



## Topological String on Local Elliptic Curve with Large Complex Structure

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### Abstract

We develop a  $2D$   $U(1)$  gauged  $\mathcal{N} = 2$  supersymmetric field theoretic realization of the local 2-torus  $\mathcal{O}(m) \oplus \mathcal{O}(-m) \rightarrow \mathbf{T}^2$  with  $\mathbf{T}^2$  in the large complex structure limit. This field representation involves a gauge invariant chiral superpotential and relies on thinking about the compact base  $\mathbf{T}^2$  as the toric boundary  $E = \partial\mathbf{P}^2$  of the complex projective plane  $\mathbf{P}^2$ . We also give generalizations of this result for local genus  $g$ -Riemann surfaces and higher complex dimensional Calabi-Yau manifolds. Evidence for a topological tetra-valent vertex is discussed.

**Key words:** *Topological string, Topological vertex and toric manifolds, 4d Black holes in type II string on local CY 3, q-Deformed 2d quiver gauge theory.*

### I. INTRODUCTION

In the last few years, topological string<sup>1,2</sup> has been shown to be a powerful method to deal with the  $4D$   $\mathcal{N} = 2$  supergravity limit of the compactification of type II superstring on Calabi-Yau (CY) threefolds  $X_3$ <sup>3,4</sup>. The OSV conjecture<sup>5</sup> relating microscopic  $4D$  black holes to  $2D$   $q$ -deformed Yang-Mills theory has given an additional impulse to this theory<sup>6-10</sup>. The interest into topological string has also become more exciting by the development of the topological tri-vertex formalism<sup>11,12</sup> and interpretation in terms of 3d-partitions of melting crystals generalizing  $U(\infty)$  Young tableau method<sup>13,14</sup>. Tri-vertex has been shown to be an important tool to compute the string partition function of topological A model for toric Calabi-Yau threefolds  $X_3$ . The key idea of topo-

logical 3-vertex method relies on the toric graph  $\Delta(X_3)$  of the toric CY 3-fold  $X_3$  and consists to: (i) partition  $\Delta(X_3)$  into tri-vertices describing local patches  $\mathcal{U}_i$  of  $X_3$  that are homeomorphic to  $\mathbf{C}^3$ , (ii) implement Lagrangian D-branes/anti D-branes at the cuts with a framing representation; and (iii) glue vertices by summing over all possible configurations.

On the other hand, attempts have been recently made in literature; in particular in<sup>15,16</sup>, to extend this method to the so called "formal" toric threefolds,

$$X_3^{(m,-m,0)} = \mathcal{O}(m) \oplus \mathcal{O}(-m) \rightarrow \mathbf{T}^2, \quad (1.1)$$

with  $\mathbf{m} \in \mathbf{Z}$  and where partial results have been obtained. Recall that, despite the important role in probing OSV conjecture; the exact results obtained in<sup>6</sup> and subsequent works for topological string on  $X_3^{(m,-m,0)}$  and local genus  $g$ -Riemann surfaces are essentially due to combination of results, on string interpretation of  $\frac{1}{N}$  expansion of the free energy in a  $U(N)$  gauge theory, derived

at different stages. The local 2-torus (1.1) is not a toric manifold since the compact base  $\mathbf{T}^2$  is a non toric curve; it has no fix points under the  $U(1)$  underlying toric action. The lack of a toric representation of eq(1.1) makes also the local 2-torus a very special local CY 3-fold. For instance,  $X_3^{(m,-m,0)}$  has no perfectly defined toric graph  $\Delta(X_3^{(m,-m,0)})$  and, to our knowledge, no precise 2D linear gauged  $\mathcal{N} = 2$  supersymmetric sigma model. These problems can however be solved by using a particular representation of  $\mathbf{T}^2$  in terms of intersecting projective lines as in the case of the compact part of affine ADE geometries as done in<sup>17,19</sup>.

In the present paper, we mainly address two questions: (1) Propose a toric representation of  $X_3^{(m,-m,0)}$  eq(1.1) and beyond. (2) Build the corresponding 2D  $\mathcal{N} = 2$  supersymmetric gauged model. To fix the ideas, notice that the solution we propose here is a particular, but remarkable, one where  $\mathbf{T}^2$  is realized, *in the large complex structure limit*, as the toric boundary  $E = \partial\mathbf{P}^2$  of the complex projective plane  $\mathbf{P}^2$ . Using the complex elliptic curve  $E$ , which is in the homology class of the 2-torus with the usual self intersection  $E \cap E = 0$ ; and thinking about eq(1.1) as  $\mathcal{O}(m) \oplus \mathcal{O}(-m) \rightarrow E$ , we propose a 2D linear  $U(1)$  gauged  $\mathcal{N} = 2$  supersymmetric model whose moduli space of supersymmetric vacua describes the topological string target space threefold (1.1). With this result at hand, we also consider solution beyond the genus  $g=1$  geometry. In particular, we study field theoretical realization of local genus  $g$ -Riemann surfaces as well as issues on higher complex  $n$  dimensional toric manifolds.

The organization of this paper is as follows: In section 2, we review briefly the  $U(1)$  gauged supersymmetric sigma model realization of the normal bundle  $N\mathbf{P}^2$ , to often designated below as local  $\mathbf{P}^2$ . In section 3, we consider the same issues for the case of local 2-torus. As the question of toric realization of  $\mathbf{T}^2$  is a crucial and an ambiguous point, we divide this section in three parts: (a) We first consider the realisation of the 2-torus as a the toric boundary of  $\mathbf{P}^2$ . (b) Then we give explicit details on the  $U(1)$  gauged supersymmetric sigma model realization of local complex surface having  $\mathbf{T}^2$  cycle as compact submanifold. (c) Next, we study the moduli space of the supersymmetric vacua. In section 4, we extend the construction to the case of local 2-torus. In section 5, we give conclusion.

## II. SIGMA MODEL FOR LOCAL $\mathbf{P}^2$ : A REVIEW

In this section, we review briefly the supersymmetric sigma model realization of local  $\mathbf{P}^2$  model. This is useful for the purpose of the this paper and also to fix convention notations. The field theoretic model is nicely formulated in the language of  $4D$ ,  $\mathcal{N} = 1$  supersymmetry. By ignoring unneeded technical details on the pure abelian gauge dynamics, this formulation is, roughly, equivalent to the usual  $2D$ ,  $\mathcal{N} = 2$  supersymmetry.

The complex two dimension projective space  $\mathbf{P}^2$  has one Kahler parameter  $t$  which, on supersymmetric gauge theory side, is interpreted as the Fayet-Iliopoulos (FI) coupling constant. So the  $U(1)$  gauged linear sigma theory describing local  $\mathbf{P}^2$ , the normal bundle of  $\mathbf{P}^2$ , involves the following  $4D$ ,  $\mathcal{N} = 1$  superfield multiplets:

(1) A  $U(1)$  gauge superfield  $V = V(x, \theta, \bar{\theta})$  which reads, in the Wess-Zumino gauge, as follows:

$$V = -\theta\sigma^\mu\bar{\theta}A_\mu - i\bar{\theta}^2\theta\lambda + i\theta^2\bar{\theta}\bar{\lambda} + \frac{1}{2}\theta^2\bar{\theta}^2D, \quad (2.1)$$

where  $(x^\mu, \theta^a, \bar{\theta}_{\dot{a}})$  stands for the  $4D$ ,  $\mathcal{N} = 1$  superspace coordinates. In this relation,  $A_\mu(x)$  and  $(\lambda_a(x), \bar{\lambda}_{\dot{a}}(x))$  are respectively the  $U(1)$  gauge vector and gaugino fields and where  $D$  is the usual auxiliary field capturing the local Calabi-Yau geometry as well as part of the scalar field potential of the gauge theory.

(2) Four chiral superfields  $\{\Phi_0, \Phi_1, \Phi_2, \Phi_3\}$ , with  $\theta$ -expansion

$$\Phi_i = z_i + \theta\psi_i + \theta^2F_i, \quad (2.2)$$

with  $z_i$  the field coordinates of local  $\mathbf{P}^2$ ,  $\psi_i$  Weyl spinors and  $F_i$  the so called F-auxiliary fields. These complex superfields carry the following  $q_i$ -charges under the  $U(1)$  gauge symmetry,

$$(q_0, q_1, q_2, q_3) = (-3, 1, 1, 1). \quad (2.3)$$

The  $q_i$ 's add exactly to zero as required by the Calabi-Yau condition

$$\sum_{i=0}^3 q_i = 0 \quad (2.4)$$

of local  $P^2$ . The superfield Lagrangian density  $\mathcal{L}_{NP^2} = \mathcal{L}(\Phi, V)$  of this model reads, in the  $\mathcal{N} = 1$ ,  $D = 4$  formalism, as follows,

$$\mathcal{L}_{NP^2} = \int d^4\theta \sum_{i=0}^3 \bar{\Phi}_i e^{2q_i V} \Phi_i - 2t \int d^4\theta V + \mathcal{L}_{gauge}(V). \quad (2.5)$$

Here  $\mathcal{L}_{gauge}(V)$  is the superspace Lagrangian density for the  $U(1)$  vector multiplet that can be found in<sup>20</sup>.  $t$  is FI coupling interpreted as the Kahler parameter of the local  $\mathbf{P}^2$ . The equation of motion of the D- auxiliary field leads to,

$$|z_1|^2 + |z_2|^2 + |z_3|^2 - 3|z_0|^2 = t, \quad (2.6)$$

it is nothing but the defining equation of  $O(-3) \rightarrow P^2$ . The compact part  $\mathbf{P}^2$  of this threefold is a complex surface given by the divisor  $z_0 = 0$ ; it is parameterized by the complex coordinates  $(z_1, z_2, z_3)$  describing a complex surface embedded in  $\mathbf{C}^3$  and has a  $U(1)$  gauge symmetry rotating the phases of the coordinates variables. Upon setting  $x_i = |z_i|^2$ , this complex surface can be represented by the planar triangle<sup>21</sup>,

$$x_1 + x_2 + x_3 = t \quad (2.7)$$

with Kahler modulus  $t$ ; see also figure 1. Because of symmetry under permutation of the  $x_i$ 's the triangle is equilateral with edge length equal to  $t$  and an area given by  $a = t^2\sqrt{3}$ . Having described the sigma model for local  $\mathbf{P}^2$ , we turn now to consider the sigma of its boundary complex surface  $\partial(NP^2)$ . Then, we extend it to the local 2-torus.

### III. FIELD MODEL FOR THE BOUNDARY OF LOCAL $\mathbf{P}^2$

We first study the toric boundaries of the manifolds  $\mathbf{P}^2$  and  $NP^2$ . These boundaries are denoted as respectively  $\partial\mathbf{P}^2$  and  $\partial(NP^2)$  and refer to the codimension one boundary subspace of these manifolds on which a 1-cycle of the torus fiber shrinks. Then we derive the corresponding gauge invariant supersymmetric field model.

#### A. Divisors of local $\mathbf{P}^2$

To begin note that local  $\mathbf{P}^2$  eq(2.6) has several divisors; i.e codimension one subspaces describing boundary patches of the normal bundle  $NP^2$ . The standard ones are obtained by setting one of the  $z_i$ 's to zero;  $z_i = 0$  with  $i = 0, 1, 2, 3$ .

##### 1. Toric boundary of $\mathbf{P}^2$

In present study, we consider the complex surfaces

$$\begin{aligned} [D_1] : & \begin{cases} |z_2|^2 + |z_3|^2 - 3|z_0|^2 = t \\ \Leftrightarrow z_1 = 0, \end{cases} \\ [D_2] : & \begin{cases} |z_3|^2 + |z_1|^2 - 3|z_0|^2 = t \\ \Leftrightarrow z_2 = 0 \end{cases}, \\ [D_3] : & \begin{cases} |z_1|^2 + |z_2|^2 - 3|z_0|^2 = t \\ \Leftrightarrow z_3 = 0 \end{cases}, \end{aligned} \quad (3.1)$$

and their union  $[D] = [D_1] \cup [D_2] \cup [D_3]$ . To see what this local geometry describes; let us set  $|z_0|^2 = 0$  in above eqs from where one sees that each relation describes a complex one dimension projective space  $\mathbf{P}^1$ . To distinguish between these complex projective lines, we use the convention notation  $\mathbf{P}_i^1$  where the subindex  $i$  refers to  $z_i = 0$ . Thus we have

$$\begin{aligned} \mathbf{P}_1^1 : & |z_2|^2 + |z_3|^2 = t, \\ \mathbf{P}_2^1 : & |z_3|^2 + |z_1|^2 = t, \\ \mathbf{P}_3^1 : & |z_1|^2 + |z_2|^2 = t. \end{aligned} \quad (3.2)$$

As we see, these lines have intersections with matrix

$$\mathbf{P}_i^1 \cap \mathbf{P}_j^1 = \begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix} \quad (3.3)$$

from which one sees that the union

$$E = \mathbf{P}_1^1 \cup \mathbf{P}_2^1 \cup \mathbf{P}_3^1, \quad (3.4)$$

is in the homology class of the 2-torus  $\mathbf{T}^2$ .

$$\begin{aligned} E \cap E &= \mathbf{P}_1^1 \cap \mathbf{P}_1^1 + \mathbf{P}_2^1 \cap \mathbf{P}_2^1 + \mathbf{P}_3^1 \cap \mathbf{P}_3^1 \\ &+ 2(\mathbf{P}_1^1 \cap \mathbf{P}_2^1 + \mathbf{P}_2^1 \cap \mathbf{P}_3^1 + \mathbf{P}_3^1 \cap \mathbf{P}_1^1) \\ &= -3 \times 2 + 2 \times 3 = 0. \end{aligned} \quad (3.5)$$

From the toric diagram of  $\mathbf{P}^2$ , one also see that  $E$  describes indeed a toric complex one dimensional curve defining the toric boundary of  $\mathbf{P}^2$ ; i.e:

$$E = \partial\mathbf{P}^2. \quad (3.6)$$

It is this curve that will be used to deal with the 2-torus and other manifolds based on  $E$ .

##### 2. Equation of the curve $E$

An interesting question is the derivation of the defining equation describing the curve  $E$ . From above analysis, it is not difficult to see that  $E$  is given by.

$$\begin{cases} |z_1|^2 + |z_2|^2 + |z_3|^2 = t \\ z_i \equiv z_i e^{iq_i \alpha}, \\ z_1 z_2 z_3 = 0. \end{cases} \quad (3.7)$$

In this relation, we have three complex variables  $(z_1, z_2, z_3)$  subject to two real constraint eqs and a complex one.

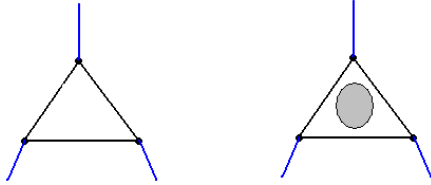


FIG. 1. (a) Left: Toric graph of  $\mathbf{P}^2$ . (b) Right: toric graph of  $E$  where we have added a hole to avoid confusion.

The two first eqs are just the defining linear sigma model eq of  $\mathbf{P}^2$ ; while the third one, which is covariant under  $U(1)$  gauge symmetry, is an extra condition implemented in order to restrict  $\mathbf{P}^2$  geometry to its toric boundary subspace  $\partial\mathbf{P}^2$ . Notice that, the implementation of the boundary condition is a new feature that has not been considered before. So our method may be viewed as a generalization of the usual approach for dealing with sigma model realization of toric manifolds. Note also that this method may be applied as well to genus  $g$  Riemann surfaces; It can be also used to describe toric boundaries of more general complex  $n$  dimensional toric Calabi-Yau manifolds. We will make a comment regarding this point in the conclusion section. Note finally that for  $t \neq 0$  the three complex variables cannot vanish simultaneously, i.e

$$(z_1, z_2, z_3) \neq (0, 0, 0) \tag{3.8}$$

In the particular case  $t = 0$ , the geometry collapses to the origin  $(0, 0, 0)$  where lives a  $\mathbf{P}^2$  singularity and an elliptic one.

### 3. Divisor $\mathcal{O}(-3) \rightarrow E$

Using above results, one can immediately write down the defining equation of the divisor  $\mathcal{O}(-3) \rightarrow E$  of local  $\mathbf{P}^2$ . We have

$$\begin{cases} |z_1|^2 + |z_2|^2 + |z_3|^2 - 3|z_0|^2 = t \\ z_i \equiv z_i e^{iq_i \alpha}, \quad i = 0, 1, 2, 3, \\ z_1 z_2 z_3 = 0, \end{cases} \tag{3.9}$$

where  $(q_0, q_1, q_2, q_3)$  are as in eq(2.3) and where the complex variable  $z_0$  parameterize the non compact direction  $\mathcal{O}(-3)$ . Note also that as we are not yet worrying about the Calabi-Yau condition, the first relation can be extended as

$$|z_1|^2 + |z_2|^2 + |z_3|^2 - m|z_0|^2 = t. \tag{3.10}$$

where  $m$  is an arbitrary positive integer.

## B. Superfield action

Here we give the supersymmetric field description of the above Kahler manifold. We first give the field realization of the toric curve  $E$ ; then we consider extensions.

### 1. Gauge invariant model for the curve $E$

To build the supersymmetric model describing this toric curve  $E$ , we start from the superfield content eq(2.6) of local  $\mathbf{P}^2$  theory and implement the constraint eq by using Lagrange multiplier method together  $U(1)$  gauge invariance. The appropriate Lagrange superfield multiplier is given by a chiral superfield  $\Upsilon$  with charge  $q_\Upsilon = -3$  under  $U(1)$  gauge symmetry so that the chiral superfield monomial

$$W(\Phi, \Upsilon) = \Phi_1 \Phi_2 \Phi_3 \Upsilon, \tag{3.11}$$

is gauge invariant. Thus the supersymmetric lagrangian describing local boundary surface is given by the density,

$$\mathcal{L}_E = \mathcal{L}_{P^2} + \left( g \int d^2\theta W(\Phi, \Upsilon) + hc \right), \tag{3.12}$$

where  $g$  is a complex coupling constant. Since  $\Upsilon$  has no kinetic term, its elimination through the equation of motion

$$\frac{\delta \mathcal{L}_E}{\delta \Upsilon} = 0 \tag{3.13}$$

gives

$$g\Phi_1 \Phi_2 \Phi_3 = 0, \tag{3.14}$$

whose lowest term is precisely  $z_1 z_2 z_3 = 0$ . Note that contrary to  $\mathbf{P}^2$ , the superfield realization of the curve  $E$  has a non trivial chiral superpotential. As we see, this result is a particular situation that can be extended to build toric realization of other toric manifolds. Note also that the Lagrange superfield multiplier  $\Upsilon$  can be given an interesting geometric interpretation. This superfield has no kinetic term  $\bar{\Upsilon}\Upsilon$  nor couplings to the gauge superfield  $V$ ; i.e no term type

$$\int d^4\theta \bar{\Upsilon} \exp(2q_\Upsilon V) \Upsilon, \tag{3.15}$$

in the Lagrange superdensity. This feature can be interpreted as corresponding to *freezing* the supersymmetric gauge invariant dynamics of  $\Upsilon$ . This

property explains why the Calabi-Yau condition for the complex toric curve  $E$  should read as,

$$\sum_{i=1}^3 q_i + q_\gamma = \sum_{i=1}^3 q_i - 3 = 0. \tag{3.16}$$

We will turn to this property when we consider the local threefold  $\mathcal{O}(m) \oplus \mathcal{O}(-m) \rightarrow E$ . Notice also that the chiral superpotential (3.11) is not the unique gauge invariant term one may have. The general form of  $W(\Phi, \Upsilon)$  is given by

$$W(\Phi, \Upsilon) = \sum g_{n_1, n_2, n_3} \Phi_1^{n_1} \Phi_2^{n_2} \Phi_3^{n_3}, \tag{3.17}$$

with  $n_1 + n_2 + n_3 = 3$ . We will discuss this point in section 5 when we study the generalization to higher dimension CY manifolds. For the moment, let us complete this discussion by giving the gauged superfield realization of the complex surface  $\mathcal{O}(-m) \rightarrow \partial \mathbf{P}^2$ .

2. Field model for the Divisor  $\mathcal{O}(-m) \rightarrow E$

In addition to the  $U(1)$  gauge superfield  $V$ , this model involves five chiral superfields  $(\Phi_0, \Phi_1, \Phi_2, \Phi_3, \Upsilon)$  with charges

$$(q_0, q_1, q_2, q_3, q_\gamma) = (-m, 1, 1, 1, -3). \tag{3.18}$$

The Lagrangian density  $\mathcal{L}_{divisor}$  is given by,

$$\begin{aligned} \mathcal{L}_{divisor} = & \int d^4\theta \sum_{i=0}^3 \bar{\Phi}_i e^{2q_i V} \Phi_i \\ & + \mathcal{L}_{gauge}(V) - 2t \int d^4\theta V \\ & + \left( g \int d^2\theta W(\Phi, \Upsilon) + hc \right) \end{aligned} \tag{3.19}$$

where the chiral superpotential  $W(\Phi, \Upsilon)$  is as in eq(3.11). Here also the first Chern class of the complex surface has a contribution coming from  $\Upsilon$  and reads as  $\sum_{i=0}^3 q_i + q_\gamma = -m$  showing, as expected, that  $\mathcal{O}(-m) \rightarrow E$  is not a Calabi-Yau surface.

C. Supersymmetric vacuum

Here we study the supersymmetric vacuum of the field model (3.19). We show that the surface (3.9) corresponds to a particular vacuum given by  $z_\gamma = 0$ .

1. Moduli space of vacua

In the supersymmetric vacuum, the vanishing condition of the scalar potential  $V = V(z)$  of the model (3.12) reads as

$$|D|^2 + |F_0|^2 + |F_1|^2 + |F_2|^2 + |F_3|^2 + |F_\gamma|^2 = 0. \tag{3.20}$$

The dependence of the scalar potential  $V$  in the scalar fields  $z$  is obtained by replacing the auxiliary fields  $D$  and  $F_i$  by their expressions

$$\begin{aligned} D &= D(z_0, z_1, z_2, z_3, z_\gamma), \\ F_i &= F_i(z_0, z_1, z_2, z_3, z_\gamma), \end{aligned} \tag{3.21}$$

using their equations of motion  $\frac{\delta \mathcal{L}}{\delta D} = 0$  and  $\frac{\delta \mathcal{L}}{\delta F_i} = 0$ . These expressions are obviously solutions to  $D = 0$  and  $F_i = 0$ . As noted before, the first part  $D = 0$  of this solution leads to

$$|z_1|^2 + |z_2|^2 + |z_3|^2 - m|z_0|^2 = t \tag{3.22}$$

and describes local  $\mathbf{P}^2$  for the particular case  $m = 3$ . The second part involves five terms describing respectively the contribution of the auxiliary fields  $F_0, F_\gamma, F_3, F_2,$  and  $F_1$ . While  $F_0$  is trivial, the remaining others lead to:

$$\begin{aligned} z_1 z_2 z_3 &= 0 \\ z_1 z_2 z_\gamma &= 0 \\ z_3 z_1 z_\gamma &= 0 \\ z_2 z_3 z_\gamma &= 0 \end{aligned} \tag{3.23}$$

where  $z_\gamma$  stands for the lowest component field of the chiral superfield  $\Upsilon$ . There are several solutions of these relations and can be classified into two sets: (1) The first one is given by

$$z_\gamma = 0. \tag{3.24}$$

Putting back into eqs(3.23), one sees that they reduce to the first equation  $z_1 z_2 z_3 = 0$ . The moduli background associated with this solution describes exactly the complex surface  $\partial(N\mathbf{P}^2)$ . (2) The second set corresponds to  $z_\gamma \neq 0$ . The solution of eqs(3.23) leads to  $|z_i|^2 - m|z_0|^2 = t$  and is contained in previous case.

Let us now consider the interesting case  $z_\gamma = 0$  and study the solution of constraint eq  $z_1 z_2 z_3 = 0$ . Here also there are several solutions which we list below:

(1)  $z_3 = 0$  but  $(z_1, z_2) \neq (0, 0)$ : the  $\partial(N\mathbf{P}^2)$  geometry reduces to its divisor

$$|z_1|^2 + |z_2|^2 - 3|z_0|^2 = t \tag{3.25}$$

and describes the local complex surface

$$\mathcal{O}(-3) \rightarrow \mathbf{P}^1 \tag{3.26}$$

which we denote as  $O(-3) \rightarrow \mathbf{P}_3^1$  where the sub-index 3 on  $\mathbf{P}_3^1$  refers to  $z_3 = 0$ . The same thing is valid for  $z_1 = 0$  and  $z_2 = 0$ . They describe respectively the local surfaces  $O(-3) \rightarrow \mathbf{P}_1^1$  and  $O(-3) \rightarrow \mathbf{P}_2^1$ .

(2)  $z_1 = 0, z_2 = 0, z_3 = \sqrt{t}$  describes the top of one of the three possible vertices of  $\partial(N\mathbf{P}^2)$ ; the two other tops are associated with the points  $z_1 = 0, z_2 = \sqrt{t}, z_3 = 0$  and  $z_1 = 0, z_2 = \sqrt{t}, z_3 = 0$ .

(3)  $z_1 = z_2 = z_3 = 0$  is the  $\mathbf{P}^2$  singularity of  $\partial(N\mathbf{P}^2)$ . It corresponds to the limit  $t = 0$  where  $(\partial\mathbf{P}^2)$ , the toric boundary of  $\mathbf{P}^2$ , and so  $\mathbf{P}^2$  itself collapse down to a point.

2. Toric pants

The above analysis is an interesting step towards the study of topological vertex of local genus g-Riemann surfaces by using toric diagrams based on the curve  $E = \partial\mathbf{P}^2$ . To that purpose, one first has to build the toric realization of basic objects of the topological vertex method. For instance, the complex coordinates associated with the vertices of the elliptic curve  $E$  are given by the local charts

$$\begin{aligned} \mathcal{U}_1 &: (z_0^{(1)}, z_1^{(1)}, z_2^{(1)}), z_3^{(1)} = 0, \\ \mathcal{U}_2 &: (z_0^{(2)}, z_1^{(2)}, z_3^{(2)}), z_2^{(2)} = 0, \\ \mathcal{U}_3 &: (z_0^{(3)}, z_2^{(3)}, z_3^{(3)}), z_1^{(3)} = 0. \end{aligned} \tag{3.27}$$

The upper index of  $z_j^{(i)}$  refers to the corresponding chart  $\mathcal{U}_i$ . Note that on each chart, we have the relation

$$z_1^{(i)} z_2^{(i)} z_3^{(i)} = 0, \quad i = 1, 2, 3 \tag{3.28}$$

The charts  $\mathcal{U}_i$  can be interpreted as the three toric pants needed for building  $E$  and may be related to the topological pant considered in<sup>13</sup>. By gluing these three charts, one reproduces the toric boundary of local  $\mathbf{P}^2$ .

3. Beyond 3-vertex

We end this section by making two comments:

(i) First note despite that the boundary toric submanifold  $O(-3) \rightarrow \partial\mathbf{P}^2$  of local  $\mathbf{P}^2$  is a complex surface, it involves tri-valent vertices in the same way as in the case of local  $\mathbf{P}^2$ . This means that tri-valent vertex is not exclusively deserved for toric threefold but also to its toric submanifolds. This result should not be taken as a strange fact; the vertices used for the mother toric threefold are the

same for its complex divisors and lower submanifolds.

(ii) Second, the boundary complex surfaces  $O(-m) \rightarrow \partial\mathbf{P}^2, m \in \mathbf{Z}$ , are not Calabi-Yau submanifolds of local  $\mathbf{P}^2$  except for the trivial  $m = 0$  case. Toric Calabi-Yau threefold based  $E$  is given by  $O(+m) \oplus O(-m) \rightarrow \partial\mathbf{P}^2$ . As we will see in next section; this local threefold involves tetra-valent vertices.

IV. LOCAL 2-TORUS

The local complex surface  $X_2 = O(-m) \rightarrow E$  we have considered so far is not a Calabi-Yau twofold. The first Chern class  $c_1(T^*X_2)$  of this variety is equal to  $-m$ . For our concern, this surface should be thought of as a divisor of the local Calabi-Yau threefold,

$$O(m) \oplus O(-m) \rightarrow E. \tag{4.1}$$

where the elliptic curve  $E \sim T^2$  is realized as  $\mathbf{P}_1^1 \cup \mathbf{P}_2^1 \cup \mathbf{P}_3^1$ .

A. Gauged field model

The supersymmetric field model relations describing the toric Calabi-Yau threefold (4.1) can be easily derived from previous study. They are given by the following system of component field equations,

$$\begin{cases} |z_1|^2 + |z_2|^2 + |z_3|^2 + 3|z_4|^2 - 3|z_0|^2 = t, \\ z_1 z_2 z_3 = 0. \end{cases} \tag{4.2}$$

Here  $z_0$  and  $z_4$  parameterise the non compact directions and  $(z_1, z_2, z_3)$  are as before. The toric graph of this local threefold is shown on figure 1 where one has three tetra-valent vertices,

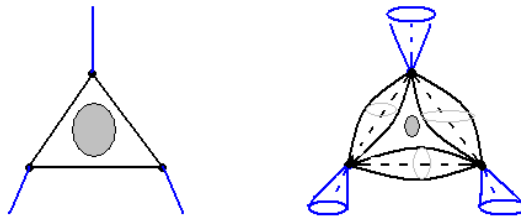


FIG. 2. Toric graph of  $O(-3) \rightarrow \mathbf{T}^2$ . The compact part is  $\mathbf{T}^2 = \partial\mathbf{P}^2$  with the usual three vertices. Its toric boundary consists of three intersecting  $\mathbf{P}^1$ 's in the homology class of 2-torus. (a) Figure (on left) represents real skeleton. (b) Figure (on right) gives its fattening.

To get the superfield Lagrangian density  $\mathcal{L}_{locT^2}$ , we think about eqs(4.2) as the field equations of motion of the  $D$  and  $F_i$  auxiliary fields. The first relation is associated with

$$\frac{\delta \mathcal{L}_{locT^2}}{\delta D} = 0, \tag{4.3}$$

while the second follows from,

$$\frac{\delta \mathcal{L}_{locT^2}}{\delta F_i} = 0. \tag{4.4}$$

The result is

$$\begin{aligned} \mathcal{L}_{locT^2} = & \int d^4\theta \sum_{i=1}^3 \bar{\Phi}_i e^{2V} \Phi_i \\ & + \int d^4\theta (\bar{\Phi}_0 e^{-2mV} \Phi_0 + \bar{\Phi}_4 e^{2mV} \Phi_4) \\ & + \mathcal{L}_{gauge}(V) - 2t \int d^4\theta V \\ & + \left( g \int d^2\theta \Phi_1 \Phi_2 \Phi_3 \Upsilon + hc \right), \end{aligned} \tag{4.5}$$

where  $\Upsilon$  is a Lagrange superfield multiplier capturing the constraint restricting the field variables to the boundary of  $\mathbf{P}^2$ . In addition to the  $U(1)$  gauge superfield  $V$ , the chiral superfields of this model are

$$(\Phi_0, \Phi_1, \Phi_2, \Phi_4, \Phi_5, \Upsilon) \tag{4.6}$$

and carry the following  $q_i$ - charges under the  $U(1)$  gauge symmetry,

$$(q_0, q_1, q_2, q_3, q_4, q_\gamma) = (-m, 1, 1, 1, m, -3). \tag{4.7}$$

where  $m$  is a priori equal to 3; but in general can take any integer value.

### B. Generalization

The construction we have developed here above can be generalized to other Calabi-Yau manifolds. Below we make a comment on two kinds of generalizations. The first extension deals with the gauged sigma model realization of local genus  $g$ -Riemann surfaces for  $g \geq 2$ . The second generalization concerns sigma model approach for higher complex dimensional compact toric Calabi-Yau manifolds.

#### 1. Local genus $g$ -Riemann surfaces

So far we have seen that for each local elliptic curve  $E$ , it is associated a  $U(1)$  gauge symmetry. This gauge symmetry is inherited from the  $\mathbf{P}^2$

model. Since local genus  $g$ -Riemann surfaces can be engineered by gluing several local elliptic  $E$ 's, we conclude that a class of local genus  $g$ -Riemann surfaces could be described by higher rank abelian  $U^n(1)$  gauged supersymmetric field model type. The rank  $n$  of the gauge symmetry depends on the way the gluing is done. To illustrate the idea, let us give the example of  $g = 2$ - Riemann surface described by a 2D  $U^2(1)$  gauged  $\mathcal{N} = 2$  supersymmetric sigma model.

The local  $g = 2$ - Riemann surface can be engineered by gluing two local elliptic curves with compact base  $E_1 = \partial\mathbf{P}_1^2$  and  $E_2 = \partial\mathbf{P}_2^2$ . In the sigma model approach, we distinguish different representations according to whether  $\mathbf{P}_1^2$  and  $\mathbf{P}_2^2$  have an intersection point or a common edge. In the first case, the sigma model involves five complex field variables,

$$(z_1, z_2, z_3, z_4, z_5) \tag{4.8}$$

and a  $U^2(1)$  gauge invariance under which these complex variables have the following gauge charges

$$\begin{aligned} (q_i^1) &= (1, 1, 1, 0, 0) \\ (q_i^2) &= (0, 0, 1, 1, 1) \end{aligned} \tag{4.9}$$

The field theoretic equations describing the compact part of the moduli space of supersymmetric vacua are given by,

$$\begin{cases} |z_1|^2 + |z_2|^2 + |z_3|^2 = t_1 \\ z_1 z_2 z_3 = 0 \end{cases}, \quad \begin{cases} |z_3|^2 + |z_4|^2 + |z_5|^2 = t_2 \\ z_3 z_4 z_5 = 0 \end{cases}, \tag{4.10}$$

where  $t_1$  and  $t_2$  are respectively the Kahler moduli of the elliptic curves  $E_1$  and  $E_2$ . The holomorphic constraint eqs  $z_1 z_2 z_3 = 0$  and  $z_3 z_4 z_5 = 0$  are implemented in the gauged supersymmetric superfield model by two chiral superfields  $\Upsilon_1$  and  $\Upsilon_2$  with gauge charges  $(q_\gamma^1, q_\gamma^2)$  equal to  $(-3, 0)$  and  $(0, -3)$  respectively. Notice that eq(4.10) describes indeed a complex curve with genus  $g = 2$ . The toric threefold based on this genus  $g = 2$  curve is parameterized by seven complex variables,

$$(z_0, z_1, z_2, z_3, z_4, z_5, z_6) \tag{4.11}$$

with gauge charges as

$$\begin{aligned} (q_i^1) &= (-3, 1, 1, 1, 0, 0, -3) \\ (q_i^2) &= (-3, 0, 0, 1, 1, 1, -3) \end{aligned} \tag{4.12}$$

The gauged supersymmetric field theoretical equation

$$\begin{aligned} |z_1|^2 + |z_2|^2 + |z_3|^2 + m|z_6|^2 - m|z_0|^2 &= t_1, \\ z_1 z_2 z_3 &= 0, \\ |z_3|^2 + |z_4|^2 + |z_5|^2 + n|z_6|^2 - n|z_0|^2 &= t_2, \\ z_3 z_4 z_5 &= 0, \end{aligned} \tag{4.13}$$

where  $m$  and  $n$  are in general arbitrary integers; but can be set equal to 3 to keep in touch with the first Chern class of the complex two dimension projective space. These relations involves seven complex variables constrained by four complex constraint eqs leaving then three complex variables free. Note also that the first relation of above eq describes  $O(m) \oplus O(-m) \rightarrow E_1$  while the second describes  $O(n) \oplus O(-n) \rightarrow E_1$ .

2. Higher dimensional toric CY manifolds

The gauged supersymmetric sigma model for the boundary surface of local  $\mathbf{P}^2$  we have considered in this paper can be extended for compact divisors of local  $\mathbf{P}^{n-1}$ . The latter is given by the following  $U(1)$  gauge invariant complex dimension  $n$  hypersurface

$$n|z_0|^2 + \sum_{i=1}^n |z_i|^2 = t, \tag{4.14}$$

embedded in  $\mathbf{C}^{n+1}$  parameterized by the local coordinates  $\{z_0, z_1, z_2, \dots, z_n\}$  with gauge charge

$$(q_0, q_1, q_2, \dots, q_n) = (-n, 1, 1, \dots, 1) \tag{4.15}$$

In eq(4.14),  $t$  is the usual Kahler parameter of  $\mathbf{P}^{n-1}$ . To describe the compact boundary  $\partial\mathbf{P}^{n-1}$  of this toric CY  $n$ -fold we supplement the hypersurface equation by the following extra gauge covariant constraint relation,

$$z_1 \dots z_n = 0. \tag{4.16}$$

Extending the analysis of section 3, the  $U(1)$  gauged supersymmetric sigma model describing  $\partial\mathbf{P}^{n-1}$  reads as follows

$$\begin{aligned} \mathcal{L}_{\partial P^{n-1}} &= \mathcal{L}_{P^{n-1}} \\ &+ \left( g \int d^2\theta W(\Phi, \Upsilon) + hc \right) \end{aligned} \tag{4.17}$$

with chiral superpotential

$$W(\Phi, \Upsilon) = \Upsilon \Phi_1 \dots \Phi_n. \tag{4.18}$$

The gauge charge  $q_\gamma$  of the Lagrange multiplier superfield  $\Upsilon$  is equal to  $(-n)$ . One of the interesting feature we have is that here also the first Chern class of  $\partial\mathbf{P}^{n-1}$  is identically zero. As noted before, this property is not a new feature since the

most general gauge invariant chiral superpotential extending eq(4.18) is given by

$$W(\Phi, \Upsilon) = \sum_{m_1 + \dots + m_n = n} g_{\{m_i\}} (\Upsilon \prod_{i=1}^n \Phi_i^{m_i}) \tag{4.19}$$

where  $g_{\{m_i\}}$  are coupling constants. The equation of motion of  $\Upsilon$  gives a degree  $n$  homogeneous polynomial describing a complex  $(n - 2)$  dimension holomorphic CY hypersurface with complex structures  $g_{\{m_i\}}$ .

V. CONCLUSION

In this paper, we have proposed a gauged supersymmetric sigma model realization for describing the local 2-torus  $\mathcal{O}(m) \oplus \mathcal{O}(-m) \rightarrow \mathbf{T}^2$ . We have also shown how this construction can be extended to local genus  $g$ -Riemann surfaces  $\mathcal{O}(m) \oplus \mathcal{O}(2 - 2g - m) \rightarrow \Sigma_g$ . The key idea in getting this solution relies on thinking about  $\mathbf{T}^2$  as given by the boundary  $E = \partial\mathbf{P}^2$  of the complex projective plane  $\mathbf{P}^2$ . In this view,  $\mathcal{O}(m) \oplus \mathcal{O}(-m) \rightarrow E$  becomes a toric threefold and so topological vertex method of<sup>13</sup> can be applied. Our proposal can be also viewed as an important step towards the use of topological vertex method to compute topological partition function for higher genus  $g$  Riemann surfaces.

Notice that because the 2-torus  $\mathbf{T}^2$  is, in general, a non toric complex curve as it has no fix point under the underlying  $U(1)$  toric action, the derivation of a gauged supersymmetric sigma model realization has been problematic. Attempts in this matter lead to partial results only. We have showed that this difficulty can be overcome by using the realization  $\mathbf{T}^2 \sim \partial\mathbf{P}^2$ . This is not the general solution; but a particular representation through which one can learn several basic properties on the local 2-torus and in general local genus  $g$ -Riemann surfaces.

Recall by the way that, though non toric, the above local 2-torus has been successfully used to probe OSV conjecture; but this time by using large  $N$  matrix model approach à la 't Hooft and Gross-Ward. Finding a way to use the powerful method of topological vertex would be then very important to get more insight in the explicit computation of A-model topological amplitudes on local (toric) 2-torus and in the engineering of toric genus-Riemann surfaces by using topological vertex. Progress in this matter will be reported in<sup>22</sup>.

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