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New Inversion Formulas for Discrete Radon Transform $\mathbf{Z^n}$ and its Dual Discrete Radon Transform

Ahmed Abouelaz

Department of Mathematics and Computer Science,
University Hassan II, Casablanca, Morocco.
E-mail: a.abouelaz@fsac.ac.ma

Abstract

In [1,2] we have defined, and study, the discrete Radon transform on the lattice $\mathbf{Z^n}$. In the following, we give a characterization of the range of the discrete Radon transform of the Schwartz's space $\mathcal{S}(\mathbf{Z^n})$ (see Theorems 3.5 and 3.7). By means of the discrete support theorem (see [1,2], we prove also a Paley -Wiener theorem for the discrete Radon transform (see Theorems 3.4 and 3.5). Finally we establish the new inversion formulas for the discrete Radon transform and its dual discrete Radon transform (see Theorems 4.6 and 5.5)
Key words: *Discrete Radon Transform, Inversion Formula, Characterization of the Image of Discrete Radon Transform, Paley-Wiener Theorem*
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I. INTRODUCTION

The Radon transform was firstly defined on $\mathbf{R^2}$ by John Radon [11] in 1917 and was afterwards generalized on the Euclidean space $\mathbf{R^n}$ by several authors particularly S.Helgason (see [8] , [9] ,[10]) and I.M Gelfand (see [6]). Let $f : \mathbf{R^n} \to \mathbf{C}$ be a function integrable on each hyperplane in $\mathbf{R^n}$. Let $\mathbf{P^n}$ denote the differentiable manifolds of all hyperplanes $H(t,\omega) = \{x \in \mathbf{R^n} \mid \mathbf{x}.\boldsymbol{\omega} = \mathbf{t}\}$ with $(t,\omega) \in \mathbf{R}\times\mathbf{S^{n-1}}$, where S^{n-1} is the united sphere of $\mathbf{R^n}$, The Radon transform of f is defined as the function $R_c f : \mathbf{P^n} \to \mathbf{C}$ given by

$$R_c f(t,\omega) = \int_{H(t,\omega)} f(x)\, d\mu(x)\,, \tag{1.1}$$

where $d\mu(x)$ is the Euclidean measure on the hyperplane $H(t,\omega)$ with $(t,\omega) \in \mathbf{R}\times\mathbf{S^{n-1}}$. In the case of lattice $\mathbf{Z^n}$, we give (see [1] , [2]) an analogue of the definition (1.1) which consists in making the average of a suitable complex - valued function f on $\mathbf{Z^n}$ over discrete hyperplane $H(a,k) = \{m \in \mathbf{Z^n} \mid \mathbf{m}.\mathbf{a} = \mathbf{k}\}$ defined by diophantine equations, with $(a,k) \in \mathcal{P}\times\mathbf{Z}$, where $\mathcal{P}$ denotes the set of elements $a = (a_1, a_2, ..., a_n) \in \mathbf{Z^n}\backslash\mathbf{0}$ such that $d(a) = 1$, with $d(a)$ is the greatest common divisor of the integers $a_1, a_2, ..., a_n$ and $a.m$ denotes the usual inner product of a and m regarded as two vectors of the Euclidean space $\mathbf{R^n}$.

Our paper is organized as follows: In the second section, we recall some properties also we fix certain notations which will be useful in the sequel of this paper, we denotes by $\mathbf{G}$ the set of all discrete hyperplanes $H(a,k) \in \mathcal{P}\times\mathbf{Z}\ /\pm1$. Recall that the function $\overline{(a,k)} \to H(a,k) = \{x \in \mathbf{Z^n} \mid \mathbf{x.a} = \mathbf{k}\}$ is a bijection of $\mathcal{P}\times\mathbf{Z}\ /\pm1$ into $\mathbf{G}$ (see[1] and [2]). $\mathbf{G}$ be called the discrete Grassmanians. We denote by $\mathcal{S}(\mathcal{Z}^{\mathbf{n}})$ the Schwartz space of $\mathbf{Z^n}$ consisting of all complex -valued rapidly decreasing functions f defined on $\mathbf{Z^n}$ (see [4]). We define a family $(p_r)_{r\in\mathbf{N}}$ of semi -norm on $\mathcal{S}(\mathbf{Z^n})$ by $p_r(f) = \sup_{m\in\mathbf{Z^n}}(1+\|m\|^2)^r\,|f(m)|$, for all $f \in \mathcal{S}(\mathbf{Z^n})$ and $N \in \mathbf{N}$. Where $\|m\|^2 = m_1^2 + m_2^2 + ... + m_n^2$ for all $m = (m_1, m_2, ..., m_n) \in \mathbf{Z^n}$. Let $\mathbf{C}^{(\ \mathbf{Z^n})}$ the subspace of $\mathcal{S}(\mathbf{Z^n})$ consisting of all complex -valued functions defined on $\mathbf{Z^n}$ with finite support. Let K be a finite set of $\mathbf{Z^n}$ and denotes by $\mathbf{C}_{\mathbf{K}}^{(\ \mathbf{z^n})}$ the subspace of $\mathbf{C}^{(\ \mathbf{Z^n})}$ constituted by the functions defined on $\mathbf{Z^n}$ with finite support included in K. We define the space $l_*^p(\mathbf{G}))$ $(1 \leq p < +\infty)$ as follows: a complex-valued function F defined on $\mathbf{G}$ belongs to $l_*^p(\mathbf{G})$ if and only if

$$\sum_{k\in\ \mathbf{Z}} |F(H(a,k)|^p < +\infty, \text{for all } a \ \in \mathcal{P}.$$

We define a family $(q_N)_{N\ \in\mathbf{N}}$ of semi- norms on the space $l_*^1(\mathbf{G})$ by

$$q_N(F) = Sup_{a\in\mathcal{P},k\in\ \mathbf{Z}} \frac{\left(1+k^2\right)^N}{\left(1+\|a\|^2\right)^N} |F(H(a,k)| \tag{1.2}$$

for all $F \in l_*^1(\mathbf{G})$ and N belongs to $\mathbf{N}$.
The Schwartz space of $\mathbf{G}$, denoted by $\mathcal{S}(\mathbf{G})$ is the subspace of $l_*^1(\mathbf{G})$ consisting of all functions $F \in l_*^1(\mathbf{G})$ such that $q_N(F) < +\infty$ for all N $\in \mathbf{N}$, we define also a subspace $\mathcal{S}_*(\mathbf{G})$ of the space $\mathcal{S}(\mathbf{G})$ constituted by the functions F such that

$$T_N(F) = Sup_{a\in\mathcal{P},k\in\ \mathbf{Z}} \left(1+\|a\|^2+k^2\right)^N |F(H(a,k)| < +\infty\ , \tag{1.3}$$

for all F$\in \mathcal{S}(\mathbf{G})$ and $N \in \mathbf{N}$. It is clear that $\mathrm{T}_N(F) \geq q_N(F)$ with $(F,N) \in \mathcal{S}_*(\mathbf{G}) \times\mathbf{N}$. We denote by $\mathcal{S}_{\mathbf{1},*}(\mathbf{G})$ the subspace of $\mathcal{S}_*(\mathbf{G})$ constituted by the functions $F \in \mathcal{S}_*(\mathbf{G})$ such that $F(H(a,k)) = 0$ whenever $|a_1| \neq 1$ with $a = (a_1, a_2, ..., a_n) \in \mathcal{P}$.
In third section, we give a characterization of the range under discrete Radon transform of $\mathbf{C}^{(\mathbf{Z^n})}$. The main result of this section is the following theorem (see theorems 3.4 and 3.5).
Let K be a finite set of $\mathbf{Z^n}$, then $R(\ \mathbf{C}_{\mathbf{K}}^{(\ \mathbf{z^n})}) = \mathcal{D}_{(\Updownarrow),\mathcal{K}}(\mathbf{G})$. This theorem can be considered as the Paley -Wiener theorem for the discrete Radon transform, and this result is a analogue to Euclidean case (see [8] ,[9] , [10]). The space $\mathcal{D}_{(m),K}(\mathbf{G})$ is the set of all functions F defined on $\mathbf{G}$ and which belongs to Schwartz's space $\mathcal{S}(\mathbf{G})$ such that $F(H) = 0$ whenever $H \cap K = \emptyset$ and satisfying a discrete moments condition, that is, for each $p \in \mathbf{N}\backslash\mathbf{0}$ the function $a \longrightarrow \sum_{k\in\mathbf{Z}} F(H(a,k))k^p$ is a homogenous polynomial of degree p in $a_1, a_2, ..., a_n$, where $a = (a_1, a_2, ..., a_n) \in \mathcal{P}$. we denote by $\mathcal{D}_{(m)}(\mathbf{G})$ the union of the spaces $\mathcal{D}_{(m),K}(\mathbf{G})$ when K cover the set of all finite subsets of $\mathbf{Z^n}$. We deduce the following Paley -Wiener theorem .

$$R\left(\mathbf{C}^{(\mathbf{Z^n})}\right) = \mathcal{D}_{(m)}(\mathbf{G}) = \lim_{\longrightarrow K\subset\mathbf{Z^n}} \mathcal{D}_{(m),\mathbf{K}}(\mathbf{G})$$

Our purpose in this section is to also study the image under discrete Radon transform of the Schwartz's space $\mathcal{S}(\mathbf{Z^n})$. We prove an important theorem which states $R(\mathcal{S}(\mathbf{Z^n})) = \mathcal{S}_{(\mathbf{m})}(\mathbf{G})$ (see theorem 3.7).
In the fourth section, we prove a inversion theorem for the dicrete Radon transform R. For this we study, at first, some properties of certain operators intervening in inversion formulas for the discrete Radon transform (see theorem 4.6) . The main result in this section can be stated as follows:
Let $f \in \mathcal{S}(\mathbf{Z^n})$ and $m_0 \in \mathbf{Z^n}$. Then for all $a \in \mathcal{P}$ the operator R can be inverted by the formula

$$f(m_0) = \sum_{k\in\mathbf{Z},\mathbf{m}\in\mathbf{Z^n}} \Delta_{(m_0,m)} Rf(H(a,k)) \tag{1.4}$$

where $\Delta_{(m_0,m)}$ is the operator of $R\left(\mathcal{S}(\mathbf{Z^n})\right)$ into $R\left(\mathcal{S}(\mathbf{Z^n})\right)$ (see Remark 4.7 and (4.3)).
In the five section, we construct and study the operator $\Delta_{(H_0,,H)}$ which permit to invert the dual discrete Radon transform R^*. We prove the following theorem (see theorem 5.5).
Let $F \in \mathcal{S}_{\mathbf{1},*}(\mathbf{G})$ and $H_0 = H\left(a_0, k_0\right)$, where $(a_0, k_0) \in \mathcal{P}\times\mathbf{Z}$. Then R^* can be inverted by the formula

$$F\left(H_0\right) = \sum_{m\in\mathbf{Z}\times\{\mathbf{0}\}} \sum_{(a,k)\in\mathcal{P}\times\mathbf{Z}} \left(\Delta_{(H_0,H)} R^* F\right)(m),$$

where $H = H\left(a,k\right) \in \mathbf{G}$. The operator $\Delta_{(H_0,,H)}$ is defined and continuous of $R^*(\mathcal{S}_{\mathbf{1},*}(\mathbf{G}))$ into $R^*(\mathcal{S}_{\mathbf{1},*}(\mathbf{G}))$

II. NOTATIONS AND PRELIMINARIES

In this section, we shall fix some notations which are useful in the sequel of this paper, and we recall certain properties of the discrete hyperplanes in the lattice $\mathbf{Z^n}(\mathbf{n} \geq \mathbf{2})$(see [1] , [2]). Let $a = (a_1, a_2, ..., a_n) \in \mathbf{Z^n}\backslash\mathbf{0}$ we denote by $d\left(a\right)$ the greatest common divisor of integers $a_1, a_2, ..., a_n$, we always denote by $\mathcal{P}$ the set$\{a \in \mathbf{Z^n}\backslash\mathbf{0}/\ \mathbf{d}\left(\mathbf{a}\right) = \mathbf{1}\}$. For $(a,k) \in \mathcal{P}\times\mathbf{Z}$ the set $\{x \in \mathbf{Z^n}/\ \mathbf{a.x} = \mathbf{k}\}$ is denoted by $H\left(a,k\right)$. Moreover we denote by $\mathbf{G}$ the set $\{H\left(a,k\right) /\ (a,k) \in \mathcal{P}\times\mathbf{Z}\}$. In the sequel of this paper, the elements of $\mathbf{G}$ will called discrete hyperplanes in $\mathbf{Z^n}$. Finally we always denote by $\mathbf{G_m}$ the set of all discrete hyperplanes containing m. By a geometrical transform, we prove that $\mathbf{G_m}$ is isomorph to the set of discrete hyperplanes through the origin, then $(\mathbf{G_{n-1,n}}) \cap (\mathcal{P}\times\mathbf{Z}\ /\pm1) \approx \mathbf{G_m}$ where $\mathbf{G_{n-1,n}} = \mathbf{RP_n} = \mathbf{S^{n-1}}/\pm\mathbf{1}$ which is the real projector space.
The following proposition is useful in the sequel and which gives us an appropriate parametrization of discrete hyperplanes in $\mathbf{Z^n}$(see[1],[2]).
Proposition 2.1 let Ψ $\mathcal{P}\times\mathbf{Z}\ /\pm1 \longrightarrow \mathbf{C}$ the function defined by $\Psi(\overline{(a,k)}) = H\left(a,k\right)$ for all $(a,k) \in \mathcal{P}\times\mathbf{Z}$, then Ψ is a bijection. Where $\overline{(a,k)}$ denote the set $\{(a,k), (-a,-k)\}$ belonging to $\mathcal{P}\times\mathbf{Z}\ /\pm1$.
For the proof see[1] , [2] .
Definition 2.2 Let $f \in l^1(\mathbf{Z^n})$. The discrete Radon transform of f is the complex -valued function Rf defined on $\mathbf{G}$ by

$$Rf\left(H\left(a,k\right)\right) = \sum_{m\in H(a,k)} f(m), \text{for all } (a,k) \in \mathcal{P}\times\mathbf{Z}. \tag{2.1}$$

Definition 2.3 Let F be a complex -valued function defined on $\mathbf{G}$ such that $\sup_{k\in\mathbf{Z}} \sum_{a\in\mathcal{P}} \left|F(H\left(a,k\right))\right|$ is finite. The discrete dual Radon transform of F is the complex-valued function R^*F defined on $\mathbf{Z^n}$by

$$R^*F(m) = \sum_{H\in\mathbf{G},\mathbf{m}\in\mathbf{H}} F(H) = \frac{1}{2} \sum_{a\in\mathcal{P},k\in\mathbf{Z},\mathbf{m}\in\mathbf{H}(\mathbf{a},\mathbf{k})} F(H(a,k)).$$

Remark 2.4 From definition 2.3, $R^*F(m)$ come down to

$$R^*F(m) = \frac{1}{2}\sum_{a\in\mathcal{P}} F(H(a,a.m)). \tag{2.3}$$

The equality (2.3) have a sens, since F verify the condition of the above definition (see [1] , [2] for more precision).
Proposition 2.5 Let $f \in \mathcal{S}(\mathbf{Z^n})$ and $p \in \mathbf{N}\backslash\mathbf{0}$. Then for all $a \in \mathcal{P}$ we have

$$\sum_{k\in\mathbf{Z}} Rf(H(a,k))k^p = \sum_{m\in\mathbf{Z^n}} f\left(m\right)(a.m)^p, \tag{2.4}$$

$$\sum_{k\in\mathbf{Z}} Rf(H(a,k)) = \sum_{m\in\mathbf{Z^n}} f\left(m\right) \tag{2.5}$$

For proof (see [1] , [2] . Let $s \in \mathbf{C}$ such that $\mathrm{Re}(s) > 1$. Define a function f_s on $\mathbf{Z^n}$ as follows

$$f_s(m) \quad = \begin{cases} \dfrac{1}{\|m\|^s} = \dfrac{1}{n^s} & \text{if } m = (n, 0, 0, ..., 0) \\ 0 & \text{elsewhere} \end{cases}$$

From formula (2.5), we have for all $a \in \mathcal{P}$,

$$\sum_{k\in\mathbf{Z}} Rf_s(H(a,k)) \quad = \zeta(s) \ ,$$

where $\zeta(s)$ is the Riemann's Zeta function. Recall that $R\left(\mathcal{S}(\mathbf{Z^n})\right)$ is the image under discrete Radon transform of $\mathcal{S}(\mathbf{Z^n})$.

We now state the continuety theorem for the discrete Radon transform R, which allows us to deduce that $R\left(\mathcal{S}(\mathbf{Z^n})\right) \subset \mathcal{S}_{(\mathbf{m})}(\mathbf{G})$, where $\mathcal{S}_{(\mathbf{m})}(\mathbf{G})$ is the subspace of $\mathcal{S}(\mathbf{G})$ constitued by the functions F which satisfyied a discrete moments condition, that is, for all $p \in \mathbf{N}\backslash 0$ the function $a \longrightarrow \sum_{k\in\mathbf{Z}} F(H(a,k)k^p$ is a homogenuous polynomial of degree p in $a_1, a_2, ..., a_n$ with $a = (a_1, a_2, ..., a_n) \in \mathcal{P}$.

Theorem 2.6 The discrete Radon transform R is a continuous linear mapping of $\mathcal{S}(\mathbf{Z^n})$ into $\mathcal{S}(\mathbf{G})$. More precisely we have for all $f \in \mathcal{S}(\mathbf{Z^n})$ all $(r, N) \in \mathbf{N} \times \mathbf{N}$ such that $\frac{n}{2} < r$ and all $a \in \mathcal{P}$, the following inequality

$$\left(1 + \|a\|^2\right)^{-N} \sum_{k\in\mathbf{Z}} (1 + k^2)^N \ |F(H(a,k))| \ \le C_{n,r} p_{N+r}(f) \,, \tag{2.6}$$

with $C_{n,r} = \sum_{k\in\mathbf{Z}} \dfrac{1}{(1+k^2)^r} < +\infty$. In particular we have

$$q_N(Rf) \le C_{n,r} p_{N+r}(f) \ . \tag{2.7}$$

For the proof (see[1]and[2]).

Remark2.7 **(a)** Let $F \in \mathcal{S}(\mathbf{G})$, we have (see introduction) for all $(a, k) \in \mathcal{P}\times\mathbf{Z}$ and $N \in \mathbf{N}$

$$\frac{(1+k^2)^{\ N}}{\left(1 + \|a\|^2\right)^N} \quad |F(H(a,k))| \le C_N,$$

where C_N is a constant which depends only the N and F, then

$$\left(\frac{k^2}{\|a\|^2}\right)^N \ |F(H(a,k))| \le C'_N,$$

where C'_N is a other constant. Thus

$$|F(H(a,k))| \le \quad C'_N \left(\frac{\|a\|^2}{k^2}\right)^N , \tag{2.8}$$

for all $H(a,k) \in \mathbf{G}$ and $F \in \mathcal{S}(\mathbf{G})$, where C'_N is a constant which depends only the N and .F

(b) we deduce from the above theorem that

$$R\left(\mathcal{S}(\mathbf{G})\right) \subset \mathcal{S}_{(\mathbf{m})}(\mathbf{G}). \tag{2.9}$$

We now state the support theorem, proved in [1] and [2], for the discrete Radon transform R which allows us to deduce that $R(\ \mathbf{C}_{\mathbf{K}}^{(\ \mathbf{Z^n})}) = \mathcal{D}_{(\mathbf{m}),\mathbf{K}}(\mathbf{G})$ for all finite set K of $\mathbf{Z^n}$.

Theorem 2.8 Let $f \in \mathbf{C}^{(\ \mathbf{Z^n})}$and $K = \{x_1, x_2, ...x_l\}$ inclued in $\mathbf{Z^n}$(with $l \in \mathbf{N}\backslash\mathbf{0}$).Then

$$Supp f \subset K \Leftrightarrow (Supp Rf \subset \bigcup_{i\in\{1,2,...,l\}} \mathbf{G}_{\mathbf{x_i}}). \tag{2.10}$$

For the proof see[1] and [2]). We deduce of the above theorem the important corollary;

Corollary 2.9 (See [1] and [2]) Let $f \in \mathbf{C}^{(\ \mathbf{Z^n})}$and $K = \{x_1, x_2, ...x_l\}$ inclued in $\mathbf{Z^n}$with $l \in \mathbf{N}\backslash\mathbf{0}$. Then

$$Supp f \subset K \Leftrightarrow (Rf(H) = 0 \text{ for all } H \in \mathbf{G} \textbf{ such that } \mathbf{H} \cap \mathbf{K} = \emptyset) \tag{2.11}$$

III. CHARACTERIZATION OF THE IMAGE UNDER DISCRETE RADON TRANSFORMATION OF $\mathbf{C}^{(\mathbf{Z}^{\mathbf{N}})}$

Recall that $\mathbf{C}^{(\mathbf{Z}^{\mathbf{n}})}$ is the subspace of $\mathcal{S}(\mathbf{Z}^{\mathbf{n}})$ consisting of all complex -valued functions defined on $\mathbf{Z}^{\mathbf{n}}$ with finite support. Let K be a finite set of $\mathbf{Z}^{\mathbf{n}}$ and denote by $\mathbf{C}_{\mathbf{K}}^{(\mathbf{Z}^{\mathbf{n}})}$ the subspace of $\mathbf{C}^{(\mathbf{Z}^{\mathbf{n}})}$ constituted by the functions defined on $\mathbf{Z}^{\mathbf{n}}$ with finite support included in K, we design by $\mathcal{D}_{(m),K}(\mathbf{G})$ the space of functions F defined on $\mathbf{G}$ satisfying a discrete moment conditions and such that $F(H)=0$ whenever $H\cap K=\emptyset$.
We shall always note K the set $\{x_1,x_2,...x_l\}$ with $l\in\mathbf{N}^*$. We design by $\mathbf{G}\left(\mathbf{K}\right)$ the set $\{H\in\mathbf{G}\ |\mathbf{H}\cap\mathbf{K}=\emptyset\}$. We have the following lemma:
Lemma 3.1 Let K be a subset of $\mathbf{Z}^{\mathbf{n}}$, then

$$\mathbf{G}\left(\mathbf{K}\right)=\bigcup_{\mathbf{i}\in\{\mathbf{1},\mathbf{2},...,\mathbf{l}\}}\mathbf{G}_{\mathbf{x_i}},\mathbf{with}\ \mathbf{K}=\{\mathbf{x_1},\mathbf{x_2},...\mathbf{x_l}\}. \tag{3.1}$$

Let $H\in\mathbf{G}$ such that $H\notin\bigcup_{i\in\{1,2,...,l\}}\mathbf{G}_{\mathbf{x_i}}$, this implies that $x_i\notin H$ for all $i\in\{1,2,...,l\}$; then $H\cap K=\phi$, consequently $H\notin\mathbf{G}\left(\mathbf{K}\right)$. It is clear that $\mathbf{G}_{\mathbf{x_i}}\subset\mathbf{G}\left(\mathbf{K}\right)$ for all $i\in\{1,2,...,l\}$, hence $(\bigcup_{i\in\{1,2,...,l\}}\mathbf{G}_{\mathbf{x_i}})\subset\mathbf{G}\left(\mathbf{K}\right)$ and this prove the lemma.
Lemma3.2 let $m_0\in\mathbf{Z}^{\mathbf{n}}$, then $\mathbf{G}_{\mathbf{m_0}}$ is isomorph to $(\mathbf{G}_{\mathbf{n-1},\mathbf{n}})\cap(\mathcal{P}\times\mathbf{Z}\ /\pm1)$, where $\mathbf{G}_{\mathbf{n-1},\mathbf{n}}=\mathbf{RP}_{\mathbf{n}}$ is the real projective space in particulary $\mathbf{G}\left(\mathbf{K}\right)\backsimeq(\mathbf{RP}_{\mathbf{n}})\cap(\mathcal{P}\times\mathbf{Z}\ /\pm1)$.

Proof. Let T_{m_0} be the translation of point $m_0\in\mathbf{Z}^{\mathbf{n}}$,that this, $T_{m_0}(x)=x-m_0$. Let $H(a,k)$ be the discrete hyperplane belongs to the set $\mathbf{G}_{\mathbf{m_0}}$

and let $x=(x_1,x_2,...,x_n)\in H(a,k)$. Putting $(X_1,X_2,...,X_n)=(x_1-m_{0,1},x_2-m_{0,1}...,x_n-m_{0,n})$. Since $x\in H(a,k)$, it follows that $(a_1X_1+...+a_nX_n)+(a_1m_{0,1}+...+a_nm_{0,n})=k$, this implies that $a_1X_1+...+a_nX_n=0$,because $m_0=(m_{0,1},...,m_{0,n})\in H(a,k)$. Consequently$T_{m_0}(H)\in(\mathbf{RP}_{\mathbf{n}})\cap(\mathcal{P}\times\mathbf{Z}\ /\pm1)$, for all $H\in\mathbf{G}_{\mathbf{m_0}}$. Thus $T_{m_0}(\mathbf{G}_{\mathbf{m_0}}.)=(\mathbf{RP}_{\mathbf{n}})\cap(\mathcal{P}\times\mathbf{Z}\ /\pm1)$. This completes the proof of lemma. ■

For $H_0\in\mathbf{G}$, let $\beta_{H_0}$the function defined on $\mathbf{G}$ by $\beta_{H_0}(H_0)=1$ and $\beta_{H_0}(H)=0$ if $H\neq H_0$ for all $H\in\mathbf{G}$. We shall be need the following lemma.
Lemma 3.3 let $K=\{x_1,x_2,...x_l\}\subset\mathbf{Z}^{\mathbf{n}}$ $(l\in\mathbf{N}\backslash\mathbf{0})$ and $F\in\mathcal{D}_{(m),K}(\mathbf{G})$. Then

$$F\left(.\right)=\sum_{H\in\mathbf{G}}F(H)\frac{1}{Card(H\cap K)}(\chi_{\mathbf{G}_{\mathbf{x_1}}}+....+\chi_{\mathbf{G}_{\mathbf{x_l}}})\left(.\right)\beta_H\left(.\right) \tag{3.2£}$$

Proof. Since $F\in\mathcal{D}_{(m),K}(\mathbf{G})$ then $SuppF\subset\ \mathbf{G}\left(\mathbf{K}\right)=\bigcup_{\mathbf{i}\in\{\mathbf{1},\mathbf{2},...,\mathbf{l}\}}\mathbf{G}_{\mathbf{x_i}}$, it follows that $F(H)=0$ if $H\cap K=\phi$. We convenient that $Card(H\cap K)^{-1}F\left(H\right)=0$ if $H\cap K=\emptyset$. Let $H_0\in\mathbf{G}$ with H_0 containing $x_1,x_2.....,x_{r_0}(0\leq r_0\leq l)$ we have

$$\sum_{H\in\mathbf{G}}F(H)\frac{1}{Card(H\cap K)}(\chi_{\mathbf{G}_{\mathbf{x_1}}}+...+\chi_{\mathbf{G}_{\mathbf{x_{r_0}}}})(H_0)\beta_H(H_0)$$
$$==F(H_0)\frac{1}{Card(H_0\cap K)}(\chi_{\mathbf{G}_{\mathbf{x_1}}}+...+\chi_{\mathbf{G}_{\mathbf{x_{r_0}}}})(H_0),$$

but

$$(\chi_{\mathbf{G}_{\mathbf{x_1}}}+....+\chi_{\mathbf{G}_{\mathbf{x_{r_0}}}})\left(H_0\right)=Card\left(H_0\cap K\right)$$

it follows from the above equality that

$$\sum_{H\in\mathbf{G}}F.(H)\frac{1}{Card(H\cap K)}(\chi_{\mathbf{G}_{\mathbf{x_1}}}+...+\chi_{\mathbf{G}_{\mathbf{x_l}}}).(H_0)\beta_H(H_0)=F\left(H_0\right),$$

if $H_0\supset\{x_1,x_2,...x_{r_0}\}$.Consequently

$$\sum_{H\in\mathbf{G}}F.(H)\frac{1}{Card(H\cap K)}(\chi_{\mathbf{G}_{\mathbf{x_1}}}+....+\chi_{\mathbf{G}_{\mathbf{x_l}}}).(H')\beta_H(H')=F(H')\text{ for all }H'\in\mathbf{G}.$$

This prove the above lemma. ■

Theorem 3.4 Let $K=\{x_1,x_2,...,x_l\}$ inclued in $\mathbf{Z^n}$, with $l\in\mathbf{N}\backslash\mathbf{0}$, then

$$R(\mathbf{C}_{\mathbf{K}}^{(\mathbf{\ z^n})})=\mathcal{D}_{(\mathbf{m}),\mathbf{K}}(\mathbf{G}) \tag{3.3}$$

Proof. Showing first all the inclusion $R(\mathbf{C}_{\mathbf{K}}^{(\mathbf{\ z^n})})\subset\mathcal{D}_{(\mathbf{m}),\mathbf{K}}(\mathbf{G})$. Let f be a element of $\mathbf{C}_{\mathbf{K}}^{(\mathbf{\ z^n})}$, that is, support of f is included in $K=\{x_1,x_2,...x_l\}\subset\mathbf{Z^n}$. By support theorem 2.8; we have $SuppRf\subset\bigcup_{i\in\{1,2,...,l\}}\mathbf{G_{x_i}}=\mathbf{G}\left(\mathbf{K}\right)$. Then $Rf\left(H\right)=0$ whenever $H\cap K=\emptyset$, thus $Rf\in\mathcal{D}_{(m),K}(\mathbf{G})$. Furthermore, from lemma 3.2 the set $\mathbf{G}\left(\mathbf{K}\right)$ is isomorph to discrete Grassmannian $(\mathbf{RP_n})\cap(\mathcal{P}\times\mathbf{Z}/\pm1)$ which is a compact. Showing now the following inclusion

$$\mathcal{D}_{(m),K}(\mathbf{G})\subset\mathbf{R}(\mathbf{C}_{\mathbf{K}}^{(\mathbf{\ z^n})})$$

From ([1] proposition 3.4) and lemma3.3, we have

$$F(H)=R\left(\frac{F\left(H\right)}{Card(H\cap K)}(\chi_{x_1}+....\chi_{x_l})\right)(H),\ \ for\ all\ H\in\mathbf{G},$$

where $\chi_{x_1},...,\chi_{x_l}$ are the characteristic functions of points $x_1,...,x_l$ respectively. Let $f_{x_1,...,x_l}\ (.,.)$ the complex- valued function defined on $\mathbf{Z^n}\times\mathbf{G}$ by

$$\begin{aligned}f_{x_1,...,x_l}\ (m,H)&=F\left(H\right)\left[Card(H\cap K)\right]^{-1}\left(\chi_{x_1}+....+\chi_{x_l}\right)(m)\\&=\begin{cases}F\left(H\right) & \text{if } m\in H\cap K\\ 0 & \text{if } m\notin H\cap K\end{cases}\end{aligned}$$

It is clear that

$$F\left(H\right)=R\left(f_{x_1,...,x_l}\left(.,H\right)\right)(H)\quad\text{for all } H\in\mathbf{G},$$

then $F\left(..\right)=R\left(f_{x_1,...,x_l}\left(.,..\right)\right)(..)$. This completes the proof of theorem ■

This theorem is analogue at classical case (see [10], [7], [9]) and this result can be considered as a Paley-Wienre theorem for the discrete Radon transform (see[3]).It is clear that

$$\mathbf{C}^{(\mathbf{\ z^n})}=\bigcup_{\mathbf{K}\subset\ \mathbf{Z^n}}\mathbf{C}_{\mathbf{K}}^{(\mathbf{\ z^n})}$$

and also

$$\mathcal{D}_{(m)}(\mathbf{G})=\bigcup_{\mathbf{K}\subset\ \mathbf{Z^n}}\mathcal{D}_{(\mathbf{m}),\mathbf{K}}(\mathbf{G})$$

We deduce from the above theorem the following theorem:
Theorem 3.5

$$R(\mathbf{C}^{(\mathbf{\ z^n})})\ =\ \mathcal{D}_{(\mathbf{m})}(\mathbf{G}) \tag{3.4}$$

Remark 3.6 $R(\mathbf{C}^{(\mathbf{\ z^n})})$ is a dense subspace of $R\left(\mathcal{S}(\mathbf{Z^n})\right)$, when we provide this last space by the induced topology of $\mathcal{S}_{(\mathbf{m})}(\mathbf{G})$. Before to prove that the space $R(\mathbf{C}^{(\mathbf{\ z^n})})$, which is equal at $\mathcal{D}_{(m)}(\mathbf{G})$, is dense subspace of $\mathcal{S}_{(\mathbf{m})}(\mathbf{G})$, we shall precise some notations useful for the sequel. For $F\in\mathcal{S}_{(\mathbf{m})}(\mathbf{G})$ we have, for $N\in\mathbf{N}^*$ and for all $(a,k)\in\mathcal{P}\times\mathbf{Z}$, the following inequality

$$(\ \frac{1}{2})^N q'\left(F\right)\leq q\left(F\right)\leq 2^N q'\left(F\right), \tag{3.5}$$

where

$$q_N'(F)=\sup_{(a,k)\in\mathcal{P}\times\mathbf{Z}}\frac{k^{2N}}{\left\|a\right\|^{2N}}\left|F(H\left(a,k\right))\right|$$

and

$$q_N(F) = \sup_{(a,k)\in\mathcal{P}\times\mathbf{Z}} \frac{(1+k^2)^N}{(1+\|a\|^2)^N} |F(H(a,k))|$$

In fact, we have for all $(a,k) \in \mathcal{P}\times\mathbf{Z}$

$$\frac{1}{2}\,\frac{k^2}{\|a\|^2} \leq \frac{(1+k^2)}{(1+\|a\|^2)} \leq 2\frac{k^2}{\|a\|^2},$$

then for all $N \in \mathbf{N}$

$$\left[\frac{1}{2}\frac{k^2}{\|a\|^2}\right]^N |F(H(a,k))| \leq \left[\frac{(1+k^2)^N}{(1+\|a\|^2)^N}\right]^N |F(H(a,k))|$$
$$\leq\leq \quad \left[2\frac{k^2}{\|a\|^2}\right]^N .\,|F(H(a,k))|$$

It follows from the above inequality that

$$\left(\frac{1}{2}\right)^N q'_N(F) \leq q_N(F) \leq 2^N q'_N(F).$$

Then $(q_N)_{N\in\mathbf{N}}$ and $(q'_N)_{N\in\mathbf{N}}$ are equivalent .
Let $K_r = B(0,r)$ the ball in $\mathbf{R^n}$ of rayon $r \in \mathbf{N}^*$ and the center zero, and let $K_r^e = \mathbf{Z^n} \cap \mathbf{B(0,r)}$.
Putting $\mathbf{G}(\mathbf{r}) = \{\mathbf{H} \in \mathbf{G} \;|\mathbf{H} \cap \mathbf{K_r^e} = \emptyset\}$. Now, we state and prove the following theorem:

Theorem 3.6 $R(\mathbf{C}^{(\mathbf{Z^n})})$ is a dense subspace of $\mathcal{S}_{(\mathbf{m})}(\mathbf{G})$

Proof. Let $H(a,k)$ a discrete hyperplane, it is well known that $d(0;H(a,k) = |k|\,\|a\|^{-1}$,where $d(0;H(a,k)$ is the distance of zero at the Euclidean hyperplane $H(a\,\|a\|^{-1}, k\,\|a\|^{-1})$ of $\mathbf{R^n}$ (see [8] , [9]). Let $F \in \mathcal{S}_{(\mathbf{m})}(\mathbf{G})$ we define a sequence $(F_r)_{r\in\mathbf{N}}$ of $\mathcal{D}_{(m)}(\mathbf{G})$ as follows

$$F_r(H(a,k)) = \begin{cases} F(H(a,k))\,\chi_{\mathbf{G_{(r)}}}(H(a,k)) & \text{if } r \geq \dfrac{|k|}{\|a\|} \\ 0 & \text{if } r < \dfrac{|k|}{\|a\|} \end{cases} \qquad (3.6)$$

Let $N \in \mathbf{N}^*$, we have

$$q'_N(F_r - F) = \sup_{(a,k))\in\mathcal{P}\times\mathbf{Z}} \frac{k^{2N}}{\|a\|^{2N}} |F_r(H(a,k)) - F(H(a,k))| \leq$$
$$\sup_{(a,k))\in\Lambda_-(r)} \frac{k^{2N}}{\|a\|^{2N}} |F_r(H(a,k)) - F(H(a,k))| +$$
$$+ \sup_{(a,k))\in\Lambda_+(r)} \frac{k^{2N}}{\|a\|^{2N}} |F_r(H(a,k)) - F(H(a,k))|,$$

where $\Lambda_+(r) = \left\{(a,k) \in \mathcal{P}\times\mathbf{Z} \;|\mathbf{r} \geq \dfrac{|\mathbf{k}|}{\|\mathbf{a}\|}\right\}$and $\Lambda_-(r) = \left\{(a,k) \in \mathcal{P}\times\mathbf{Z} \;|\mathbf{r} < \dfrac{|\mathbf{k}|}{\|\mathbf{a}\|}\right\}$.
Setting

$$l_-(N,r,F_r - F) = \sup_{(a,k))\in\Lambda_-(r)} \frac{k^{2N}}{\|a\|^{2N}} |F_r(H(a,k)) - F(H(a,k))|$$

and

$$l_+(N,r,F_r - F) = \sup_{(a,k))\in\Lambda_+(r)} \frac{k^{2N}}{\|a\|^{2N}} |F_r(H(a,k)) - F(H(a,k))|.$$

From (3.6), we obtain $l_+(N, r, F_r - F) = 0$, then

$$q'_N(F_r - F) \le l_-(N, r, F), \qquad (3.7)$$

since $F_r(H(a,k)) = 0$ whenever $(a,k) \in \Lambda_-(r)$ and $F \in \mathcal{S}_{(\mathbf{m})}(\mathbf{G})$. Let be b and j two integer numbers enough greater and let $N = b + j$, there exists a constant $C(N)$ which depends of N such that $q'_N(F) \le C(N)$ for all $F \in \mathcal{S}_{(\mathbf{m})}(\mathbf{G})$, since $F \in \mathcal{S}_{(\mathbf{m})}(\mathbf{G})$. It follows from the above inequality that

$$\left(\frac{k^2}{\|a\|^2}\right)^b |F(H(a,k))| \le C(N)\left(\frac{\|a\|^2}{k^2}\right)^j \le C(N)\frac{1}{r^{2j}},$$

since $r < |k| \, \|a\|^{-1}$ .Thus

$$l_-(b, r, F) \le C(N)\frac{1}{r^{2j}}.$$

Therefore if $r \to \infty$, $l_-(b, r, F) \to 0$. Consequently by (3.7), $q'_N(F_r - F)$ converges to zero when $r \to \infty$, for all $N \in \mathbf{N}$. This completes the proof of theorem. ∎

We deduce from theorem 3.6, the following properties

(a) $\overline{R(\mathbf{C}^{(\mathbf{Z^n})})} = \mathcal{S}_{(\mathbf{m})}(\mathbf{G})$

(b) $R(\mathbf{C}^{(\mathbf{Z^n})}) = \mathcal{D}_{(m)}(\mathbf{G})$

(c) $R(\mathbf{C}^{(\mathbf{Z^n})}) \subset \mathbf{R}\left(\mathcal{S}(\mathbf{Z^n}).\right) \subset \mathcal{S}_{(\mathbf{m})}(\mathbf{G})$

where $\overline{R(\mathbf{C}^{(\mathbf{Z^n})})}$ is the closure of the set $R(\mathbf{C}^{(\mathbf{Z^n})})$. From (a),(b),(c),it follows that $\overline{R(\mathcal{S}(\mathbf{Z^n}))} = \mathcal{S}_{(\mathbf{m})}(\mathbf{G})$.

To prove that $R\left(\mathcal{S}(\mathbf{Z^n}).\right) = \mathcal{S}_{(\mathbf{m})}(\mathbf{G})$, it suffice to show that $R\left(\mathcal{S}(\mathbf{Z^n})\right)$ is closed in $\mathcal{S}_{(\mathbf{m})}(\mathbf{G})$ when $\mathcal{S}_{(\mathbf{m})}(\mathbf{G})$ is equipped with the topology defined by the family of semi -norms $(q_N)_N$ (see introduction). Now, we prove and state the following theorem:

Theorem 3.7: $R\left(\mathcal{S}(\mathbf{Z^n})\right) = \mathcal{S}_{(\mathbf{m})}(\mathbf{G})$

Proof. It suffice to prove that $R(\mathcal{S}(\mathbf{Z^n}))$ is closed space of $\mathcal{S}_{(\mathbf{m})}(\mathbf{G})$.

Let $F \in \overline{R(\mathcal{S}(\mathbf{Z^n}))}$, there exists $F_j = R(f_j)$ with $f_j \in \mathcal{S}(\mathbf{Z^n})$ for all $j \in \mathbf{N}$, such that for all $N \in \mathbf{N}$

$$q_N(F_j - F) \to 0, \qquad (3.8)$$

when $j \to \infty$. In particulary $q_0(F_j - F) \to 0$ when $j \to \infty$. Let $H_r = H(a_r, a_r m)$ a discrete hyperplane where $a_r = (1, r, r^2, ..., r^{n-1})$ and $(m, r) \in \mathbf{Z^n} \times \mathbf{N}^*$. It is clear that

$$|R(f_j)(H_r) - F(H_r)| \le q_0(R(f_j) - F), \text{ for all } r \in \mathbf{N}^*.$$

Since $q_0(F_j - F)$ converges to zero when $j \to \infty$, we deduce

$$\lim_{j \to +\infty} R(f_j)(H_r) = F(H_r), \quad \text{for all } r \in \mathbf{N}^*. \qquad (3.9)$$

Where $H(a_r, a_r m)$ a sequence of discrete hyperplanes containing m. By the inversion formula of discrete Radon transform (see [1]) we have

$$\begin{aligned} \lim_{r\to+\infty}\lim_{j\to+\infty} R(f_j)(H_r) &= \lim_{r\to+\infty} F(H_r) \\ &= \lim_{j\to+\infty}(\lim_{r\to+\infty} R(f_j)(H_r)) \qquad (3.10) \\ &= \lim_{j\to+\infty} f_j(m) \end{aligned}$$

Let $y_r(m) = F(H(a_r, a_r.m))$. Since $F \in \mathcal{S}(\mathbf{G})$ the sequence $(y_r(m))_{r\in\mathbf{N}}$ is bounded for all $m \in \mathbf{Z^n}$, that is, $|y_r(m)| \le C$ where C is a constant which depends only of F (see introduction for definition of $\mathcal{S}(\mathbf{G})$). Since $y_r(m)$ is bounded there exists then a closed ball in

$\mathbf{R^n}$ namely $B(0, r_0)$ such that $y_r(m) \in B(0, r_0)$ for all $(r, m) \in \mathbf{N}^* \times \mathbf{Z}$. By Bolzano Weirstrass's theorem $y_{r_j}(m)$ converges to $f(m)$ when $j \to \infty$. Where f is a function defined in $\mathbf{Z^n}$ and which belongs to $\mathcal{S}(\mathbf{Z^n})$, because for all $N \in \mathbf{N}^*$ we have/

$$|y_r(m)| \leq C_N \frac{||a_r||^{2N}}{|a_r m|^{2N}} \leq C_N \frac{1}{||m||^{2N} \cos^{2N}(a_r, m)}$$

We note that $\cos^{2N}(a_r, m) \neq 0$ for all r enough great. Indeed if $\cos(a_r, m) = 0$ this implies that $m_1 + r.m_2 + ... + r^{n-1}m_n = 0$ it follows that r divide m_1 which is absurd (since r enough great). The equality (3.10) implies that $f_j \to f$, where $(f(m) = \lim_{r \to +\infty} F(H_r(a_r, a_r.m)))$, simply when $j \to \infty$. Since $f_j \in \mathcal{S}(\mathbf{Z^n})$ for all $j \in \mathbf{N}$, by Fisher- Riez's theorem (see[5]) there exists a subsequence f_{j_r} such that f_{j_r} converges to f for the norm $||.||_1$. From theorem 2.6, we obtain $R(f_{j_r})$ converges simply to $R(f)$. It follows from (3.8) that $F_{j_r} = R(f_{j_r})$ converges simply to F when $r \to \infty$, consequently $F = R(f)$. This prove the theorem. ■

IV. INVERSION FORMULA FOR THE DISCRETE RADON TRANSFORM OF $\mathbf{Z^N}$

We shall now show how a function $f \in \mathcal{S}(\mathbf{Z^n})$ can be recovered explicitly from its discrete Radon transform, precisely. In this section we shall establish a new inversion formula for the discrete Radon transform via the operators $\nabla_m (m \in \mathbf{Z^n})$, where ∇_m is a linear continuous operator of $R(\mathcal{S}(\mathbf{Z^n}))$ into $R(\mathcal{S}(\mathbf{Z^n}))$. We recall that $R(\mathcal{S}(\mathbf{Z^n})) = \mathcal{S}_{(\mathbf{m})}(\mathbf{G})$ (see theorem 3.7, section 3) and that $R(\mathcal{S}(\mathbf{Z^n}))$ is the image under discrete Radon transform of $\mathcal{S}(\mathbf{Z^n})$(see[1]). The operator ∇_m ($m \in \mathbf{Z^n}$) are called the multiplication's operators by the piquant functions.
Let $m_0 \in \mathbf{Z^n}$ a element of $\mathbf{Z^n}$ which shall be fixed once for all. Define the operator ∇_{m_0} by:

$$\nabla_{m_0} Rf = \sum_{m \in \mathbf{Z^n}} (\exp(\|m\|^2)\chi_{m_0}(m) + 1) f(m)\, \chi_{\mathbf{G_m}} \qquad (4.1)$$

Where $\mathbf{G_m}$ is the set of all $H(a, k)$ containing m, and f is a element of Schwartz's space $\mathcal{S}(\mathbf{Z^n})$ andχ_{m_0}(resp.$\chi_{\mathbf{G_m}}$) the characteristic function of m_0 (resp.$\mathbf{G_{m_0}}$). We note g_{m_0} the function

$$g_{m_0}(m) = (\exp(\|m\|^2)\chi_{m_0}(m) + 1) \qquad (4.2)$$

It is clear that $g_{m_0} \in l^\infty(\mathbf{Z^n})$, because $g_{m_0}(m_0) = \exp(\|m_0\|^2) + 1$ and $g_{m_0}(m) = 1$ if $m \neq m_0$.
Remark 4.1: **(a)** The expression (4.1) can be transformed as follows

$$\nabla_{m_0} Rf = R(g_{m_0}.f), \text{ for all } f \in \mathcal{S}(\mathbf{Z^n}) \qquad \mathbf{(\ 4.3)}$$

(b) We deduce from (4.3) that $\nabla_{m_0} Rf \in R(\mathcal{S}(\mathbf{Z^n}))$, since the function $g_{m_0} f$ belongs to $\mathcal{S}(\mathbf{Z^n})$, (because $f \in \mathcal{S}(\mathbf{Z^n})$ and $g_{m_0} \in l^\infty(\mathbf{Z^n})$).
Proposition 4.2 The space $R(\mathcal{S}(\mathbf{Z^n}))$ is normed space by the norm

$$|||R(f)|||_1 = \sup_{a \in \mathcal{P}} \sum_{k \in \mathbf{Z}} R(|f|)(H(a,k)), \text{for all } f \in \mathcal{S}(\mathbf{Z^n}) \qquad \mathbf{(4.4)}$$

Proof. At first, the formula (4.4) have a sense. Indeed, from ([1] corollary 3.9, formula 3.7) we have

$$\sum_{k \in \mathbf{Z}} R(|f|)(H(a,k)) \leq \sum_{k \in \mathbf{Z}} \sum_{m \in H(a,k)} |f|(m)$$
$$\leq \sum_{m \in \mathbf{Z^n}} |f(m)| < \infty \qquad (\ 4.5)$$

Consequently

$$|||R(f)|||_1 \leq ||f||_1$$

The others properties of the norm are easy. ■

Remark 4.3 Let $H(a,k) \in \mathbf{G}$
(a) If $m_0 \notin H(a,k)$, then

$$\begin{aligned}\nabla_{m_0} Rf\left(H(a,k)\right) &= \sum_{m\in H(a,k), m\neq m_0} \left(g_{m_0}.f\right)(m) \\ &= \sum_{m\in H(a,k)} f(m) \\ &= Rf\left(H(a,k)\right),\end{aligned}$$

since $m_0 \notin H(a,k)$ implies that $g_{m_0}(m) = 1$, for all $m \in H(a,k)$.
(b) If $m_0 \in H(a,k)$, we have

$$\begin{aligned}\nabla_{m_0} Rf\left(H(a,k)\right) &= R\left(g_{m_0}.f\right)\left(H(a,k)\right) \\ &= \sum_{m\in H(a,k)} (\exp(\|m\|^2)\chi_{m_0}(m) + 1) f(m) \\ &= (\exp(\|m_0\|^2) \;\; + 1) f(m_0) \; + \sum_{m\in H(a,k)\;, m\neq m_0} f(m) \\ &= \exp(\|m_0\|^2)\; f(m_0) \;\; + Rf\left(H(a,k)\right).\end{aligned}$$

It follows from (a) and (b)

$$\nabla_{m_0} Rf\left(H(a,k)\right) = \begin{cases} Rf(H(a,k)) & \text{if } \; m_0 \notin H(a,k) \\ \exp(\|m_0\|^2)\; f(m_0) \;\; + Rf\left(H(a,k)\right) & \text{if } m_0 \in H(a,k)\end{cases} \tag{4.6}$$

(c) We obtain from (4.6) the following inequality

$$\begin{aligned}\nabla_{m_0} R(|f|)\left(H(a,k)\right) &\leq R(|f|)\left(H(a,k)\right) + (\exp(\|m_0\|^2) + 1) R(|f|)\left(H(a,k)\right) \\ &\leq R(|f|)\left(H(a,k)\right)\left(\exp(\|m_0\|^2) + 2\right),\end{aligned}$$

The above inequality implies

$$\sup_{a\in\mathcal{P}} \sum_{k\in\mathbf{Z}} \nabla_{m_0} R(|f|)(H(a,k)) \leq \; |||R(f)|||_1 \left[\exp(\|m_0\|^2) + 2\right]$$

Hence

$$|||\nabla_{m_0} R(f)|||_1 \leq \left(\exp(\|m_0\|^2) + 2\right) |||Rf|||_1 . \tag{4.7}$$

Now we state and prove the following proposition useful for the sequel.

Proposition 4.4 the operator ∇_{m_0} is linear continuous of $R\left(\mathcal{S}(\mathbf{Z^n})\right)$ into $R\left(\mathcal{S}(\mathbf{Z^n})\right)$ when we provide $R\left(\mathcal{S}(\mathbf{Z^n})\right)$ by the norm $|||.|||_1$.

Proof. It is clear that ∇_{m_0} is a linear operator of $R\left(\mathcal{S}(\mathbf{Z^n})\right)$ into $R\left(\mathcal{S}(\mathbf{Z^n})\right)$ see formula (4.1).Showing that ∇_{m_0} is a continuous operator. Let($Rf_j)_{j\in\mathbf{N}}$ a sequence of $R\left(\mathcal{S}(\mathbf{Z^n})\right)$ such that $|||Rf_j.|||_1$ converges to zero when $j \to \infty$. By remark 4.3 (see inequality (4.7)) we obtain

$$|||\nabla_{m_0} R(f_j)|||_1 \leq \left(\exp(\|m_0\|^2) + 2\right) |||Rf_j|||_1 . \tag{4.8}$$

Then $|||\nabla_{m_0} R(f_j)|||_1 \to 0$ when $j \to \infty$ .This completes the proof of proposition. We will be need the following lemma ■

Lemma (4.5) Let $f \in \mathcal{S}(\mathbf{Z^n})$ and $m_0 \in \mathbf{Z^n}$. Then we have, for all $a \in \mathcal{P}$, the following formula

$$\begin{aligned}f(m_0) = \exp(-\|m_0\|^2) \sum_{k\in\mathbf{Z}} \nabla_{m_0} R(f)(H(a,k)) - \\ - \sum_{m\in\mathbf{Z^n}} \sum_{k\in\mathbf{Z}} \exp(-\|m_0\|^2)\lambda(m)\, \nabla_m R(f)\left(H(a,k)\right)\end{aligned} \tag{4.9}$$

Where $\lambda(m) = \dfrac{1}{1+T}\exp\Big(-\|m\|^2\Big)$ with $T = \sum_{m\in\mathbf{Z^n}}\exp(-\|m\|^2)$

Proof. Before to show this lemma, proving that the series

$$\sum_{m\in\mathbf{Z^n}}\sum_{k\in\mathbf{Z}}\lambda(m)\nabla_m R(f)(H(a,k)),$$

namely A, have a sense for all $a\in\mathcal{P}$. Indeed

$$\begin{aligned} A &= \frac{1}{1+T}\sum_{m\in\mathbf{Z^n}.}\exp(-\|m\|^2)\left(\sum_{k\in\mathbf{Z}}R(g_m f)(H(a,k))\right) \\ &= \frac{1}{1+T}\sum_{m\in\mathbf{Z^n}.}\exp(-\|m\|^2)\left(\sum_{t\in\mathbf{Z^n}}g_m(t)f(t)\right), \end{aligned}$$

since $\sum_{k\in\mathbf{Z}}R(g_m f)(H(a,k)) = \sum_{t\in\mathbf{Z^n}}g_m(t)f(t)$, (see formula (2.5)). It follows that

$$A = \frac{1}{1+T}\sum_{m\in\mathbf{Z^n}.}\exp(-\|m\|^2)\left(\sum_{t\in\mathbf{Z^n}}f(t)+\exp(\|m\|^2)f(m)\right),$$

then

$$|A| \le \frac{T}{1+T}\,||f||_1 + \frac{1}{1+T}\,||f||_1 \le ||f||_1$$

■

Proof. of lemma 4.5 Let $f\in\mathcal{S}(\mathbf{Z^n})$ and $m_0\in\mathbf{Z^n}$ fixing a element $a\in\mathcal{P}$. Applying the formula (2.5) at the function $g_m f$, and using also the definition of ∇_{m_0}, we obtain

$$\begin{aligned} \sum_{k\in\mathbf{Z}}\nabla_{m_0}Rf(H(a,k)) &= \sum_{k\in\mathbf{Z}}R(\;g_{m_0}f\;)(H(a,k)) \\ &= \sum_{m\in\mathbf{Z^n}}g_{m_0}(m)f(m)\;. \end{aligned}$$

Decomposing the right -hand side we have

$$\sum_{k\in\mathbf{Z}}\nabla_{m_0}Rf(H(a,k)) = g_{m_0}(m_0)f(m_0) + \sum_{m\in\mathbf{Z^n},\mathbf{m}\neq\mathbf{m_0}}g_{m_0}(m)f(m)\;.$$

From (4.2) the above equality becomes

$$\sum_{k\in\mathbf{Z}}\nabla_{m_0}Rf(H(a,k)) = \exp(\|m_0\|^2)f(m_0) + \sum_{m\in\mathbf{Z^n}}f(m),$$

which gives

$$f(m_0) = \exp(-\|m_0\|^2)\left[\sum_{k\in\mathbf{Z}}\nabla_{m_0}Rf(H(a,k)) - \sum_{m\in\mathbf{Z^n}}f(m)\right]. \qquad (4.10)$$

By summation over $m_0\in\mathbf{Z^n}$,we obtain

$$\sum_{m'\in\mathbf{Z^n}}f(m') = \sum_{m'\in\mathbf{Z^n}}\left(\exp(-\|m'\|^2)\sum_{k\in\mathbf{Z}}\nabla_{m'}Rf(H(a,k))\right) - T\sum_{m\in\mathbf{Z^n}}f(m), \qquad (4.11)$$

recall that $T = \sum_{m\in\mathbf{Z^n}}\exp(-\|m\|^2)$. The equality (4.11) gives

$$\sum_{m\in\mathbf{Z^n}}f(m) = \frac{1}{1+T}\sum_{k\in\mathbf{Z},\mathbf{m}\in\mathbf{Z^n}}\exp(-\|m\|^2)\nabla_m Rf(H(a,k)), \qquad (4.12)$$

remplacing in the formula (4.10) $\sum_{m\in\mathbf{Z^n}} f(m)$ by its expression (see (4.12)); we obtain

$$f(m_0) = \exp(-\|m_0\|^2)\sum_{k\in\mathbf{Z}} \nabla_{m_0} Rf(H(a,k)).$$
$$-\frac{1}{1+T}\sum_{k\in\mathbf{Z},\mathbf{m}\in\mathbf{Z^n}} \exp(-\|m\|^2-\|m_0\|^2)\nabla_m Rf(H(a,k)). \quad (4.13)$$

Thus

$$f(m_0) = \exp(-\|m_0\|^2)\sum_{k\in\mathbf{Z}} \nabla_{m_0} Rf(H(a,k)) -$$
$$-\sum_{k\in\mathbf{Z},\mathbf{m}\in\mathbf{Z^n}} \exp(-\|m_0\|^2)\lambda(m)\nabla_m Rf(H(a,k)) \quad (4.14)$$

This proves the lemma ■

We now introduce a other operator $\Delta_{m_0,m}$ with $(m,m_0)\in(\mathbf{Z^n})^{\mathbf{2}}$ which shall be defined as follows

$$\Delta_{m_0,m} = \begin{cases} \exp(-\|m_0\|^2)(1-\lambda(m_0))\nabla_{m_0} & \text{if } m_0 = m \\ -\exp(-\|m_0\|^2)\,\lambda(m)\nabla_m & \text{if } m_0 \neq m\,, \end{cases} \quad (4.15)$$

we now state and prove the inversion formula for the discrete Radon transform.

Theorem 4.6 let $f\in\mathcal{S}(\mathbf{Z^n})$ and $m_0\in\mathbf{Z^n}$ then, for all $a\in\mathcal{P}$, the operator R can be inverted by the following formula

$$f(m_0) = \sum_{k\in\mathbf{Z},\mathbf{m}\in\mathbf{Z^n}} (\Delta_{m_0,m})Rf(H(a,k)) \quad (4.16)$$

Proof. From equality (4.14), we have

$$f(m_0) = \sum_{k\in\mathbf{Z}} \exp(-\|m_0\|^2)\nabla_{m_0} Rf(H(a,k)) -$$
$$-\sum_{k\in\mathbf{Z},\mathbf{m}\in\mathbf{Z^n}} \exp(-\|m_0\|^2)\lambda(m)\nabla_m Rf(H(a,k)).$$

But

$$\sum_{m\in\mathbf{Z^n}} \lambda(m)\nabla_m Rf(H(a,k)) = \lambda(m_0)\nabla_{m_0} Rf(H(a,k)) +$$
$$+\sum_{m\in\mathbf{Z^n},\mathbf{m}\neq\mathbf{m_0}} \lambda(m)\nabla_m Rf(H(a,k)).$$

It follows from (4.14)

$$f(m_0) = \sum_{k\in\mathbf{Z}} \exp(-\|m_0\|^2)(1-\lambda(m_0))(\nabla_{m_0} Rf(H(a,k)) -$$
$$-\exp(-\|m_0\|^2)\sum_{k\in\mathbf{Z}}\sum_{m\in\mathbf{Z^n},\mathbf{m}\neq\mathbf{m_0}} \lambda(m)\nabla_m Rf(H(a,k))$$

Thus

$$f(m_0) = \sum_{k\in\mathbf{Z}} \exp(-\|m_0\|^2)\left\{(1-\lambda(m_0))\nabla_{m_0} Rf - \sum_{m\in\mathbf{Z^n},\mathbf{m}\neq\mathbf{m_0}} \lambda(m)\nabla_m Rf\right\}(H(a,k)$$
$$= \sum_{k\in\mathbf{Z},}\sum_{m\in\mathbf{Z^n}} (\Delta_{m_0,m})Rf(H(a,k)) \quad \text{, see equality (4.15)}$$

This proves the theorem ■

Remark 4.7 (a) The operator $(\Delta_{m_0,m})$ have the same properties that the operator ∇_{m_0} (see (4.15)), then $(\Delta_{m_0,m})$ is linear continuous of $R\left(\mathcal{S}(\mathbf{Z^n})\right)$ into $R\left(\mathcal{S}(\mathbf{Z^n})\right)$
(b)The inversion formula (4.16) hold if we replace the operator $(\Delta_{m_0,m})$ by the operator

$$\Delta^{(\varphi)}_{m_0,m} = \begin{cases} \dfrac{1}{\varphi(m_0)}\left(1-\lambda_\varphi(m_0)\right)\nabla^{(\varphi)}_{m_0} & \text{if } m_0 = m \\ -\dfrac{1}{\varphi(m_0)}\lambda_\varphi(m)\nabla^{(\varphi)}_{m} & \text{if } m_0 \neq m \end{cases}$$

Where φ is a real -valued function defined on $\mathbf{Z^n}$ such that $\varphi(m) \neq 0$ for all $m \in \mathbf{Z^n}$ and $m \to \dfrac{1}{\varphi(m)}$ is rapidly decreasing function, with $c(\varphi) = \sum_{m\in\mathbf{Z^n}} \dfrac{1}{\varphi(m)} \neq -1$ and $\lambda_\varphi(m) = \dfrac{1}{\varphi(m)(1+c(\varphi))}$; finally $\nabla^{(\varphi)}_{m_0}Rf = R(g_{m_0,\varphi}.f)$ with $g_{m_0,\varphi}(m) = (\varphi(m)\chi_{m_0}(m)+1)$

V. INVERSION FORMULA FOR THE DUAL DISCRETE RADON TRANSFORM

In this section we prove a inversion formula for the dual discrete Radon transform R^*. Throughout this section we note $H_0 = H(a_0,k_0)$ with $a_0 = (a_{0,1}, a_{0,2}, ..., a_{0,n}) \in \mathcal{P}$ and $k_0 \in \mathbf{Z}$. We begin in this section by study the properties of the operator ∇_{H_0} which shall be useful in the sequel. Define the operator ∇_{H_0} as

$$\nabla_{H_0}R^*F(m) = \frac{1}{2}\exp\left(\|a_0\|^2 + k_0^2\right)\chi_{H_0}(m)F(H_0) + R^*F(m), \qquad (5.1)$$

for all $m \in \mathbf{Z^n}$ and $F \in \mathcal{S}_{\mathbf{1},*}(\mathbf{G})$. Let ξ_{H_0} be the function defined by

$$\xi_{H_0}(H) = \exp\left(\|a\|^2 + k^2\right)\beta_{H_0}(H) + 1, \qquad (5.2)$$

where $H = H(a,k)$ and χ_{H_0}(resp.β_{H_0}) the characteristic function of the set H_0 (resp.β_{H_0} is the function of the section 3, in the end of proof's lemma3.1). We design, in the sequel, by $\delta(a,k)$ the real number $\exp\left(-\|a\|^2 - k^2\right)$.

Proposition 5.1 Let F be a element of $\mathcal{S}_{\mathbf{1},*}(\mathbf{G})$,then

$$\nabla_{H_0}R^*F = R^*(\xi_{H_0}F) \qquad (5.3)$$

Proof. Let $m \in \mathbf{Z^n}$, distinguishing two cases::
(1) Suppose that $m \in H_0$, and calculating $R^*(\xi_{H_0}F)(m)$

$$\begin{aligned} R^*(\xi_{H_0}F)(m) &= \frac{1}{2}\sum_{a\in\mathcal{P}} F(H(a,am))\,\xi_{H_0}(H(a,am)) \\ &= \frac{1}{2}F(H(a_0,a_0m))\left[\exp\left(\|a_0\|^2 + (a_0m)^2\right) + 1\right] + \\ &+ \frac{1}{2}\sum_{a\in\mathcal{P},a\neq a_0} F(H(a,am))\,\xi_{H_0}(H(a,am)). \ (5.4) \end{aligned}$$

Since $a \neq a_0$ (then $H(a,am) \neq H(a_0,k_0)$) the above equality come down to

$$R^*(\xi_{H_0}F)(m) = \frac{1}{2}F(H(a_0,a_0m))\exp\left(\|a_0\|^2 + (a_0m)^2\right) + R^*(F)(m)$$

(2) case $m \notin H(a_0,k_0) = H_0$. If $m \notin H(a_0,k_0)$, then $k_0 \neq a_0m$ this implies that $\xi_{H_0}(H(a,am)) = 1$, since $\beta_{H_0}(H) = 0$. It follows that

$$\begin{aligned} R^*(\xi_{H_0}F)(m) &= \frac{1}{2}\sum_{a\in\mathcal{P}} F(H(a,am))\,\xi_{H_0}(H(a,am)) \\ &= R^*F(m.) \end{aligned}$$

Hence

$$(5.5)\qquad R^*(\ \xi_{H_0}F)\,(m) = \begin{cases} \frac{1}{2}F\left(H\left(a_0, a_0m\right)\right)\ \delta(a_0,(a_0m))^{-1}\ + R^*(\ F)\,(m) & \text{if } m \in H_0 \\ R^*F\,(m) & \text{if } m \notin H_0 \end{cases}$$

This equality can be transformed as follows

$$R^*(\ \xi_{H_0}F)\,(m) = \frac{1}{2}\exp\left(\left\|a_0\right\|^2 + k_0^2\right)\right) F\left(H_0\right)\chi_{H_0}\left(m\right) + R^*F\left(m.\right). \quad (5.6)$$

From (5.1) we deduce that $R^*(\ \xi_{H_0}F) = \nabla_{H_0}\left(\ R^*F\right)$. This proves the proposition ■

It is clear that $\xi_{H_0} \in l^\infty\left(\mathbf{G}\right)$,because $\xi_{H_0}\left(H_0\right) = \exp\left(\left\|a_0\right\|^2 + k_0^2\right)\right) + 1$ and if $H \neq H_0$ we have $\xi_{H_0}\left(H\right) = 1$ then $\xi_{H_0}F \in \mathcal{S}_{\mathbf{1},*}(\mathbf{G})$ whenever $F \in \mathcal{S}_{\mathbf{1},*}(\mathbf{G})$. By the above proposition, we have also $\nabla_{H_0}\left(R^*F\right) \in R^*\left(\mathcal{S}_{\mathbf{1},*}(\mathbf{G})\right)$ and ∇_{H_0} is a linear operator defined on $R^*\left(\mathcal{S}_{\mathbf{1},*}(\mathbf{G})\right)$ into $R^*\left(\mathcal{S}_{\mathbf{1},*}(\mathbf{G})\right)$. Let now $\Box_{H_0}$ the operator defined by

$$\begin{aligned}(\ \Box_{H_0}F)\,(H) &= \left[\exp\left(\left\|a\right\|^2 + k^2\right)\beta_{H_0}\left(H\right) + 1\right] F\left(H\right) \qquad (5.7)\\ &= \ \xi_{H_0}\left(H\right) F\left(H\right)\end{aligned}$$

$\Box_{H_0}$ will be called multiplication's operator .It follows from the proposition 5.1 and 5.5 that

$$\begin{aligned}R^*\left(\Box_{H_0}F\right) &= \nabla_{H_0}R^*F\\ &= R^*(F.\xi_{H_0}) \qquad (\ 5.8)\end{aligned}$$

The formula (5.8) is analogue to the operator transmutation 's formula. We shall now state and prove the following lemmas which are useful for the sequel

Lemma 5.2 Let $F \in \mathcal{S}_{\mathbf{1},*}(\mathbf{G})$,then

$$\sum_{m\in\mathbf{Z}^*} R^*F\left(m\right) \quad =\frac{1}{2} \sum_{a\in\mathcal{P},k\in\mathbf{Z}} F\left(H\left(a,k\right)\right) \qquad (5.9)$$

Proof. First of all, the series $\sum_{a\in\mathcal{P},k\in\mathbf{Z}} F\left(H\left(a,k\right)\right)$ is absolutely convergent, in fact

$$\begin{aligned}\sum_{a\in\mathcal{P},k\in\mathbf{Z}} \left|F\left(H\left(a,k\right)\right)\right| &\leq \sum_{a\in\mathcal{P},k\in\mathbf{Z}} (1+\left\|a\right\|^2+k^2)^{2N}(1+\left\|a\right\|^2+k^2)^{-2N}\left|F\left(H\left(a,k\right)\right)\right|\\ &\leq C_N\left(\sum_{a\in\mathcal{P}}\frac{1}{(1+\left\|a\right\|^2)^N}\right)\left(\sum_{k\in\mathbf{Z}}\frac{1}{(1+k^2)^N}\right),\end{aligned}$$

since $F \in \mathcal{S}_{\mathbf{1},*}(\mathbf{G})$ and $(1+\left\|a\right\|^2+k^2)^2 \geq \left(1+\left\|a\right\|^2\right)\left(1+k^2\right)$. Where C_N is a constant which depend only of N and F. The number N is chosen enough great. Consequently

$$\sum_{a\in\mathcal{P},k\in\mathbf{Z}} \left|F\left(H\left(a,k\right)\right)\right| < \infty, \qquad (5.11)$$

By definition of R^*F (see(2.2)) we have

$$\sum_{m\in\mathbf{Z}\times\{\mathbf{0}\}} R^*F\left(m\right) \ = \sum_{m\in\mathbf{Z}\times\{\mathbf{0}\}}\left[\frac{1}{2}\ \sum_{a\in\mathcal{P}} F\left(H\left(a,am\right)\right)\right].$$

When $(m,a) \in (\mathbf{Z}\times\{\mathbf{0}\})\times\mathcal{P}$ vary in $(\mathbf{Z}\times\{\mathbf{0}\})\times\mathcal{P}$ then $a.m$ describe $\mathbf{Z}$ (since $F \in \mathcal{S}_{\mathbf{1},*}(\mathbf{G})$). So

$$\sum_{m\in\mathbf{Z}\times\{\mathbf{0}\}} R^*F\left(m\right) = \frac{1}{2} \sum_{a\in\mathcal{P},k\in\mathbf{Z}} F\left(H\left(a,k\right)\right), \qquad (5.12)$$

and this completes the proof of lemma ■

The formula (5.12) is analogue to Poisson summation and it is useful to invert the dual discrete Radon transform.

Lemma 5.3 let $F \in \mathcal{S}_{\mathbf{1},*}(\mathbf{G})$, then the series

$$\sum_{a\in\mathcal{P},k\in\mathbf{Z},\mathbf{m}\in\mathbf{Z}^*} \gamma\left(H\left(a,k\right)\right)\ \nabla_{H_{(a,k)}} R^* F(m)$$

is absolutely covergent, where $\gamma\left(H\left(a,k\right)\right) = \frac{1}{1+L}\exp\left(-\left\|a\right\|^2-k^2\right)$ with $L = \sum_{a\in\mathcal{P},k\in\mathbf{Z}} \exp\left(-\left\|a\right\|^2-k^2\right)$.

Proof. Let $A = 2\sum_{a\in\mathcal{P},k\in\mathbf{Z},\mathbf{m}\in\mathbf{Z}^*} \gamma\left(H\left(a,k\right)\right)\ \nabla_{H_{(a,k)}} R^* F(m)$.From equality (5.9) and proposition 5.1 we have

$$\begin{aligned} A &= 2\sum_{a\in\mathcal{P},k\in\mathbf{Z}} \gamma\left(H\left(a,k\right)\right)\left(\sum_{m\in\mathbf{Z}\times\{\mathbf{0}\}} R^*\left(\xi_{H(a,k)}.F\right)(m)\right) \\ &= \sum_{a\in\mathcal{P},k\in\mathbf{Z}} \gamma\left(H\left(a,k\right)\right)\left(\sum_{(a',k')\in\mathcal{P}\times\mathbf{Z}} \xi_{H(a,k)}(H\left(a',k'\right)F(H\left(a',k'\right))\right) \end{aligned}$$

Using, now, the equality(5.2) we obtain

$$\begin{aligned} A = &\sum_{a\in\mathcal{P},k\in\mathbf{Z}} \gamma\left(H\left(a,k\right)\right)(\xi_{H(a,k)}(H\left(a,k\right)F(H\left(a,k\right)+ \\ &+\sum_{a\in\mathcal{P},k\in\mathbf{Z}} \gamma\left(H\left(a,k\right)\right)\sum_{(a',k')\in\mathcal{P}\times\mathbf{Z},\mathbf{a}'\neq\mathbf{a},\mathbf{k}'\neq\mathbf{k}} \xi_{H(a,k)}(H\left(a',k'\right)F(H\left(a',k'\right))). \end{aligned}$$

It follows that

$$\begin{aligned} A &= \sum_{a\in\mathcal{P},k\in\mathbf{Z}} \gamma\left(H\left(a,k\right)\right)\left[(\exp\left(\left\|a\right\|^2+k^2\right)+1)F(H\left(a,k\right)+\sum_{a'\neq a,k'\neq k} F\left(H(a',k')\right)\right] \\ &= \sum_{a\in\mathcal{P},k\in\mathbf{Z}} \frac{1}{(1+L)}F(H\left(a,k\right))+\gamma\left(H\left(a,k\right)\right)\left(F\left(H\left(a,k\right)\right)+\sum_{a'\neq a,k'\neq k} F\left(H(a',k')\right)\right) \\ &= \sum_{a\in\mathcal{P},k\in\mathbf{Z}}\left[\frac{1}{(1+L)}FH\left(a,k\right)+\gamma\left(H\left(a,k\right)\right)\sum_{a'\in\mathcal{P},k'\in\mathbf{Z}} F\left(H(a',k')\right)\right] \\ &= \frac{1}{(1+L)}\sum_{a\in\mathcal{P},k\in\mathbf{Z}} F\left(H(a,k)\right)+\left[\sum_{a'\in\mathcal{P},k'\in\mathbf{Z}} F\left(H(a',k')\right)\right]\left[\sum_{a\in\mathcal{P},k\in\mathbf{Z}} \gamma\left(H\left(a,k\right)\right)\right] \end{aligned}$$

A can be transformed as follows

$$A = \left(\frac{1}{(1+L)}+\frac{L}{(1+L)}\right)\sum_{a\in\mathcal{P},k\in\mathbf{Z}} F\left(H(a,k)\right)$$

Thus, $|A| \leq \sum_{a\in\mathcal{P},k\in\mathbf{Z}} \left|F\left(H(a,k)\right)\right| < \infty$ (see the proof of lemmas 5.2 and 5.11), the lemma is proved ■

Lemma 5.4 Let $F \in \mathcal{S}_{\mathbf{1},*}(\mathbf{G})$, then

$$F(H_0) = 2\delta(a_0,k_0) \sum_{m\in\mathbf{Z_1}\times\{\mathbf{0}\}} \nabla_{H_0} R^* F(m) -$$
$$- 2\delta(a_0,k_0) \sum_{(a,k)\in\mathcal{P}\times\mathbf{Z},\mathbf{m}\in\mathbf{Z}\times\{\mathbf{0}\}} \gamma(H(a,k)) \nabla_{H_{H(a,k)}} R^* F(m), \qquad (5.13)$$

where $\gamma(H(a,k)) = \delta(a,k)(1+L)^{-1}$ with $L = \sum_{(a,k)\in\mathcal{P}\times\mathbf{Z}} \delta(a,k)$,and $H_0 = H(a_0,k_0)$ the discrete hyperplane

Proof. From(5.9) and (5.3) we have

$$\sum_{m\in\mathbf{Z}\times\{\mathbf{0}\}} \nabla_{H_0} R^* F(m) = \sum_{m\in\mathbf{Z}\times\{\mathbf{0}\}} R^*(\xi_{H_0}F)(m)$$
$$= \frac{1}{2} \sum_{(a,k)\in\mathcal{P}\times\mathbf{Z},} (\xi_{H_0}F)(H(a,k)) \qquad (5.14)$$

We decompose the right -hand side of the above equality, we obtain

$$\sum_{m\in\mathbf{Z}\times\{\mathbf{0}\}} R^*(\xi_{H_0}F)(m) = \frac{1}{2}\left(\delta(a_0,k_0)^{-1}+1\right) F(H_0) + \frac{1}{2} \sum_{a\in\mathcal{P},k\in\mathbf{Z},\mathbf{a_0}\neq\mathbf{a},\mathbf{k_0}\neq\mathbf{k}} F(H(a,k))$$
$$= \frac{1}{2}\delta(a_0,k_0)^{-1} F(H_0) + \frac{1}{2} \sum_{a\in\mathcal{P},k\in\mathbf{Z}} F(H(a,k)). \qquad (5.15)$$

The equality (5.15) leads to

$$F(H_0) = \delta(a_0,k_0) \left\{ 2 \sum_{m\in\mathbf{Z_1}\times\{\mathbf{0}\}} \nabla_{H_0} R^* F(m) - \sum_{a\in\mathcal{P},k\in\mathbf{Z}} F(H(a,k)). \right\} .(5.16)$$

By summation over $H_0 \in \mathbf{G}$, the formula (5.16) can be transformed as

$$\sum_{H'\in\mathbf{G}} F(H') = \frac{1}{2} \sum_{a'\in\mathcal{P},k'\in\mathbf{Z}} F(H(a',k'))$$
$$= \sum_{(a',k')\in\mathcal{P}\times\mathbf{Z},\mathbf{m}\in\mathbf{Z_1}\times\{\mathbf{0}\}} \delta(a',k') \nabla_{H_{H(a',k')}} R^* F(m) -$$
$$- \frac{L}{2} \sum_{a\in\mathcal{P},k\in\mathbf{Z}} F(H(a,k)). \qquad (5.17)$$

Thus

$$\sum_{a'\in\mathcal{P},k'\in\mathbf{Z}} F(H(a',k')) = 2(L+1)^{-1} \sum_{(a,k)\in\mathcal{P}\times\mathbf{Z},\mathbf{m}\in\mathbf{Z}\times\{\mathbf{0}\}} \delta(a,k) \nabla_{H_{H(a,k)}} R^* F(m). \ (5.18)$$

Remplacing $\sum_{a'\in\mathcal{P},k'\in\mathbf{Z}} F(H(a',k'))$ by its expression (see (5.18)) in the formula (5.16) we obtain

$$F(H_0) = 2\delta(a_0,k_0) \sum_{m\in\mathbf{Z}\times\{\mathbf{0}\}} \nabla_{H_0} R^* F(m) -$$
$$- 2\delta(a_0,k_0) \sum_{(a,k)\in\mathcal{P}\times\mathbf{Z},\mathbf{m}\in\mathbf{Z}\times\{\mathbf{0}\}} \gamma(H(a,k)) \nabla_{H_{H(a,k)}} R^* F(m).$$

This proves the lemma.
Before to prove the inversion theorem for R^*. We define the following operator ■

$$\Delta_{(H_0,H)} = \begin{cases} 2\delta(a_0,k_0)(1-\gamma(H(a_0,k_0))) \nabla_{H_0} & \text{if } H = H_0 \\ -2\delta(a_0,k_0)\gamma(H(a,k)) \nabla_H & \text{if } H \neq H_0 \end{cases},$$

recall that $H_0 = H(a_{0.}, k_0)$ and $H = H(a_. k)$.
Theorem 5.5 Let $F \in \mathcal{S}_{\mathbf{1},*}(\mathbf{G})$ and $H_0 = H(a_{0.}, k_0) \in \mathbf{G}$, then R^* can be inverted as follows:

$$F(H_0) = \sum_{m \in \mathbf{Z_1} \times \{\mathbf{0}\}} \sum_{(a,k) \in \mathcal{P} \times \mathbf{Z}} (\Delta_{(H_0,H)}) R^* F(m). \qquad (5.21)$$

Where $H_0 = H(a_{0.}, k_0)$ and $H = H(a_. k)$.

Proof. From lemma 5.4, the expression (5.13) can be written in the form

$$\begin{aligned} F(H_0) &= \sum_{m \in \mathbf{Z_1} \smallsetminus \{\mathbf{0}\}} 2\delta(a_0, k_0) \left[\nabla_{H_0} R^* F - \sum_{(a,k) \in \mathcal{P} \times \mathbf{Z}} \gamma(a,k) \nabla_{H_{H(a,k)}} R^* F \right](m) \\ &= 2\delta(a_0, k_0) \sum_{m \in \mathbf{Z_1} \smallsetminus \{\mathbf{0}\}} \nabla_{H_0} R^* F(m) - 2\delta(a_0, k_0) \gamma(a_0, k_0) \sum_{m \in \mathbf{Z_1} \times \{\mathbf{0}\}} \nabla_{H_0} R^* F(m) \\ &- 2\delta(a_0, k_0) \sum_{m \in \mathbf{Z_1} \smallsetminus \{\mathbf{0}\}} \sum_{(a,k) \in \mathcal{P} \times \mathbf{Z}, \mathbf{a_0} \neq \mathbf{a}, \mathbf{k_0} \neq \mathbf{k}} \gamma(a,k) \nabla_{H_{H(a,k)}} R^* F(m) \end{aligned}$$

It follows that

$$\begin{aligned} F(H_0) = 2\delta(a_0, k_0)(1 - \gamma(H_0)) \sum_{m \in \mathbf{Z_1} \smallsetminus \{\mathbf{0}\}} \nabla_{H_0} R^* F(m) \\ - 2\delta(a_0, k_0) \sum_{m \in \mathbf{Z} \times \{\mathbf{0}\}} \sum_{(a,k) \in \mathcal{P} \times \mathbf{Z}, \mathbf{a_0} \neq \mathbf{a}, \mathbf{k_0} \neq \mathbf{k}} \gamma(a,k) \nabla_{H(a,k)} R^* F(m) \end{aligned}$$

By means of the operator $\Delta_{(H_0,H)}$(see(5.20)) the above equality becomes

$$F(H_0) = \sum_{m \in \mathbf{Z_1} \times \{\mathbf{0}\},} \sum_{(a,k) \in \mathcal{P} \times \mathbf{Z}} (\Delta_{(H_0,H)}) R^* F(m)$$

This completes the proof of theorem ■

Remark 5.6 Let φ and $\Psi_{H_{0.}}$ be two functions defined by

$$\begin{aligned} \varphi(H(a,k)) &= 2\delta(a,k)(1 - \gamma(H(a,k)) \\ \Psi_{H_{0.}}(H(a,k)) &= -2\delta(a_0, k_0)\gamma(H(a,k)) \end{aligned}$$

Putting

$$\theta_{H_0}(H) = \varphi(H).\beta_{H_0}(H) \ \ \Psi_{H_{0.}}(H)(1 - .\beta_{H_0}(H))$$

where $H_0 = H(a_{0.}, k_0)$ and $H = H(a_. k)$ the formula (5.20) becomes

$$\Delta_{(H_0,H)} = \theta_{H_0}(H) \nabla_H$$

then $\Delta_{(H_0,H)} R^* F = \theta_{H_0}(H) R^*(\xi_H F)$.
Now, we shall study the properties of the operator $\Delta_{(H_0,H)}$ with $H_0 = H(a_{0.}, k_0) \in \mathbf{G}$ and $H = H(a_. k) \in \mathbf{G}$. By the above remark, it suffice to study only the properties of the operator ∇_{H_0}.
Proposition 5.6

$$R^*(\mathcal{S}_*(\mathbf{G})) \subset l^\infty(\mathbf{Z^n}) \qquad (5.22)$$

Proof. Let $F \in \mathcal{S}_*(\mathbf{G})$, showing that $R^* F \in l^\infty(\mathbf{Z^n})$. By the definition of R^*, we have $R^*(|F|)(m) = \frac{1}{2}\sum_{a \in \mathcal{P}} |F(H(a, am))|$. Since $F \in \mathcal{S}_*(\mathbf{G})$, we obtain

$$\begin{aligned} |F(H(a, am))| &\le T_N(F)\left(1 + ||a||^2 + (am)^2\right)^{-N} \\ &\le q_N(F)\left(1 + ||a||^2\right)^{-N} \quad \text{for all } N \in \mathbf{N} \end{aligned}$$

Choosing N enough great, we have

$$\sum_{a\in\mathcal{P}} |F(H(a,am))| \le q_N(F) \sum_{a\in\mathcal{P}} \left(1+||a||^2\right)^{-N}$$
$$\le C_N\, q_N(F),$$

where $T_N(F)$ is a constant which depend of N and F, and $C_N = \sum_{a\in\mathcal{P}} \left(1+||a||^2\right)^{-N}$. Thus

$$\sup_{m\in\mathbf{Z^n}} R^*(|F|)(m) < \infty$$

It follows that $R^*(|F|) \in l^\infty(\mathbf{Z^n})$. Then $\sup_{m\in\mathbf{Z^n}} |R^*(F)|(m) < \infty$, therefore $R^*(\mathcal{S}_*(\mathbf{G})) \subset l^\infty(\mathbf{Z^n})$. This completes the proof of proposition ■

Remark 5.7 the space $R^*(\mathcal{S}_{1,*}(\mathbf{G}))$ is a normed space by the norm

$$||R^*(F)||_{\infty,*} = \sup_{m\in\mathbf{Z^n}} R^*(|F|)(m) \qquad (5.23)$$

The proof of this remark is easy.
Proposition 5.8 Let $H_0 = H(a_{0.}, k_0)$ and $F \in \mathcal{S}_{\mathbf{1},*}(\mathbf{G})$. Then

$$\nabla_{H_0} \quad R^* \quad (\mathcal{S}_{\mathbf{1},*}(\mathbf{G})) \quad \to \quad R^* \quad (\mathcal{S}_{\mathbf{1},*}(\mathbf{G}))$$

is a linear continuous operator of the normed space $(R^*(\mathcal{S}_{\mathbf{1},*}(\mathbf{G})), ||.||_{\infty,*})$ into $(R^*(\mathcal{S}_{\mathbf{1},*}(\mathbf{G})), ||.||_{\infty,*})$ precisely we have

$$||\nabla_{H_0} R^* F||_{\infty,*} \le \left[\exp\left(||a_0||^2 + k^2\right) + 2\right] ||R^* F||_{\infty,*} \qquad (5.24)$$

Proof. Let $H_0 = H(a_{0.}, k_0) \in \mathbf{G}$, we know that $\nabla_{H_0} R^* F = R^*(\xi_{H_0} F)$ for all $F \in \mathcal{S}_{\mathbf{1},*}(\mathbf{G})$, then

$$\nabla_{H_0} \qquad R^* \quad (\mathcal{S}_{\mathbf{1},*}(\mathbf{G})) \quad \to \quad R^* \quad (\mathcal{S}_{\mathbf{1},*}(\mathbf{G}))$$

is will defined and ∇_{H_0} is linear. Showing now, that ∇_{H_0} is a continuous operator for the norm $||.||_{\infty,*}$ .From (5.3) and (5.6) we have

$$R^*(\xi_{H_0}|F|)(m) = \frac{1}{2}\exp\left(||a_0||^2 + k^2\right) |F|(H_0)\, \chi_{H_0}(m) + R^*(|F|)(m)$$

It follows that

$$||\nabla_{H_0} R^* F||_{\infty,*} \le \left[\exp\left(||a_0||^2 + k_0^2\right) + 2\right] ||R^* F||_{\infty,*}$$

Thus

$$\nabla_{H_0} \qquad R^* \quad (\mathcal{S}_{\mathbf{1},*}(\mathbf{G})) \quad \to \quad R^* \quad (\mathcal{S}_{\mathbf{1},*}(\mathbf{G}))$$

is linear continuous operator for the norm $||.||_{\infty,*}$. This completes the proof of the above proposition ■

We end this section by giving an analogue of the Hardy's theorem in the case of the discrete Radon transform of $\mathbf{Z^n}$.
Proposition 5.9 Let $f \in l^1(\mathbf{Z^n})$ satisfying at the following inequality

$$|Rf(H(a,k))| \le C\exp\left[-\alpha\left(||a||^2 + k^2\right)\right], \qquad (5.25)$$

for all $(a,k) \in \mathcal{P}\times\mathbf{Z}$. Then $f = 0$, where $(c,\alpha) \in (\mathbf{R}_+^*)^{\mathbf{2}}$ which are the absolute constants .

Proof. let $m \in \mathbf{Z^n}$ and $a_j = \left(1, j, j^2, ..., j^{n-1}\right)$ with $j \in \mathbf{N}^*$.The inequality (5.25) is valid for all $H(a,k) \in \mathbf{G}$, in particulary for the family $H(a_j, a_j m)$, it follows from (5.25) that

$$|Rf\left(H(a_j, a_j m)\right)| \leq C \exp\left[-\alpha\left(||a_j||^2 + (a_j m)^2\right)\right]$$

The above inequality implies that

$$\lim_{j\to\infty} Rf\left(H(a_j, a_j m)\right) = 0,$$

since $||a_j||^2 = 1 + j^2 + j^4 + ... + j^{2(n-1)} \to \infty$ when j$\to \infty$.But, by the inversion formula for the discrete Radon transform (*see* [1] Theorem 4.1) we have

$$\lim_{j\to\infty} Rf\left(H(a_j, a_j m)\right) = f\left(m\right).$$

Thus $f\left(m\right) = 0$ for all $m \in \mathbf{Z^n}$.This prove the proposition ■

In the classical case the above proposition is false, indeed taking the function $f\left(x\right) = \exp\left(-\left||x|\right|^2\right)$, we have $|Rf(t,\omega)| \leq C \exp\left(-t^2\right)$. C is a absolute constant and $(t,\omega) \in \mathbf{R} \times \mathcal{S}^{\mathbf{n-1}}$.

We end this study by noting that the inversion formulas for the discrete d-plane Radon transform and its dual may be obtained by using the same technics as above (sections 4 and 5, see also[2]). The proof is exactly as for the discrete hyperplane Radon transform.

VI. REFERENCES

[1] A. Abouelaz and A. Ihsane, Diophantine Integral Geometry, Medeterranean Journal Mathematics first issue n$\underline{^0}$1Vol 5 (2008) pp 77-99

[2] A. Abouelaz and A. Ihsane, Integral Geometry on Discrete Grassmannians **G**(d,n),Submitted for publicution in Medeterranean Journal of Mathematics.

[3] A. Abouelaz, Integral geometry in the sphere S^d ,Chapman and Hall /CRC Res.Notes Math, 422, Boca Raton,FL, 2001, 83-125

[4] J. Dieudonné, éléments d'analyse TomeVI ChapitreXXII (collection Cahiers Scientifiques) .Fascicule XXXIX Gauthiers Villars Paris 1975.

[5] J. Dieudonné, élements d'analyse TomeII(Collection Cahiers scientifiques).Fascicule XXX Gauthiers.Villars,Paris1969.

[6] I.M.Gelfand, M.I Graev, N.A Vilennkin, Generalized Functions, Vol 5.Integral geometry and Representations theory Academic.Pren, New York(1966).

[7] F.Gonzalez, on the range of Radon transform and its dual.Trans. Math.Soc.327(1991) 601-619.

[8] S.Helgason, Groups and Geometry.Invariant Differential operators and Spherical Functions.Academic Press,New-York 1984.

[9] S.Helgason, Geometric Analysis on Symmetric Spaces,Math.Surveys and Monographs,Vol 39,Amer-Math.Soc Providence,R1,(1994).

[10] S.helgason, The Radon transform, second edition, Birkhäser, Boston, Progress in Mathematics,(1999).

[11] J.Radon, über die Bestimmung Von Functionen durch ihre Integralwerte längs gewisser Manniglaltigkeiten, Ber.Verh.Akad.Wiss. Leïpziz.Math.Nat.kl.69 (1917), 262-277.

[12] R.Strichartz, Radon inversion-Variations on a theme, Amer-Math.Monthly, June (1982) 372-384.