition* ($z \in \mathbb{C}^n, z = r\omega$, $r > 0$, $\omega \in S^{2n-1}$):

$$H_\nu = (1+r^2)\{(1+r^2)\frac{d^2}{d^2r} + (\frac{2n-1}{r}+r)\frac{d}{dr} + \frac{1}{r^2}\Delta_{S^{2n-1}} - 4L_\omega^2 - 8\nu L_\omega\} - 4\nu^2 r^2, \tag{3.4}$$

where L_ω *is the spherical part of the complex Euler operator* $E = \sum_{j=1}^n z_j\frac{\partial}{\partial z_j}$ *on* S^{2n-1} *defined by:*

$$E = \frac{1}{2}(r\frac{d}{dr}) + L_\omega.$$

and where $\Delta_{S^{2n-1}}$ *is the usual Laplacian on the sphere* S^{2n-1} *and for which we have:*

$$L_\omega h_{p,q}(\omega,\overline{\omega}) = \frac{1}{2}(p-q)h_{p,q}(\omega,\overline{\omega}), \quad \Delta_{S^{2n-1}}h_{p,q} = -(p+q)(p+q+2n-2)h_{p,q}.$$

ii) Let λ *be a complex number. Then every function* $F(z)$ *satisfying the eigenfunction equation:* $(H_\nu F)(z) = (n^2-\lambda^2+4\nu^2)F(z)$ *can be expanded in the following form:*

$$F(z) = (1+|z|^2)^{\frac{n+\lambda}{2}}\sum_{pq}{}_2F_1(\frac{n+\lambda}{2}+p+\nu, \frac{n+\lambda}{2}+q-\nu, n+p+q; -|z|^2)h_{pq}(z,\overline{z}).$$

for some polynomials $h_{pq}(z,\overline{z})$ *in* $H(p,q)$.

Lemma III.2 *For $\lambda \in \mathbb{C}$; $p, q \in \mathbb{Z}_+$ let $b^{\nu}_{\lambda,pq}(r)$ the functions given by:*

$$b^{\nu}_{\lambda,pq}(r) = r^{p+q}(1+r^2)^{\frac{\lambda+n}{2}}{}_2F_1(\frac{n+\lambda}{2}+p+\nu, \frac{n+\lambda}{2}+q-\nu, n+p+q; -r^2)$$

Then we have:
i) $b^{\nu}_{\lambda,pq}(r) = b^{\nu}_{-\lambda,pq}(r)$.
ii) For $\lambda \notin D_\nu$, the functions $b^{\nu}_{\lambda,pq}(r)$ are not bounded on $[0,\infty[$ for all $p, q \in \mathbb{Z}_+$.
iii) Let $\lambda \in D_\nu$. Then for $b^{\nu}_{\lambda_l,pq}(r)$ to be bounded on $[0,\infty[$ it is necessary and sufficient that λ is of the form $\lambda_l = \pm(2(l+\nu)+n)$ for some $l \in \mathbb{Z}_+$ and p, q satisfies $0 \le p \le l, 0 \le q \le l+2\nu$.

Now, using the above lemmas, we can prove theorem 3.1 as follows. Let $F(z)$ be a bounded eigenfunction of H_ν: $H_\nu F(z) = (n^2 - \lambda^2 + 4\nu^2)F(z)$. Then, by ii) of lemma 3.1 $F(z)$ can be expanded as

$$F(z) = F(r\omega) = \sum_{p,q\ge 0}\sum_{1\le j\le d} a^j_{\lambda_l,pq} b^{\nu}_{\lambda_l,pq}(r) h^j_{pq}(\omega,\bar{\omega}),$$

where $a^j_{\lambda_l,pq}$ are complex numbers and $b^{\nu}_{\lambda_l,pq}(r)$ are as given in lemma 3.1 which are essentially the Fourier cœfficients of $\omega \mapsto F(r\omega)$. More precisely, we have

$$a^j_{\lambda_l,pq} b^{\nu}_{\lambda_l,pq}(r) = \int_{S^{2n-1}} F(r\omega)\overline{h^j_{pq}(\omega,\bar{\omega})}d\omega, \quad r > 0.$$

Now, since $F(z)$ is bounded in $\mathbb{C}^n$, we see from above the non vanishing Fourier cœfficients $a^j_{\lambda_l,pq} b^{\nu}_{\lambda_l,pq}(r)$ must be bounded in $r \in [0,+\infty[$.
Hence, using lemma 3.2 we get the desired result stated in i) and ii) of theorem 3.1.
Finally, by using (3.3) in ii) of the theorem 3.1 and by letting $r \mapsto +\infty$, we can check that iii) holds.

Proof of lemma 3.1. i) can be easily established. For ii) let F be an eigenfunction of H_ν with $n^2 - \lambda^2 + 4\nu^2$ as eigenvalue, i.e.,

$$\left(4(1+|z|^2)\{\sum_{ij}[\delta_{ij} + z_i\overline{z_j}]\frac{\partial^2}{\partial z_i \partial \overline{z_j}} - \nu(E - \overline{E})\} - 4\nu^2|z|^2\right) F(z) = (n^2 - \lambda^2 + 4\nu^2)F(z)$$

Since H_ν is elliptic on $\mathbb{C}^n$, we know that such F is in $C^\infty(\mathbb{C}^n)$. Thus for fixed $r > 0$ we can expand the function $\omega \mapsto F(r\omega)$ into its harmonic spherical expansion with respect to $\mathcal{H}(p,q)$ to get:

$$F(r\omega) = \sum_{p,q\ge 0}\sum_{1\le j\le d(n,p,q)} a^{\nu,j}_{\lambda,pq}(r) h^j_{pq}(\omega,\overline{\omega}) \quad , r \in \mathbb{R}^+, \quad \omega \in S^{2n-1}$$

where $a^{\nu,j}_{\lambda,pq}(r)$ are the "Fourier coefficients" of $\omega \mapsto F(r\omega)$. Using (3.1) and (3.6) in (3.5) as well as the fact that the $h_{pq}(\omega)'$s form a basis of $L^2(S^{2n-1}, d\sigma)$, we get, for every fixed $(p,q) \in \mathbb{Z}_+ \times \mathbb{Z}_+$, the following ordinary differential equation that the Fourier coefficients $a^{\nu,j}_{\lambda,pq}(r)$ must satisfy:

$$[(1+r^2)\{r^2(1+r^2)\tfrac{d^2}{d^2r} + (2n-1+r^2)r\tfrac{d}{dr} - (p+q)(p+q+2n-2) - (p-q)^2 - 4\nu(p-q)\} +$$
$$(\lambda^2 - n^2 - 4\nu^2 - 4\nu^2r^2)r^2]a^{\nu,j}_{\lambda,pq}(r) = 0$$

Hence, by setting $a^{\nu,j}_{\lambda,pq}(r) = r^{p+q}(1+r^2)^{\frac{n+\lambda}{2}}g(t)$ with $t = -r^2$, and using a direct computation we find out that $g(t)$ satisfies the following hypergeometric differential equation:

$$\{t(1-t)\frac{d^2}{d^2t} + [n+p+q+(n+p+q+\lambda+1)t]\frac{d}{dt} - (\frac{\lambda+n}{2}+p+\nu)(\frac{\lambda+n}{2}+q-\nu)\}g(t) = 0,$$

and whose regular solution at the point $t = 0$ is given, up to a multiplicative constant, by the usual Gauss hypergeometric function (see [7]):

$${}_2F_1(\frac{n+\lambda}{2}+p+\nu, \frac{n+\lambda}{2}+q-\nu, n+p+q; -r^2).$$

That is, the Fourier coefficients $a^{\nu,j}_{\lambda,pq}(r)$ can be written as follows:

$$a^{\nu,j}_{\lambda,pq}(r) = a^{\nu,j}_{p,q} r^{p+q}(1+r^2)^{\frac{n+\lambda}{2}} {}_2F_1(\frac{n+\lambda}{2}+p+\nu, \frac{n+\lambda}{2}+q-\nu, n+p+q; -r^2)$$

where $a^{\nu,j}_{p,q}$ are some constant complex numbers.

Proof of lemma (3.2). We should notice first that the involved function $b^{\nu}_{\lambda,p,q}(r)$ in lemma 3.2 can be rewritten in the following equivalent forms

$$b^{\nu}_{\lambda,p,q}(r) = r^{p+q}(1+r^2)^{-p-\nu} {}_2F_1(\frac{n+\lambda}{2}+p+\nu, \frac{n-\lambda}{2}+p+\nu, n+p+q; \frac{r^2}{1+r^2}), \tag{3.5}$$

or

$$b^{\nu}_{\lambda,p,q}(r) = r^{p+q}(1+r^2)^{\nu-p} {}_2F_1(\frac{n+\lambda}{2}+q-\nu, \frac{n-\lambda}{2}+q-\nu, n+p+q; \frac{r^2}{1+r^2}), \tag{3.6}$$

Hence, using this we see that $b^{\nu}_{\lambda,p,q}(r) = b^{\nu}_{-\lambda,p,q}(r)$; thus i) holds.

Second, to show ii) of the lemma, we can use the well known asymptotic behavior of the hypergeometric function ${}_2F_1(a,b,c,x)$ when $x \mapsto 1$ to see that the functions $b^{\nu}_{\lambda,p,q}(r)$ as given in (3.6) or (3.7) behave, when $r \to \infty$, as follows (see [7]):

i) $\frac{\Gamma(n+p+q)\Gamma(q-(p+2\nu))}{\Gamma(\frac{n+\lambda}{2}+q-\nu)\Gamma(\frac{n-\lambda}{2}+q-\nu)} r^{q-(p+2\nu)}$ if $q > p+2\nu$,

ii) $\frac{\Gamma(n+p+q)\Gamma(p+2\nu-q)}{\Gamma(\frac{n+\lambda}{2}+p+\nu)\Gamma(\frac{n-\lambda}{2}+p+\nu)} r^{p+2\nu-q}$ if $q < p+2\nu$,

iii) $\frac{\Gamma(n+2p+2\nu)}{\Gamma(\frac{n+\lambda}{2}+p+\nu)\Gamma(\frac{n-\lambda}{2}+p+\nu)}(2\psi(1) - \psi(\frac{n-\lambda}{2}+p+\nu) - \psi(\frac{n+\lambda}{2}+p+\nu) - log(1+r^2))$ if $q = p+2\nu$,

where $\Gamma(z)$ is the Euler function and $\psi(z) = \frac{\Gamma'(z)}{\Gamma(z)}$.

Hence using the above asymptotic behavior, we see easily that for $\lambda \notin D_\nu$ the function $b^{\nu}_{\lambda,p,q}(r)$ are not bounded on $[0,+\infty[$ for every $(p,q) \in \mathbb{Z}_+ \times \mathbb{Z}_+$. Now let $\lambda \in D_\nu$. Then if $\lambda = \pm(2(l+\nu)+n)$ for some $l \in \mathbb{Z}_+$, we can argue similarly using again i), ii) and iii) to see that in fact $b^{\nu}_{\lambda,p,q}(r)$ are bounded if and only if (p,q) are satisfying $0 \le p \le l, \quad 0 \le q \le l+2\nu$. But the case of $\lambda = \pm(2(l'-\nu)+n)$ for some $l' \in \mathbb{Z}_+$ is quite less obvious than the above case. But making careful checking, we end that the corresponding functions

$$b^{\nu}_{l',pq}(r) = r^{p+q}(1+r^2)^{\nu-l'} {}_2F_1(p+\nu-l', q-l'^2)$$

are bounded if and only if $\quad 0 \le q \le l'$, $0 \le p \le l'-2\nu$,$(l' \ge 2\nu)$. But putting $l'-2\nu = l$, we see that the corresponding λ is exactly of the form as given in ii) of theorem 3.1.
In bellow, we give the proof of theorem 3.2 of this letter,

Proof of theorem (3.2). Let $\Phi^j_{l,pq}(z)$ be the polynomials given by

$$\Phi^j_{l,pq}(z) = (1+|z|^2)^{-(l+\nu)} {}_2F_1(p-l, q-l-2\nu, n+p+q; -|z|^2) h^j_{pq}(z,\bar{z}).$$

The polynomials $\{\Phi^j_{l,pq}, 0 \le p \le l, 0 \le q \le l+2\nu, 1 \le j \le d(n,p,q)\}$ on $\mathbb{C}^n$ form an orthogonal system in the Hilbert space $L^2(\mathbb{C}^n, d\mu(z))$:

$$\begin{aligned} &L^2(\mathbb{C}^n, d\mu(z)) \\ &= \{F : \mathbb{C}^n \to \mathbb{C}, <\mathrm{F},\mathrm{F}> = \|\mathrm{F}\|^2 \\ &= \int_{\mathbb{C}^n} |\mathrm{F(z)}|^2 \frac{\mathrm{dm(z)}}{(1+|\mathrm{z}|^2)^{\mathrm{n}+1}} < \infty\}. \end{aligned}$$

Therefore, by standard fact the reproducing kernel $K^{\nu}_l(z,w)$ of $(E^{\nu}_b(\lambda_l), <>)$ is given by:

$$K^{\nu}_l(z,w) = \sum_{0\le p\le l, 0\le q\le l+2\nu} \sum_{1\le j\le d(n,p,q)} \|\Phi^j_{l,pq}\|^{-2} \Phi^j_{l,pq}(z)\overline{\Phi^j_{l,pq}(w)}.$$

But using the fact that the $\{h^j_{pq}(\omega,\bar\omega\}_{1\leq j\leq d}$ is an orthonormal basis, we see that the norms $||\Phi^j_{l,pq}||$ are independent of j, hence using Koorwinder formulae, we can rewrite the kernel K^ν_l in the following simplified form:

$$K^\nu_l(z,w) = [(1+|z|^2)(1+|w|^2)]^{-(l+\nu)} \sum_{0\leq p\leq l, 0\leq q\leq l+2\nu} \|\Phi_{l,pq}\|^{-2}$$
$$\times\, {}_2F_1(p-l, q-l-2\nu, n+p+q; -|z|^2)\, {}_2F_1(p-l, q-l-2\nu, n+p+q; -|w|^2) G^n_{pq}(z,w).$$

Hence, to prove the formulae given in theorem 3.2, we see that this amounts to explicit the square of the norms $\|\Phi^j_{l,pq}\|^2$ together with some invariance properties of the reproducing kernel $K^\nu_l(z,w)$with respect to the $SU(n+1)$-action $T^\nu(g)$. More precisely, we establish the following lemma:

Lemma III.3 *i) The square of the norms of polynomials $\Phi^j_{l,pq}$ in $L^2(\mathbb{C}^n, d\mu(z))$ is given by:*

$$\|\Phi^j_{l,pq}\|^2 = \frac{\Gamma(l-p+1)\Gamma^2(n+p+q))\Gamma(l+2\nu-q+1)}{2(2l+2\nu+n)\Gamma(l+q+n)\Gamma(l+2\nu+p+n)}.$$

ii) Let $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ in $SU(n+1)$ and let $(z,w) \in \mathbb{C}^n \times \mathbb{C}^n$ such that $Cz+D \neq 0$ and $Cw+D\neq 0$. Then we have the following invariance properties:

$$K^\nu_l(g.z, g.w) = \left[\frac{\overline{Cz+D}}{Cz+D}\right]^{-\nu} \left[\frac{\overline{Cw+D}}{Cw+D}\right]^{\nu} K^\nu_l(z,w)$$

Proof of lemma (3.3). i) is reduced to calculate the following integral:

$$\|\Phi^j_{l,p,q}\|^2 = \int_{\mathbb{C}^n} |\Phi^j_{l,pq}(z)|^2 (1+|z|^2)^{-(n+1)} dm(z).$$

Using polar coordinates $z = r\omega$ with variable change $t = r^2$ we obtain:

$$\|\Phi^j_{l,p,q}\|^2 = \frac{1}{2}\int_0^\infty t^{n+p+q-1}(1+t)^{-2\nu-2l-n-1} |{}_2F_1(p-l, q-l-2\nu, n+p+q; -t)|^2 dt$$

Further, using the following formulae of Rodrigues type [7]:

$$(\frac{d}{dt})^m [t^{c+m-1}(1-t)^{b-c}] = (c)_m t^{c-1}(1-t)^{b-c-m} {}_2F_1(-m, b, c; t)$$

for $m = l-p$, $b = q-l-2\nu$, $c = n+p+q$, we have:

$$\begin{aligned} &{}_2F_1(p-l, q-l-2\nu, n+p+q; t) \\ &= \frac{\Gamma(n+p+q)}{\Gamma(n+l+q)} t^{1-(n+p+q)}(1-t)^{2l+2\nu+n} (\frac{d}{dt})^{l-p} [t^{n+q+l-1}(1-t)^{-l-2\nu-n-p}] \end{aligned}$$

Then, replacing this in (3.9) after changing t by $-t$, we get

$$\|\Phi^j_{l,p,q}\|^2 = \frac{(-1)^{n+p+q-1}\Gamma(n+p+q)}{2\Gamma(n+l+q)} \times$$
$$\int_{-\infty}^0 (\frac{d}{dt})^{l-p} [t^{n+q+l-1}(1-t)^{-l-2\nu-n-p}](1-t)^{-1} {}_2F_1(p-l, q-l-2\nu, n+p+q; t) dt$$

Furthermore, by using an integration by parts we get:

$$\|\Phi^j_{l,p,q}\|^2 = \frac{(-1)^{n+l+q-1}\Gamma(n+p+q)}{2\Gamma(n+l+q)} \times$$
$$\int_{-\infty}^0 t^{n+q+l-1}(1-t)^{-l-2\nu-n-p} (\frac{d}{dt})^{l-p} [(1-t)^{-1} {}_2F_1(p-l, q-l-2\nu, n+p+q; t)] dt$$

and by applying the hypergeometric relation of type:

$$(\frac{d}{dt})^m [(1-t)^{a+m-1} {}_2F_1(a,b,c;t)] = (-1)^m \frac{(a)_m (c-a)_m}{(c)_m} (1-t)^{a-1} {}_2F_1(a+m,b,c+m;t)$$

with $a = p-l$, $b = q-l-2\nu$, $c = n+p+q$ and $m = l-p$, we obtain finally that

$$\|\Phi_{l,p,q}\|^2 = \frac{(-1)^{l-p}(p-l)_{l-p}(n+p+l+2\nu)_{l-p}\Gamma(n+p+q)}{2\Gamma(n+l+q)(n+p+q)_{l-p}} \int_0^\infty t^{n+q+l-1}(1+t)^{-2l-n-2\nu-1}dt$$

But, the last above integral is an Euler-Beta integral and its value is given by:

$$\int_0^\infty t^{n+q+l-1}(1+t)^{-2l-n-2\nu-1}dt = \int_0^1 u^{n+q+l-1}(1-u)^{l+2\nu-q}du = \frac{\Gamma(n+q+l)\Gamma(l+2\nu-q+1)}{\Gamma(2l+n+2\nu+1)}.$$

Now for proving ii) of the lemma (3.3), we apply iii) of proposition 2.1 to see that the eigenspaces $E_b^\nu(\lambda_l)$ are $T^\nu(g)$-invariant spaces and since the $T^\nu(g)$ are unitary operators in $L^2(\mathbb{C}^n, d\mu(z))$ we get the desired result because of the uniqueness of the reproducing kernel $K_l^\nu(z,w)$.

Proof of the corollary (3.1). For i) we use the closed explicit formula of the reproducing kernel $K_l^\nu(z,w)$ to compute the dimension of the eigenspace $E_b^\nu(\lambda_l)$. Indeed, the integral operator K_l^ν on the space $E_b^\nu(\lambda_l)$ is the identity map. Hence, its trace is exactly the dimension of $E_b^\nu(\lambda_l)$. Hence we have:

$$Tr(K_l^\nu) = dim(E_b^\nu(\lambda_l)) = \int_{\mathbb{C}^n} K_l(z,z)\frac{dm(z)}{(1+|z|^2)^{n+1}},$$

where $dm(z)$ is the usual Lebesgue measure on $\mathbb{C}^n$. But using the closed explicit formula of $K_l^\nu(z,w)$ and polar coordinates change of variable, we see that the above integral can be written as:

$$Tr(K_l^\nu) = (2l+n+2\nu)\frac{\Gamma(l+n)\Gamma(l+n+2\nu)}{\Gamma^2(n)\Gamma(l+1)\Gamma(l+2\nu+1)} \int_0^\infty \frac{2r^{2n-1}}{(1+r^2)^{n+1}}dr$$

Further, making the following change of variable $u = \frac{r^2}{1+r^2}$, we obtain finally that we have:

$$Tr(K_l^\nu) = (2l+n+2\nu)\frac{\Gamma(l+n)\Gamma(l+n+2\nu)}{n\Gamma^2(n)\Gamma(l+1)\Gamma(l+2\nu+1)}.$$

Hence the proof of i) is finished.
The proof of ii) is standard and can be easily established.

IV. FINAL REMARKS .

We end this paper by the following remarks and comments

Remark IV.1 *We mention that the results given in theorem (3.1) and (3.2) include those obtained in [8] in the case $n = 1$ provided that we make the convention that $pq = 0$ when $n = 1$. But the techniques used in this letter are different from those used by the authors of [8].*

Remark IV.2 *Let H_ν be the differential operator on $\mathbb{C}^n$ given by:*

$$H_\nu = 4(1+|z|^2)\{\sum_{ij}[\delta_{ij} + z_i\overline{z_j}]\frac{\partial^2}{\partial z_i \partial \overline{z_j}} - \nu(E - \overline{E})\} - 4\nu^2|z|^2.$$

Then, one may discuss in $L^2(\mathbb{C}^n, d\mu(z))$ the eigenfunction equation

$$H_\nu F(z) = (\lambda^2 - n^2 + 4\nu^2)F(z), \quad F \in L^2(\mathbb{C}^n, d\mu(z)).$$

But in this case, the operator H_ν will no longer have a discrete point spectrum in $L^2(\mathbb{C}^n, d\mu(z))$. Indeed, for fixed complex number $\lambda = x + iy$ with $xy \neq 0$ (i.e. $\lambda^2 \notin \mathbb{R}$), one can check that the functions given by

$$F_{\lambda,p}(z) = (1+|z|^2)^{\frac{n+\lambda}{2}}\,{}_2F_1(\frac{n+\lambda}{2}+p+\nu, \frac{n+\lambda}{2}+p+\nu, n+2p+2\nu, -|z|^2)h_{p,p+2\nu}(z,\overline{z})$$

are in $L^2(\mathbb{C}^n, d\mu(z))$ and they satisfies $H_\nu F_{\lambda,p}(z) = (\lambda^2 - n^2 + 4\nu^2)F_{\lambda,p}(z)$ for every $p \in \mathbb{Z}_+$. Also, since for such λ, we have $\lambda^2 \notin \mathbb{R}$, we deduce that the operator H_ν cannot admit a unique self-adjoint realization in $L^2(\mathbb{C}^n, d\mu(z))$. Furthermore, for $n \geq 2$ and within such λ we get that the eigenspace

$$\begin{aligned} E^\nu_\lambda = \{F\ :\mathbb{C}^n \to \mathbb{C}^n; \mathrm{H}_\nu \mathrm{F(z)} \\ = (\lambda^2 - \mathrm{n}^2 + 4\nu^2)\mathrm{F(z)}, \quad \int_{\mathbb{C}^n} |\mathrm{F(z)}|^2 \mathrm{d}\mu(\mathrm{z}) < +\infty\} \end{aligned}$$

is of infinite dimension contrarly to the case of $n = 1$.

Remark IV.3 *Recall from corollary that the multiplicity of the first eigenvalue of our Magnetic Laplacian H_ν is given by*

$$d_{n,\nu}(0) = (n+2\nu)\frac{\Gamma(n+2\nu)}{n\Gamma(n)\Gamma(2\nu+1)} = \frac{1}{\Gamma(n+1)}(n+2\nu)(n+2\nu-1)......(2\nu-1).$$

Then for $n = 1$, we have $d_{1,\nu} = 1+2\nu$. Hence, in this case the Shrödinger operators H_ν are almost enough to produce arbitrary multiplicity of the first eigenvalue. However, for $n \geq 2$, the multiplicity of the first eigenvalue of our Magnetic Laplacian H_ν, as given above cannot range all the non negative integers by varying $\nu \in \mathbb{Z}_+$ or $\nu \in \frac{1}{2}\mathbb{Z}_+$.

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V. REFERENCES

[1] Besson, G., Colbois, B. and Courtois, G., Transactions of the american mathematical society, volume 350, number 1, january 331(1998).
[2] Folland,G.B., Pro. Am. Math. Soc.**47**(2),401-407(1975) Funct. Anal. 129, 168(1995).
[3] Gilkey,P. B., Leahy, J. V. and Park, J. H., Spinors, Spectral Geometry, and Riemannian Submersions(Monograph in Internet).
[4] Griffiths,P., Harris, J., Principles of algebraic geometry, Pure & Applied Mathematics, A Wiley-Interscience Series Of Texts, Monographs and Tracts, New York, (1978).
[5] Haldane, F. D. M., Phys. Rev. Lett. **51**, 605(1983).
[6] Li, D., Nucl. Phys.**B336**, 411(1993).
[7] Magnus, W., Oberhettinger, F. and Soni R. P., Formulas and Theorems for The special functions of Mathematical Physics. Third enlarged edition Springer-Verlag Berlin Heidelbg New York 1966A.
[8] Peetre, J. and Zhang, G., Internat. J. Math. Sci. Vol. 16 NO. 2, 225(1993).