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Integrable Systems: Bosonic and Supersymmetric Cases

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Abstract

The main task of this work concerns integrable models and supersymmetric extensions of the Gelfand-Dickey algebra of pseudo differential operators. The consistent and systematic study that we perform consists in describing in detail the relation existing between the algebra of super(pseudo-)differential operators on the ring of superfields $u_{\frac{s}{2}}(z, \theta), s \in Z$ and the higher and lower spin extensions of the conformal algebra. In relation to integrable systems, the supersymmetric GD bracket play a pioneering role as it gives in some sense a guarantee of integrability of the associated non linear supersymmetric systems.
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I. INTRODUCTION

For several years, two dimensional integrable models[1] in connection with conformal field theories[2,3] and their underlying lower $(s \leq 2)^{4-7}$ and higher $(s \geq 2)^{8,9}$ spin symmetries, have occupied a central position in various areas of research. More particularly, a lot of interest has been paid to w-symmetries[8], which are infinite dimensional algebras extending the conformal invariance (Virasoro algebra) by adding to the energy momentum operator $T(z) \equiv W_2$, a set of conserved currents $w_s(z)$, of conformal spin $s > 2$ with some composite operators necessary for the closure of the algebra.
In the language of $2d$ conformal field theory, the above mentioned currents w_s are taken in general as primary satisfying the OPE[2]

$$T(z)W_s(\omega) = \frac{s}{(z-\omega)^2}W_s(\omega) + \frac{W_s'(\omega)}{(z-\omega)}, \tag{I.1}$$

or equivalently,

$$W_s = J^s \tilde{W}_s \tag{I.2}$$

under a general change of coordinate (diffeomorphism) $x \to \tilde{x}(x)$ with $J = \frac{\partial \tilde{x}}{\partial x}$ is the associated Jacobian. These w-symmetries exhibit among others a non linear structure and are not Lie algebra in the standard way as they incorporate composite fields in their OPE. In integrable models these higher spin symmetries appear such that the Virasoro algebra corresponds to the second Hamiltonian structure (Gelfand Dickey Poisson Bracket) for the KdV hierarchy[?,?], w_3 for the Boussinesq[12] and $W_{1+\infty}$ for the KP hierarchy[13] and so on. These correspondences are achieved naturally in terms of pseudo-differential Lax operators[14]

$$\mathcal{L}_n = \sum_{j\in Z} u_{n-j}\partial^j, \tag{I.3}$$

allowing both positive as well as nonlocal powers of the differential ∂^j. The fields u_j of arbitrary conformal spin j did not define a primary basis. The construction of primary fields from the u_j one's is originated from the well-known covariantization method of Di-Francesco -Itzykson-Zuber (DIZ)[?] showing that the primary w_j fields are given by adequate polynomials of u_j and their k-th derivatives $u_j^{(k)}$.

More recently there has been a growth in the study of the supersymmetric version of conformal and w-symmetries in connection to integrable systems from the point of view of field theory[6] and through the Lax formalism and the theory of pseudo differential operators[7,?,?]. Much attention has been paid also to derive the supersymmetric extension of the Gelfand-Dickey Poisson bracket. The importance of this bracket is that it can reproduce successfully the classical form of the superconformal (and super w-) algebra.

Besides its crucial role in string theories[18] and the theory of representation[19], the importance of Lie superlagbera in relation with supersymmetrization and integrability is motivated by the following: given a set of simple roots for some Lie algebra, one can construct an associated integrable bosonic conformal field theory (Toda field theory)[20]. If the algebra is finite-dimensional then the resulting theory is massless and exhibits an extended conformal symmetry[21,?] whilst if the algebra is of affine Kac-Moody type, then the resulting theory is massive. In seeking to generalize this construction, it is important to stress that there is no obvious way to supersymmetrize a given bosonic Toda theory whilst maintaining integrability[23]. One can, however, write down integrable Toda theories based on Lie superalgebras which contain both bosons and fermions but which are not supersymmetric in general. For superalgebras, unlike conventional Lie algebras, there can exist inequivalent bases of simple roots and each of these inequivalent bases leads to a distinct Toda theory. Each root of a superalgebra carries a Z_2-grading which makes it either of 'bosonic' or 'fermionic' type and it turns out that it is precisely those simple root systems which are purely fermionic which give rise to supersymmetric Toda theories .

Many important aspects of integrable models with extended conformal symmetries including the fractional supersymmetry[24] and the noncommutativity of coordinate[25] are of great interest to this study. All these aspects with some applications of the GD Poisson bracket to non trivial symmetries and geometries will be in the center of our future works. For reviews, see refs [1, 2, 3, 7, 23].

II. THE SPACE OF DIFFERENTIAL LAX OPERATORS

This section is devoted to a brief account of the basic properties of the space of differential Lax operators in the bosonic case. Presently we know that any differential operator is completely specified by a conformal spin s, $s \in Z$, two integers p and $q = p + n$, $n \geq 0$ defining the lowest and the highest degrees respectively and finally $(1 + q - p) = n + 1$ analytic fields $u_j(z)$[7]

We recall that the space $\mathcal{A}$ of all local and non local differential operators admits a Lie algebra's structure with respect to the commutator build out of the Leibnitz product. Moreover we find that $\mathcal{A}$ splits into $3 \times 2 = 6$ subalgebras $\mathcal{A}_{|+}$ and $\mathcal{A}_{j-}$, $j = 0, \pm 1$ related to each others by two types of conjugations namely the spin and the degrees conjugations. The algebra $\mathcal{A}_{++}$ and its dual $\mathcal{A}_{--}$ are of particular interest as they are incorporated into the construction of the Gelfand-Dickey (G.D) Poisson bracket of 2d integrable models.

Let us first consider the algebra $\mathcal{A}$ of all local and non local differential operators of arbitrary conformal spins and arbitrary degrees, one may expand $\mathcal{A}$ as

$$\mathcal{A} = \oplus_{p\leq q}\mathcal{A}^{(p,q)} = \oplus_{p\leq q}\oplus_s \mathcal{A}_s^{(p,q)},\ p, q, s \in Z, \tag{II.1}$$

where we have denoted by (p, q) the lowest and thee highest degrees respectively and by s the conformal spin. The vector space $\mathcal{A}^{(p,q)}$ of differential operators with given degrees (p, q) but undefined spin exhibits a Lie algebra's structure with respect to the Lie bracket for $p \leq q \leq 1$. To be explicit, consider the space $\mathcal{A}_s^{(p,q)}$ of differential operators

$$d_s^{(p,q)} = \sum_{i=p}^{q} u_{s-i}(z)\partial^i. \tag{II.2}$$

It's straightforward to check that the commutator of two operators of $\mathcal{A}_s^{(p,q)}$ is an operator of conformal spin $2s$ and degrees $(p, 2q-1)$. Since the Lie bracket $[.,.]$ acts as

$$[.,.] : \mathcal{A}_s^{(p,q)} \times \mathcal{A}_s^{(p,q)} \longrightarrow \mathcal{A}_{2s}^{(p,2q-1)}, \tag{II.3}$$

imposing the closure, one gets strong constraints on the spin s and the degrees parameters (p, q) namely

$$s = 0 \quad \text{and} \quad p \leq q \leq 1. \tag{II.4}$$

From these equations we learn in particular that the spaces $\mathcal{A}_0^{(p,q)}$, $p \leq q \leq 1$ admit a Lie algebra's structure with respect to the bracket Eq(2.3) provided that the Jacobi identity is fulfilled. This can be ensured by showing that the Leibnitz product is associative. Indeed given three arbitrary differential operators $d_{m_1}^{(p_1,q_1)}, d_{m_2}^{(p_2,q_2)}$ and $d_{m_3}^{(p_3,q_3)}$ we find that associativity follows by help of the identity

$$\sum_{l=0}^{i} \begin{pmatrix} i \\ l \end{pmatrix} \begin{pmatrix} j \\ k-l \end{pmatrix} = \begin{pmatrix} i+j \\ k \end{pmatrix} \tag{II.5}$$

where $\begin{pmatrix} i \\ j \end{pmatrix}$ is the usual binomial coefficient. The spaces $\mathcal{A}_0^{(p,q)}, p \leq q \leq 1$ as well as the vector space $\mathcal{A}_0^{(0,1)}$ are in fact subalgebra of the Lie algebra $\mathcal{A}_0^{(-\infty,1)}$ which can be decomposed as

$$\mathcal{A}_0^{(-\infty,1)} = \mathcal{A}_0^{(-\infty,-1)} \oplus \mathcal{A}_0^{(0,1)} \tag{II.6}$$

$\mathcal{A}_0^{(-\infty,-1)}$ is nothing but the Lie algebra of Lorentz scalar pure pseudo-differential operators of higher degree $q = -1$ and $\mathcal{A}_0^{(0,1)}$ is the central extension of the Lie algebra $\mathcal{A}_0^{(1,1)}$ of vector fields $Diff(S^1)$,

$$\mathcal{A}_0^{(0,1)} = \mathcal{A}_0^{(0,0)} \oplus \mathcal{A}_0^{(1,1)} \tag{II.7}$$

and where $\mathcal{A}_0^{(0,0)}$ is the one dimensional trivial ideal. The infinite dimensional huge space $\mathcal{A}$ is the algebra of differential operators of arbitrary spins and arbitrary degrees. It's obtained from the space $\mathcal{A}^{(p,q)}$ by summing over all allowed degrees

$$\mathcal{A} = \oplus_{p\leq q}\mathcal{A}^{(p,q)} \tag{II.8}$$

or equivalently

$$\begin{aligned} \mathcal{A} &= \quad \oplus_{p\in Z}[\oplus_{\ n\in N}\mathcal{A}^{(p,p+n)}] \\ &= \oplus_{p\in Z}[\oplus_{n\in N}[\oplus_{s\in Z}\mathcal{A}_s^{(p,p+n)}]] \end{aligned} \tag{II.9}$$

This infinite dimensional space which is the combined conformal spin and degrees tensor algebra is closed under the Lie bracket without any constraint. A remarkable property of $\mathcal{A}$ is that it can splits into six infinite subalgebras $\mathcal{A}_{j+}$ and $\mathcal{A}_{j-}$, $j = 0, \pm 1$ related to each others by conjugation of the spin and degrees. Indeed given two integers p and $q \geq p$ it is not difficult to see that the vector spaces $\mathcal{A}^{(p,q)}$ and $\mathcal{A}^{(-q-1,-p-1)}$ are dual with respect to the pairing product $(.,.)$ defined as

$$(d^{(r,s)}, d^{(p,q)}) = \delta_{0,1+r+q}\delta_{0,1+s+p}res[d^{(r,s)} \times d^{(p,q)}], \tag{II.10}$$

where $d^{(r,s)}$ are differential operators with fixed degrees $(r, s; s \geq r)$ but arbitrary spin and where the residue operation res is defined as

$$res(\partial^i) = \delta_{0,i+1} \tag{II.11}$$

This equation shows that the operation res exhibits a conformal spin $\Delta = 1$. using the properties of this operation and the pairing product eq(2.10) one can decompose $\mathcal{A}$ as follows

$$\mathcal{A} = \mathcal{A}_+ \oplus \mathcal{A}_- \tag{II.12}$$

with

$$\mathcal{A}_+ = \oplus_{p\geq 0}[\oplus_{n\in N}\mathcal{A}^{(p,p+n)}] \tag{II.13}$$

$$\mathcal{A}_- = \oplus_{p\geq 0}[\oplus_{n\in N}\mathcal{A}^{(-p-n-1,-p-1)}] \tag{II.14}$$

The indices $+$ and $-$ carried by $\mathcal{A}_+$ and $\mathcal{A}_-$ refer to the positive (local) and negative (non local) degrees respectively. On the other hand one can decomposes the space $\mathcal{A}^{(p,p+n)}, n \geq 0$ as

$$\mathcal{A}^{(p,p+n)} = \Sigma_-^{(p,p+n)} \oplus \Sigma_0^{(p,p+n)} \oplus \Sigma_+^{(p,p+n)} \tag{II.15}$$

$\Sigma_-^{(p,p+n)}$ and $\Sigma_+^{(p,p+n)}$ denotes the spaces of differential operators of negative and positive definite spin. They read as

$$\Sigma_-^{(p,p+n)} = \oplus_{s>0}\mathcal{A}_{-s}^{(p,p+n)} \tag{II.16}$$

$$\Sigma_0^{(p,p+n)} = \mathcal{A}_0^{(p,p+n)} \tag{II.17}$$

$$\Sigma_+^{(p,p+n)} = \oplus_{s>0}\mathcal{A}_s^{(p,p+n)} \tag{II.18}$$

$\Sigma_0^{(p,p+n)}$ is just the vector space of Lorenz scalar differential operators. Combining eqs(2.12-18) one sees that $\mathcal{A}$ decomposes into $6 = 3 \times 2$ subalgebras

$$\mathcal{A} = \oplus_{j=0,\pm}[\mathcal{A}_{j+} \oplus \mathcal{A}_{j-}] \tag{II.19}$$

with

$$\mathcal{A}_{j+} = \oplus_{p\geq 0}[\oplus_{n\in N}\Sigma_j^{(p,p+n)}] \tag{II.20}$$

$$\mathcal{A}_{j-} = \oplus_{p\geq 0}[\oplus_{n\in N}\Sigma_j^{(-p-n-1,-p-1)}] \tag{II.21}$$

The duality of these $6 = 3 \times 2$ subalgebras is described by the combined scalar product $\ll ., . \gg$ built out of the product Eq(2.10) and conformal spin pairing

$$< u_k, u_l >:= \int dz\ u_k(z)u_{1-k}(z)\delta_{k+l,1} \tag{II.22}$$

as follows[7].

$$\ll d_m^{(r,s)}, d_n^{(p,q)} \gg := \delta_{0,n+m}\delta_{0,1+q+r}\delta_{0,1+p+s}\int dz\ res[d_m^{(r,s)} \times d_{-m}^{(-s-1,-r-1)}] \tag{II.23}$$

with respect to this new product; $\mathcal{A}_{++}$, $\mathcal{A}_{0+}$ and $\mathcal{A}_{-+}$ behaves as the dual algebras of $\mathcal{A}_{--}$, $\mathcal{A}_{0-}$ and $\mathcal{A}_{+-}$ respectively while $\mathcal{A}_{0-}$ is just the algebra of Lorenz scalar pure pseudo-operators. This algebra and its dual $\mathcal{A}_{0+}$, the space of Lorenz scalar local differential operators, are very special subalgebras as they are systematically used to construct new realizations of the w_i-symmetry, $i \geq 2$ by using scalar differential operators type.

$$l^{(k)}(a) = a_{-k}(a)\partial^k \tag{II.24}$$

To close this short recapitulating section, we note that the space $\mathcal{A}_{++}$ is the algebra of local differential operators of positive definite spins and positive degrees. $\mathcal{A}_{--}$ however, is the Lie algebra of pure pseudo-differential operators of negative degrees and spins. It is these two kinds of algebras which are usually considered in the construction of the G.D Poisson bracket in the bosonic case as it's explicitly shown in[7].

III. SUPERSYMMETRIC LAX OPERATORS

A. Basics definitions

The aim of this section is to describe the supersymmetric extension of the space of bosonic Lax operators introduced previously. This supersymmetric generalization which is straightforward and natural in the fist steps, exhibits some non trivial properties and make the fermionic study more fruitful. Using the space of supersymmetric Lax operators, one can derive the Hamiltonian structure of non linear two dimensional super integrable models obtained by extending the bosonic Hamiltonian structure defined on the algebra $\mathcal{A}_{++} \oplus \mathcal{A}_{--}$.
Let's first consider the ring of all analytic super fields $u_{\frac{k}{2}}(\hat{z}),\ k \in Z$, which depend on (1|1) superspace coordinates $\hat{z} = (z, \theta)$. In this super commutative Z_2-graded ring $R\,[u(\hat{z})]$, one can define an odd super derivation $D = \partial_\theta + \theta\partial$, the N=1 supercovariant derivatives which obeys $N = 1$ supersymmetric algebra $D^2 = \partial$ with $\theta^2 = 0$ and $\partial_\theta \equiv \int d\theta$. Note that the supersymmetric G.D bracket, which we shall discuss in the sequel, defines a Poisson bracket on the space of functional of the superfields $u_{\frac{k}{2}}(\hat{z})$ defined on the ring $R\,[u(\hat{z})]$.
We define the ring $\Sigma\,[D]$ of differential supersymmetric operators as polynomials in D with coefficients in R. Using our previous notation, one set

$$\Sigma[D] = \oplus_{m\in Z}\ \oplus_{p\leq q}\Sigma^{(p,q)}_{\frac{m}{2}}[D] \quad p,q \in Z \tag{III.1}$$

where $\Sigma^{(p,q)}_{\frac{m}{2}}[D]$ is the space of supersymmetric operators type

$$\pounds^{(p,q)}_{\frac{m}{2}}[u] = \sum_{i=p}^{q} u_{\frac{m-i}{2}}(\hat{z})D^i \quad p,q \in Z \tag{III.2}$$

$\Sigma^{(p,q)}_{\frac{m}{2}}$ behaves as a $(1+q-p)$ dimensional superspace. Note also that the ring R of all graded superfields can be decomposed as

$$R \equiv R^{(0,0)} := \oplus_{k\in Z} R^{(0,0)}_{\frac{k}{2}} \tag{III.3}$$

where $R^{(0,0)}_{\frac{k}{2}}$ is the set of superfield $u_{\frac{k}{2}}(\hat{z})$ indexed by half integer conformal spin $\frac{k}{2}, k \in Z$. Thus, the one dimensional objects $u_{\frac{m-i}{2}}(\hat{z})D^i$ are typical elements of the superspace

$$\Sigma^{(i,i)}_{\frac{m}{2}} = R^{(0,0)}_{\frac{m-i}{2}} \times D^i \equiv R^{(0,0)}_{\frac{m-i}{2}} \otimes \Sigma^{(i,i)}_{\frac{i}{2}} \tag{III.4}$$

which is fundamental in the construction of supersymmetric operators type Eq.(3.2). The expression Eq.(3.2) means also that

$$\Sigma^{(p,q)}_{\frac{m}{2}}[D] \equiv \oplus_{i=p}^{q} \Sigma^{(i,i)}_{\frac{m}{2}} \tag{III.5}$$

Indeed these definitions are important in the sense that one can easily identify all objects of the huge superspace Σ. An element $\pounds$ of $\Sigma[D]$ is called a local supersymmetric Lax operator if it is homogeneous under the Z_2-grading

$$|x| := \begin{cases} 0 & ,\ x\ \ even \\ 1 & ,\ x\ \ odd \end{cases} \tag{III.6}$$

and have the following form at order $n, n \in N$

$$\pounds^{(0,n)}_{\frac{n}{2}} := \sum_{i=0}^{n} u_{\frac{i}{2}}(\hat{z})D^{n-i} \tag{III.7}$$

The homogeneity condition simply states that the Z_2-grading of the N=1 superfield $u(\hat{z})$ is defined as

$$\left|u_{\frac{i}{2}}(\hat{z})\right| = i\,(mod\,2) \tag{III.8}$$

The space of supersymmetric Lax operators is refereed hereafter to as $\Sigma^{(0,n)}_{\frac{n}{2}}$ and exhibits a dimension $n+1$. We recall that the upstairs integers $(0,n)$ are the lowest and the highest degrees of $\pounds$ and the down stair index $\frac{n}{2}$ is the spin of $\pounds$. To define a Lie algebraic structure on the superspace Σ one need to introduce a graded commutator defined for two arbitrary operators X and Y as

$$[X,Y]_i = XY - (-)^i YX \tag{III.9}$$

where the index $i = \overline{0}$ or $\overline{1}$ refer to the commutator [,] or anticommutator {,} respectively. As shown in section 2, this index is related to the graduation of the super operators X and Y as follows

$$i = |X| \, . \, |Y| \tag{III.10}$$

Moreover, the super commutator Eq.(3.9) must satisfy the

$$[X,Y]_i = -(-)^i \, [Y,X]_i \tag{III.11}$$

with

$$\begin{array}{rcccl} [X,Y]_{\overline{0}} & = & XY - YX & = & [X,Y] \\ [X,Y]_{\overline{1}} & = & XY + YX & = & \{X,Y\} \end{array} \tag{III.12}$$

and the super-Jacobi identity

$$[[X,Y]_i, Z]_j + [[Z,X]_i, Y]_j + [[Y,Z]_i, X]_j = 0 \tag{III.13}$$

Next we introduce the multiplication of operators in the superspace Σ and derive some crucial algebraic properties. Let $\Sigma^{(0,0)}_{\frac{j}{2}} \equiv R^{(0,0)}_{\frac{j}{2}}$ be the ring of analytic superfields $\phi_{\frac{j}{2}}(\hat{z}) \equiv \phi(\hat{z})$ of conformal spin $\frac{j}{2}$ and $\Sigma^{(p,q)}_{\frac{j}{2}}$ be a superspace endowed with a super derivative D such that

$$D^{(l)}\left(\Sigma^{(p,q)}_{\frac{j}{2}}\right) \subseteq \Sigma^{(p,q+l)}_{\frac{j+l}{2}} \tag{III.14}$$

We will denote the superfield derivatives $(D\phi)$, $\left(D^2\phi\right)$, ...,$\left(D^i\phi\right)$ simply as ϕ', ϕ'', ...,$\phi^{(i)}$ respectively. The multiplication of operators in Σ is defined with respect to the super Leibnitz rule given by the following mapping

$$D^{(l)} : R^{(0,0)}_{\frac{j}{2}} \longrightarrow \Sigma^{(p,l)}_{\frac{j+l}{2}} \tag{III.15}$$

such that[4]

$$D^{(l)}\phi\,(\hat{z}) = \sum_{i=0}^{\infty} \left[\begin{array}{c} l \\ l-i \end{array}\right] (-)^{j(l-i)} \phi^{(i)} D^{(l-i)} \tag{III.16}$$

where l is an arbitrary integer and the super binomial coefficients $\left[\begin{array}{c} l \\ k \end{array}\right]$ are defined by

$$\left[\begin{array}{c} l \\ k \end{array}\right] = \left\{\begin{array}{l} 0, \text{for } k > l \text{ and for } (k,l) = (0,1) mod 2 \\ \qquad \left(\begin{array}{c} \left[\frac{l}{2}\right] \\ \left[\frac{k}{2}\right] \end{array}\right), \qquad \text{otherwise} \end{array}\right. \tag{III.17}$$

The lowest degree p of the superspace $\Sigma^{(p,l)}_{\frac{j+l}{2}}$ Eq.(3.15) is given by

$$p = \left\{\begin{array}{ll} 0, & if \;\; l \geq 0 \\ -\infty & if \;\; l \leq -1 \end{array}\right. \tag{III.18}$$

The symbol $[x]$ stands for the integer part of $x \in \frac{Z}{2}$ and $\begin{pmatrix} i \\ j \end{pmatrix}$ is the usual binomial coefficient. The binomial and super binomial coefficients satisfy among others, the following useful properties

a/

$$\begin{pmatrix} p \\ q \end{pmatrix} = \begin{matrix} 1 & \text{for } q = 0 \text{ or } q = p \\ \frac{p(p-1)..(q+1)}{(p-q)!} & \text{for } q \prec p \\ 0 & \text{otherwise} \end{matrix}$$

$$q \begin{pmatrix} p \\ q \end{pmatrix} (-)^q = p \begin{pmatrix} -q \\ -p \end{pmatrix} (-)^p$$

$$\begin{pmatrix} -p \\ q \end{pmatrix} = (-)^q \begin{pmatrix} p+q-1 \\ q \end{pmatrix} \quad \text{(III.19)}$$

b/

$$\begin{bmatrix} 2p \\ 2q \pm 1 \end{bmatrix} = 0, \qquad p, q \in Z$$

$$\begin{bmatrix} 2p+1 \\ 2p \end{bmatrix} = \begin{bmatrix} p \\ 0 \end{bmatrix} = \begin{bmatrix} p \\ p \end{bmatrix} = \begin{bmatrix} 2p+1 \\ 1 \end{bmatrix} = 1$$

$$\begin{bmatrix} 2p \\ 2q \end{bmatrix} = \begin{bmatrix} 2p+1 \\ 2q \end{bmatrix} = \begin{bmatrix} 2p+1 \\ 2q+1 \end{bmatrix} = \begin{pmatrix} p \\ q \end{pmatrix} \quad \text{(III.20)}$$

$$\begin{bmatrix} p \\ q \end{bmatrix} = (-)^{[\frac{q}{2}]} \begin{bmatrix} q-p-1 \\ q \end{bmatrix}$$

The first local super derivatives of the superfield $\phi = \phi_{j/2}(\hat{z})$ are given explicitly by

$$D\phi = \phi' + (-)^j \phi D \quad \text{(III.21)}$$

$$D^2\phi = \phi'' + \phi D^2$$

$$D^3\phi = \phi''' + (-)^j \phi'' D + \phi' D^2 + (-)^j \phi D^3$$

$$D^4\phi = \phi^{(4)} + 2\phi'' D^2 + \phi D^4$$

$$D^5\phi = \phi^{(5)} + (-)^j \phi^{(4)} D + 2\phi''' D^2 + 2(-)^j \phi''^{D^3} D^3 + \phi' D^4 + (-)^j \phi D^5$$

$$D^6\phi = \phi^{(6)} + 3\phi^{(4)} D^2 + 3\phi'' D^4 + \phi D^6$$

More generally one have

$$D^{2i}\phi = \phi^{(2i)} + i\phi^{(2i-2)}D^2 + \tfrac{i(i-1)}{2}\phi^{(2i-4)}D^4 + ...$$
$$= \phi^{(2i)} + \textstyle\sum_{n=1}^{i} \tfrac{i(i-1)...(i+1-n)}{n!}\phi^{(2i-2n)}D^{2n}$$

$$D^{2i+1}\phi = \textstyle\sum_{k=1}^{2i+1} \begin{bmatrix} 2i+1 \\ k \end{bmatrix} (-)^{j(2i+1-k)}\,\phi^{(k)}D^{2i+1-k} \tag{III.22}$$
$$= \textstyle\sum_{p=0}^{i} \begin{pmatrix} i \\ p \end{pmatrix} \left(\phi^{(2p+1)} + (-)^j\,\phi^{(2p)}D\right) D^{2i-2p},$$

where i is a positive integer and j is the graduation of superfield $\phi(\hat{z})$. Similar formulas can be written for non local super derivatives of the superfield $\phi(\hat{z})$, we have for example

$$D^{-1}\phi = (-)^j\phi D^{-1} + \phi' D^{-2} - (-)^j\phi'' D^{-3} - \phi''' D^{-4} + (-)^j\phi^{(4)}D^{-5} + \phi^{(5)}D^{-6} + .. \tag{III.23}$$
$$D^{-2}\phi = \phi D^{-2} - \phi'' D^{-4} + \phi^{(4)}D^{-6} + ...$$
$$D^{-3}\phi = (-)^j\phi D^{-3} + \phi' D^{-4} - 2(-)^j\phi'' D^{-5} - 2\phi''' D^{-6} + ...$$
$$D^{-4}\phi = \phi D^{-4} - 2\phi'' D^{-6} + ...$$
$$D^{-5}\phi = (-)^j\phi D^{-5} + \phi' D^{-6} + ...$$
$$D^{-6}\phi = \phi D^{-6} + ...$$

Setting

$$D^{-k}\phi = \textstyle\sum_{l=0}^{\infty} m_{kl}(\phi)D^{-k-l}$$
$$m_{kl}(\phi) = \begin{bmatrix} -k \\ -k-l \end{bmatrix} (-)^{j(k+l)}\,\phi^{(l)} \tag{III.24}$$

One have then to define an infinite matrix $M(\phi)$ whose entries $m_{kl}(\phi)$ are functions which depend on the superfield ϕ and its derivatives. The integers k and l indicate respectively the index of "row" and "coulumn" of the huge matrix $[M(\phi)]_{kl}$, $k = -\infty, ..., -2, -1$ and $l = 0, 1, 2, ..., \infty$.

In this context, we express the non locality property of the super derivatives $D^{-k}\phi\,(\hat{z})$ by an infinite order higher triangular matrix $[M(\phi)]_{kl}$ which acts as follows

$$[M]_{k=i,l} \equiv (m_{i0}, m_{i1}, .., ..) : \begin{pmatrix} D^{-i} \\ D^{-i-1} \\ \vdots \end{pmatrix} \mapsto D^{-i}\phi \tag{III.25}$$

for a fixed row's index $k = i, i \geq 1$.

An important aim of this formulation is to define Poisson brackets on the superspace

$$\oplus_{n\geq 0}\left[\Sigma^{(0,n)}_{\frac{n}{2}} \oplus \left[\Sigma^{(0,n)}_{\frac{n}{2}}\right]^*\right] \tag{III.26}$$

where $\left[\Sigma^{(0,n)}_{\frac{n}{2}}\right]^*$ is a subspace of the super Volterra algebra of pseudo-differential operators which is dual to $\Sigma^{(0,n)}_{\frac{n}{2}}$. We note that the algebra Eq.(3.26) is just the supersymmetric analogue of the bosonic algebra $\mathcal{A}_{++}\oplus\mathcal{A}_{--}$ introduced previously in the construction of the bosonic Gelfand-Dickey bracket. The functional involved in the definition of Gelfand-Dickey(G.D) super bracket are of the form

$$F\left[u\left(\hat{z}\right)\right]=\int_{B}f(u), \tag{III.27}$$

where $f(u)$ is an homogenous differential polynomial of the u's and $\int_B$ is the well known Berezin integral $\int d\hat{z}=\int dz.d\theta$ which is usually defined in the following way: for any $u\left(\hat{z}\right)=a+\theta b$ and $f(u)=A(a,b)+\theta B(a,b)$ we have $\int_B f(u)=\int dzB(a,b)$.

Next we introduce the notions of super-residue and super trace which are necessary for the present study. Given a super-pseudo operator $\mathcal{P}$ in a super Volterra basis

$$\mathcal{P}=\sum_{i\in Z}D^{i}f_{i}(\hat{z}) \tag{III.28}$$

The super-residue is defined as

$$Sres\mathcal{P}=\int_{B}\left(-\right)^{|f_i|}f_{-1}(\hat{z}) \tag{III.29}$$

Note that the residue operation (res) introduced in the bosonic case Eq.(2.11), exhibits a spin $\Delta(res)=1$, while the spin of the super-residue operation $\Delta(Sres)=\frac{1}{2}$, fact which is immediate if we remark that $\Delta(d\hat{z})=\Delta(dzd\theta)=\frac{-1}{2}$. One can also easily show that the super residue of a graded supercommutator is a total derivative, so that its supertrace is a vanishing number

$$Sres\left[\pounds,P\right\}=total\ \ derivative \tag{III.30}$$

$$Str\left[\pounds,P\right\}=0, \tag{III.31}$$

for every $\pounds\in\mathit{\Sigma}$ and $P\in\Sigma^{*}$. Since $\left[\pounds,R\right\}=\pounds P-(-)^{|\pounds||P|}P\pounds$, the property Eq.(3.31) means that it's possible to define a graded superbilinear form

$$Str\left(\pounds P\right)=(-)^{|\pounds||P|}Str\left(P\pounds\right) \tag{III.32}$$

on the superspace

$$\Sigma_{++}\oplus\Sigma_{--}=\oplus_{n\geq0}\left[\Sigma_{\frac{n}{2}}^{(0,n)}\oplus\left[\Sigma_{\frac{n}{2}}^{(0,n)}\right]^{*}\right] \tag{III.33}$$

This form pairs the super differential operators $\pounds$ of Σ_{++} and the super pseudo-operators P of Σ_{--} as follows

$$\ll\pounds_{\frac{n}{2}}^{(0,n)},P_{\frac{m}{2}}^{(r,s)}\gg:=\delta_{n+m,0}\delta_{n+r+1,0}\delta_{s+1,0}\int_{B}Sres\left(\pounds_{\frac{n}{2}}^{(0,n)}\circ P_{\frac{-n}{2}}^{(-n-1,-1)}\right) \tag{III.34}$$

This supersymmetric scalar product, which connect both the super-residue and degrees pairing, can be rewriting in a similar form

$$\ll\pounds_{\frac{n}{2}}^{(0,n)},P_{\frac{-n}{2}}^{(-n-1,-1)}\gg=\int_{B}\sum_{i=1}^{n}(-)^{i+1}\left(u_{\frac{i}{2}}\left(\hat{z}\right)\chi_{\frac{1-i}{2}}\left(\hat{z}\right)\right), \tag{III.35}$$

where

$$\pounds_{\frac{n}{2}}^{(0,n)}=\sum_{i=0}^{n}u_{\frac{i}{2}}\left(\hat{z}\right)D^{n-i} \tag{III.36}$$

$$P_{\frac{-n}{2}}^{(-n-1,-1)}=\sum_{i=1}^{n+1}D^{-i}\chi_{\frac{i-n}{2}}\left(\hat{z}\right) \tag{III.37}$$

The supersymmetric Lax operators usually are those for which $u_0(\hat{z}) = 1$. This simple choice which is consistent with the definition of (supersymmetric) w-symmetries imply a constraint on the corresponding dual superfield $\chi_{\frac{1}{2}}(\hat{z})$, namely

$$\chi_{\frac{1}{2}}(\hat{z}) = 0 \tag{III.38}$$

On the other hand, using the supersymmetric combined product Eq.(3.35), it is not difficult to see that

$$\left[\Sigma_{\frac{n}{2}}^{(0,n)}\right]^* = \Sigma_{\frac{-n}{2}}^{(-n-1,-1)} \tag{III.39}$$

So, the super Lax operator $\pounds$ and its dual P Eqs.(3.36-37) read

$$\pounds = D^n + \sum_{i=1}^{n} u_{\frac{i}{2}}(\hat{z}) D^{n-i} \tag{III.40}$$

$$P = \sum_{i=1}^{n} D^{-i} \chi_{\frac{i-n}{2}}(\hat{z}) \tag{III.41}$$

For the formal sum Eq(3.37), note that only a finite number of the superfields $\chi_{\frac{i-n}{2}}(\hat{z})$ are non zero. The super-residue duality imply that it is possible to realize the pseudo-superfields $\chi_{\frac{i}{2}}(\hat{z})$ in terms of $u_{\frac{j}{2}}(\hat{z})$. Indeed, let us consider a functional $f[u_{\frac{1}{2}}, u_1, u_{\frac{3}{2}}, ..., u_n]$ acting on the ring $R_{\frac{k}{2}}^{(0,0)}$ of chiral super fields $u_{\frac{k}{2}}(\hat{z})$. We have

$$P_f = \sum_{j=1}^{n} D^{-j}.\chi_{\frac{j-n}{2}}(\hat{z}), \tag{III.42}$$

with

$$\chi_{\frac{1-k}{2}}(\hat{z}) := (-)^k \frac{\delta f[u]}{\delta u_{\frac{k}{2}}}, k = 1, 2, ... \tag{III.43}$$

(III.44)

$$\Delta\left(\frac{\delta f}{\delta u_{\frac{k}{2}}}\right) = \frac{1-k}{2} \tag{III.45}$$

Note by the way that, in addition to the functionals $f[u]$, other geometrical objects that are necessary to construct the super symmetric Gelfand-Dickey brackets are given by vector fields (1-forms) and the map sending a function to its associated Hamiltonian vector field, ie the coadjoint supersymmetric operator . Here we will not need to follow this procedure, however we shall concentrate in the next part of this work on the infinitesimal (coadjoint) supersymmetric operator, which is fundamental in the definition of G.D super bracket and the derivation of higher spin extensions of the conformal symmetry.

IV. THE SUPER-HAMILTONIAN OPERATOR $V_P(L)$:

First of all we remind that the supersymmetric G.D Poisson bracket is of the form

$$\left\{F[u_{\frac{i}{2}}], G[u_j]\right\} = \int_B Sres\left\{V_{P_F}(L) \circ P_G\right\}, \tag{IV.1}$$

where F and G are functionals of the superfield $u_i(\hat{z})$. This definition of the super G.D bracket is based on the super-residue duality Eq.(3.29) of super differential operators L and super volterra ones P. The map which combines these two kind of dual operators is given by the hamiltonian operator:

$$V_P(L) = L(PL)_+ - (LP)_+ L \ , \tag{IV.2}$$

where the subscripts + indicate the restriction to the local part, ie

$$\left[\sum_{i\epsilon Z} a_i(\hat{z}) D^i\right]_+ = \sum_{i\geq 0} a_i(\hat{z}) D^i \tag{IV.3}$$

Before one turns to the supersymmetric analysis of the G.D algebra and its induced superconformal and super w-symmetry, we give here below an explicit description of the hamiltonian operator $V_P(L)$ which is defined as the infinitesimal action of super- pseudo operators P on the space of supersymmetric Lax operators L. In order to simplify the notation we have considered $L^{(0,n)}_{n/2} \equiv L$ and $P^{(-n-1,-1)}_{-n/2} \equiv P$. An important property of $V_P(L)$ is its fundamental role in describing both the first and the second hamiltonian G.D Poisson brackets occurring in the definition of super integrable systems. Knowing that the first G.D bracket is constructed by using the local graded commutator $[P, L]_+$ while the second G.D bracket Eq.(4.1) is generated by $V_P(L)$, the following shift for example:

$$L \longrightarrow L + \lambda \equiv \tilde{L}, \tag{IV.4}$$

shows clearly the relevance of the second GD bracket. Indeed remark that $V_P(L)$ transform with respect to Eq.(4.4) like:

$$V_P(L) \longrightarrow V_P(L) + \lambda\, [P, L]_+ = V_P(\tilde{L}), \tag{IV.5}$$

showing in turn how the first GD bracket can be described by the second one. Next we focus to work out supersymmetric hamiltonian operators $V_P(L)$ for super Lax operators L of degree three and five.

1. The $N = 2$ superconformal algebra.

Let's consider the following super (pseudo) operators

$$L = D^3 + UD^2 + VD + W, \qquad L \in \Sigma^{(0,3)}_{\frac{3}{2}} \tag{IV.6}$$

$$P = D^{-1}X + D^{-2}Y + D^{-3}Z, \qquad P \in \Sigma^{(-3,-1)}_{-\frac{3}{2}} \tag{IV.7}$$

where (U, V, W) and (X, Y, Z) are superfields of spin $\left(\frac{1}{2}, 1, \frac{3}{2}\right)$ and $\left(-1, -\frac{1}{2}, 0\right)$ respectively. The superfields are constrained by the super residue duality

$$\begin{aligned} Sres\,(L.P) &= \sum_{i=1}^{3} (-)^{i+1} u_{\frac{i}{2}}(\hat{z}) v_{\frac{1-i}{2}}(\hat{z}) \\ &= UZ - VY + WX \end{aligned} \tag{IV.8}$$

with the convention notation $\left(u_{\frac{1}{2}}, u_1, u_{\frac{3}{2}}\right) \equiv (U, V, W)$ and $\left(v_{-1}, v_{-\frac{1}{2}}, v_0\right) \equiv (X, Y, Z)$. Note that we have to set $U \equiv u_{\frac{1}{2}}(\hat{z}) = 0$ which is the traceless condition required by the $sl(2|2)$ Lie super algebra structure. This condition is equivalent to the following coset superspace operation

$$\Sigma^{(0,3)}_{\frac{3}{2}} \longrightarrow \Sigma^{(0,3)}_{\frac{3}{2}} \ / \ \Sigma^{(2,2)}_{\frac{3}{2}} \tag{IV.9}$$

where $\Sigma_{\frac{3}{2}}^{(2,2)}$ is the one dimensional subspace of $\Sigma_{\frac{3}{2}}^{(0,3)}$ which is generated by half spin superfield $U \equiv u_{\frac{1}{2}}(\hat{z})$. At first sight, it seems difficult how to determine the superfield $Z \equiv v_0$ of spin zero, dual to the vanishing superfield $U \equiv u_{\frac{1}{2}}$. To do this, one must compute $V_P(L)$ for

$$L = D^3 + VD + W \tag{IV.10}$$
$$P = D^{-1}X + D^{-2}Y + D^{-3}Z \tag{IV.11}$$

and require that L is invariant under the coadjoint action Eq.(4.2). Straightforward computations lead to

$$(LP)_+ = XD^2 - YD + X^{''} + Y^{'} + VX + Z \tag{IV.12}$$

$$(PL)_+ = XD^2 + (X^{'} + Y)D - X^{''} + XV + Z, \tag{IV.13}$$

implying in turns the following

$$L\,(P.L)_+ = \sum_{i=0}^{5} A_i(\hat{z})D^i \tag{IV.14}$$

with

$$\begin{aligned}
A_5(\hat{z}) &= X \\
A_4(\hat{z}) &= -Y \\
A_3(\hat{z}) &= 2VX + Z + Y^{'} + X^{''} \\
A_2(\hat{z}) &= WX + X^{'}V + XV^{'} - VY + Z^{'} - Y^{''} - X^{'''} \\
A_1(\hat{z}) &= WX^{'} + WY + VY^{'} + VZ + XV^2 + X^{''}V \\
&\quad + XV^{''} + Z^{''} + Y^{'''} \\
A_0(\hat{z}) &= WXV + VXV^{'} + VX^{'}V + WZ - WX^{''} \\
&\quad + XV^{'''} + X^{''}V^{'} + X^{'}V^{''} + VZ^{'} + Z^{'''} - X^{(5)}
\end{aligned} \tag{IV.15}$$

where

$$\Delta\left(A_i(\hat{z})\right) = \frac{3-i}{2}, i = 0, 1, ..., 5 \tag{IV.16}$$

Similar computations give

$$(LP)_+\, L = \sum_{i=0}^{5} B_i(\hat{z})D^i \tag{IV.17}$$

with

$$\begin{aligned}
B_5(\hat{z}) &= X \\
B_4(\hat{z}) &= -Y \\
B_3(\hat{z}) &= 2VX + Z + Y^{'} + X^{''} \\
B_2(\hat{z}) &= XW - YV \\
B_1(\hat{z}) &= XV^2 + ZV + YW - YV^{'} + Y^{'}V + X^{''}V + XV^{''} \\
B_0(\hat{z}) &= VXW + Y^{'}W - W^{'}Y + X^{''}W + XW^{''} + ZW
\end{aligned} \tag{IV.18}$$

we find

$$\begin{aligned} V_P(L) &= \textstyle\sum_{i=0}^{2} (A_i - B_i)(\hat{z})D^i \\ &= \textstyle\sum_{i=0}^{2} C_i(\hat{z})D^i \end{aligned} \tag{IV.19}$$

with

$$\begin{aligned} C_0(\hat{z}) &= VXV' + VX'V - 2WX'' + VZ' \\ &\quad +X'V'' + XV''' + X''V' - Y'W \\ &\quad +W'Y - XW'' + Z''' - X^{(5)} \\ C_1(\hat{z}) &= WX' + 2WY - YV' + Y''' + Z''' \\ C_2(\hat{z}) &= X'V + XV' + Z' - Y'' - X''' \end{aligned} \tag{IV.20}$$

It is important to remark that the operator $V_P(L)$ don't preserve the Lie algebra's structure of the supersymmetric coset space Eq.(4.9) generated by the super Lax operators L Eq.(4.10). In other words, $V_P(L)$ is not an affine $A(1|1)^{(1)}$ operator because its D^2-term: $C_2(\hat{z})$ does not vanish in general. However, knowing that $V_P(L)$ share with L the properties of locality (positive degrees), grading and spin $\frac{3}{2}$, we can require that

$$C_2(\hat{z}) = 0 \tag{IV.21}$$

or

$$Z(\hat{z}) = X'' + Y' - XV \tag{IV.22}$$

Note also that the dual constraint Eq.(4.8) which describes the trace zero is equivalent to:

$$Sres\,\{L, P\} = 0 \tag{IV.23}$$

Injecting the constraint equation Eq.(4.22) into the expressions of $C_0(\hat{z})$ and $C_1(\hat{z})$ one finds

$$\begin{aligned} C_0(\hat{z}) &= Y^{(4)} + VY'' - 2WX'' - XW'' \\ C_1(\hat{z}) &= X^{(4)} + 2Y''' - (XV)'' + WX' + 2WY - YV' \end{aligned} \tag{IV.24}$$

which imply in turns

$$\begin{aligned} V_P(L) &= \Big[-\left(\tfrac{\delta f}{\delta W}\right)^{(4)} + 2\left(\tfrac{\delta f}{\delta V}\right)''' + \left(V.\tfrac{\delta f}{\delta W}\right)'' \\ &\quad -W\left(\tfrac{\delta f}{\delta W}\right)' + 2W\tfrac{\delta f}{\delta V} - \tfrac{\delta f}{\delta V}.V' \Big] D \\ &\quad + \Big[\left(\tfrac{\delta f}{\delta V}\right)^{(4)} + V\left(\tfrac{\delta f}{\delta V}\right)'' \\ &\quad +2W\left(\tfrac{\delta f}{\delta W}\right)'' + \tfrac{\delta f}{\delta W}.W'' \Big] \end{aligned} \tag{IV.25}$$

where: $X = \frac{\delta f}{\delta W}$ and $Y = \frac{\delta f}{\delta V}$ for $f = f\,[U, V, W]$.
The super Gelfand-Dickey algebra of the second kind takes then the following form:

$$\begin{aligned} \{f(\hat{z}), g(\hat{z})\} &= \textstyle\int_B Sres\left[V_{P_f}(L).P_g\right] \\ &= \textstyle\int d\hat{\sigma} \Big\{ \Big[\left(\partial_\sigma^2 \tfrac{\delta f}{\delta W}\right) - 2\left(D\partial_\sigma.\tfrac{\delta f}{\delta V}\right) - \left(\partial_\sigma(V.\tfrac{\delta f}{\delta W})\right) - W\left(D\tfrac{\delta f}{\delta W}\right) - 2W.\tfrac{\delta f}{\delta V} + \tfrac{\delta f}{\delta V}.(DV) \Big] \tfrac{\delta g}{\delta V} \\ &- \Big[\left(\partial_\sigma^2 \tfrac{\delta f}{\delta V}\right) + V\left(\partial_\sigma \tfrac{\delta f}{\delta V}\right) + 2W\left(\partial_\sigma \tfrac{\delta f}{\delta W}\right) + \tfrac{\delta f}{\delta W}(\partial_\sigma W) \Big] \tfrac{\delta g}{\delta W} \Big\} \end{aligned} \tag{IV.26}$$

Before we give the super G.D poisson bracket of the superfields $V(z,\theta)$ of conformal spin 1 and $\frac{3}{2}$ respectively, let's first discuss their covariantization. In this example we have used the $N=1$ supersymmetry to derive the $N=1$ superconformal algebra. The latter is best described in an $N=1$ superspace with local analytic coordinates (z,θ). The superanalytic map is given by

$$\hat{z}=(z,\theta)\longrightarrow\tilde{\hat{z}}=\left(\tilde{z}(z,\theta)\ ;\ \tilde{\theta}(z,\theta)\right), \tag{IV.27}$$

with $\tilde{\theta}^2=\theta^2=0$. The $N=1$ superderivative transforms with respect this map like:

$$D=\left(D\tilde{z}-\tilde{\theta}D\tilde{\theta}\right)\tilde{D}^2+\left(D\tilde{\theta}\right)\tilde{D} \tag{IV.28}$$

The superanalytic map defines a superconformal transformation if the superderivative D transforms homogenously

$$D=\left(D\tilde{\theta}\right)\tilde{D} \tag{IV.29}$$

which is equivalent also to the following constraint equation

$$D\tilde{z}=\tilde{\theta}D\tilde{\theta} \tag{IV.30}$$

The transformation of the super Lax operators $L\equiv L_{\frac{1}{2}}^{(0,n)}$ with respect to the superconformal transformation is given by:

$$L\quad\longrightarrow\quad\tilde{L}=\left(D\tilde{\theta}\right)^{\frac{-n-1}{2}}L\left(D\tilde{\theta}\right)^{\frac{1-n}{2}} \tag{IV.31}$$

This transformation is important in the sense that it allows to determine the correct transformations of all the superfields $u_{\frac{i}{2}}(\hat{z})$. The present example gives

$$\tilde{D}^3+\tilde{V}\tilde{D}+\tilde{W}=\left(D\tilde{\theta}\right)^{-2}\left[D^3+VD+W\right]\left(D\tilde{\theta}\right)^{-1} \tag{IV.32}$$

identifying both sides in Eq.(4.32) we obtain the following transformations laws for the superfields $V(z,\theta)$ and $W(z,\theta)$,

$$\begin{aligned}V(z,\theta)&=\tilde{V}\left(\tilde{z},\tilde{\theta}\right)\left(D\tilde{\theta}\right)^2\\ W(z,\theta)&=\tilde{W}\left(\tilde{z},\tilde{\theta}\right)\left(D\tilde{\theta}\right)^3+\tilde{V}.D\tilde{\theta}.D^2\tilde{\theta}+S(\tilde{z},\tilde{z})\end{aligned} \tag{IV.33}$$

where $S(\tilde{z},\tilde{z})$ is the super Schwarzian derivative given by

$$S(\tilde{z},\tilde{z})=\frac{\partial^2\tilde{\theta}}{D\tilde{\theta}}-2\frac{D\partial\tilde{\theta}}{D\tilde{\theta}}\frac{\partial\tilde{\theta}}{D\tilde{\theta}} \tag{IV.34}$$

The superfield $V(z,\theta)$ transforms covariantly as a field of conformal spin one, while $W(z,\theta)$ does not have the correct transformation property for a field with conformal dimension $\frac{3}{2}$. If we consider the redefinition

$$W(z,\theta)\quad\longrightarrow\quad\widehat{W}(z,\theta)=W(z,\theta)-\frac{1}{2}\left(DV(z,\theta)\right) \tag{IV.35}$$

We can easily check that $\widehat{W}(z,\theta)$ transforms covariantly as a field of conformal spin $\frac{3}{2}$. Therefore we can identify $V(z,\theta)$ and $\widehat{W}(z,\theta)$ with

$$\begin{aligned}V(z,\theta)&\equiv\Gamma(z,\theta)=J(z)+\theta G_1(z)\\ \widehat{W}(z,\theta)&\equiv\Sigma(z,\theta)=G_2(z)+\theta T(z)\end{aligned} \tag{IV.36}$$

By virtue of the second hamiltonian super GD bracket, the fields $J(z)$, $G_1(z)$, $G_2(z)$ and $T(z)$ form then an $N=2$ supermultiplet $\left(1,(\frac{3}{2})^2,2\right)$ and satisfy an $N=2$ supersymmetric algebra spanned by the following Poisson brackets

$$\begin{aligned}
\left\{J(z),J(z^{'})\right\} &= 2\partial_{z^{'}}\delta(z-z^{'}) \\
\left\{J(z),G_1(z^{'})\right\} &= -2G_2(z)\delta(z-z^{'}) \\
\left\{G_1(z),G_1(z^{'})\right\} &= -2\partial_{z^{'}}\delta(z-z^{'})-2T(z)\delta(z-z^{'}) \\
\left\{J(z),G_2(z^{'})\right\} &= -\tfrac{1}{2}G_1(z)\delta(z-z^{'}) \\
\left\{T(z),J(z^{'})\right\} &= (\partial_{z^{'}}J)\delta(z-z^{'})+J(z^{'})\partial_{z^{'}}\delta(z-z^{'})
\end{aligned} \tag{IV.37}$$

$$\begin{aligned}
\left\{G_2(z),G_1(z^{'})\right\} &= \tfrac{1}{2}\partial_{z^{'}}J\delta(z-z^{'})+J(z^{'})\partial_{z^{'}}\delta(z-z^{'}) \\
\left\{T(z),G_1(z^{'})\right\} &= \tfrac{3}{2}G_1(z^{'})\partial_{z^{'}}\delta(z-z^{'})+\partial_{z^{'}}G_1(z^{'})\delta(z-z^{'}) \\
\left\{G_2(z),G_2(z^{'})\right\} &= \tfrac{1}{2}\partial_{z^{'}}\delta(z-z^{'})+\tfrac{1}{2}T(z^{'})\delta(z-z^{'}) \\
\left\{T(z),T(z^{'})\right\} &= \tfrac{1}{2}\partial^3_{z^{'}}\delta(z-z^{'})+2T(z^{'})\partial_{z^{'}}\delta(z-z^{'})+\partial_{z^{'}}T(z^{'})\delta(z-z^{'}) \\
\left\{T(z),G_2(z^{'})\right\} &= \tfrac{3}{2}G_2(z^{'})\partial_{z^{'}}\delta(z-z^{'})+\partial_{z^{'}}G_2(z^{'})\delta(z-z^{'})
\end{aligned} \tag{IV.38}$$

The compact form of the above Poisson brackets reads

$$\begin{aligned}
\left\{\Gamma(z,\theta),\Gamma(z^{'},\theta^{'})\right\} &= -2\left(D^{'}\partial_{z^{'}}\Delta\right)-2\Sigma(z^{'},\theta^{'})\Delta \\
\left\{\Sigma(z,\theta),\Gamma(z^{'},\theta^{'})\right\} &= -\Gamma(z^{'},\theta^{'})\left(D^{'2}\Delta\right)-\tfrac{1}{2}(D^{'}\Gamma)(D^{'}\Delta)+(D^{'2}\Gamma)\Delta \\
\left\{\Sigma(z,\theta),\Sigma(z^{'},\theta^{'})\right\} &= -\tfrac{1}{2}(D^{'5}\Delta)-\tfrac{3}{2}\Sigma(z^{'},\theta^{'})\left(D^{'2}\Delta\right)-\tfrac{1}{2}(D^{'}\Sigma)(D^{'}\Delta)-(D^{'2}\Sigma)\Delta
\end{aligned} \tag{IV.39}$$

with $\Delta=\delta(z-z^{'}).(\theta-\theta^{'})$ and $D^{'}=\partial_{\theta^{'}}+\theta^{'}\partial_{z^{'}}$.

2. The $N=2$ super W_3-algebra

Here we describe the infinitesimal coadjoint operator $V_P(L)$ associated to $N=2$ super W_3-algebra which is an extension of the supersymmetric $N=2$ Virasoro algebra. The affine graded superalgebra considered is $A(2\mid 2)^{(1)}$ generated by the superfields $(J,Q,T,W)\equiv\left(U_1,U_{\frac{3}{2}},U_2,U_{\frac{5}{2}}\right)$. These fields are the coefficients of the following five order's super Lax operator

$$\begin{aligned}
L &= D^5+JD^3+QD^2+TD+W \\
L &\in \Sigma^{(0,5)}_{\frac{5}{2}}/\Sigma^{(4,4)}_{\frac{5}{2}}
\end{aligned} \tag{IV.40}$$

The superpseudo operator corresponding to L is

$$P = D^{-5}X_5 + D^{-4}X_4 + D^{-3}X_3 + D^{-2}X_2 + D^{-1}X_1$$
$$P \in \Sigma^{(-5,-1)}_{-\frac{5}{2}} \tag{IV.41}$$

with $\Delta(X_i) = \frac{i-5}{2}$, $i = 1, 2, ..., 5$ and $|X_i| = (i + 1)$ mod 2 .

The functional realization of the pseudo superfields is given by

$$X_1 = -\frac{\delta f}{\delta W} \tag{IV.42}$$
$$X_2 = \frac{\delta f}{\delta T} \tag{IV.43}$$
$$X_3 = -\frac{\delta f}{\delta Q} \tag{IV.44}$$
$$X_4 = \frac{\delta f}{\delta J}, \tag{IV.45}$$

where $f = f\ [R, Q, T, W]$. The scalair superfield $X_5(\hat{z})$ dual to the vanishing coefficient $U_{\frac{1}{2}}(\hat{z})$ of the D^4-term of L is requested to satisfy the traceless condition

$$Sres\,\{L, P\} = 0 \tag{IV.46}$$

with

$$Sres\,(L, P) = -JX_4 + QX_3 - TX_2 + WX_1 \tag{IV.47}$$

After a long computation we find

$$(L \circ P)_+ L = \sum_{i=0} \Gamma_i D^{(i)} L \tag{IV.48}$$
$$= \sum_{i=0}^{9} \gamma_i D^i \tag{IV.49}$$

where Γ_i , $i = 0, ..., 4$ are superfields of spin $\Delta(\Gamma_i) = -\frac{i}{2}$ expressed in term of the pseudo-superfields X_i as follows

$$\begin{aligned}
\Gamma_0 &= -J\left(\tfrac{\delta f}{\delta W}\right)'' + J\left(\tfrac{\delta f}{\delta T}\right)' - J\left(\tfrac{\delta f}{\delta Q}\right) + Q\left(\tfrac{\delta f}{\delta T}\right) - T\left(\tfrac{\delta f}{\delta W}\right) - \left(\tfrac{\delta f}{\delta W}\right)^{(4)} + \left(\tfrac{\delta f}{\delta T}\right)''' - \left(\tfrac{\delta f}{\delta Q}\right)'' \\
&\quad + \left(\tfrac{\delta f}{\delta R}\right)' + X_5 - Q\left(\tfrac{\delta f}{\delta W}\right)' \\
\Gamma_1 &= -J\tfrac{\delta f}{\delta T} - Q\tfrac{\delta f}{\delta W} - \left(\tfrac{\delta f}{\delta T}\right)'' - \tfrac{\delta f}{\delta R} \\
\Gamma_2 &= -J\left(\tfrac{\delta f}{\delta W}\right) - 2\left(\tfrac{\delta f}{\delta W}\right)'' + \left(\tfrac{\delta f}{\delta T}\right)' - \tfrac{\delta f}{\delta Q} \\
\Gamma_3 &= -\tfrac{\delta f}{\delta T} \\
\Gamma_4 &= -\tfrac{\delta f}{\delta W}
\end{aligned} \tag{IV.50}$$

An explicit calculations gives

$$
\begin{aligned}
\gamma_0 &= \Gamma_0 W + \Gamma_1 W^{'} + \Gamma_2 W^{''} + \Gamma_3 W^{'''} + \Gamma_4 W^{(4)} \\
\gamma_1 &= \Gamma_0 T + \Gamma_1 \left(T^{'} - W\right) + \Gamma_2 T^{''} + \Gamma_3 \left(T^{'''} - W^{''}\right) + \Gamma_4 T^{(4)} \\
\gamma_2 &= \Gamma_0 Q + \Gamma_1 \left(Q^{'} + T\right) + \Gamma_2 \left(Q^{''} + W\right) + \Gamma_3 \left(Q^{'''} + W^{'} + T^{''}\right) + \Gamma_4 \left(Q^{(4)} + 2W^{''}\right) \\
\gamma_3 &= \Gamma_0 J + \Gamma_1 \left(J^{'} - Q\right) + \Gamma_2 \left(J^{''} + T\right) + \Gamma_3 \left(J^{'''} - Q^{''} + T^{'} - W\right) + \Gamma_4 \left(J^{(4)} + 2T^{''}\right) \\
\gamma_4 &= \Gamma_1 J + \Gamma_2 Q + \Gamma_3 \left(J^{''} + Q^{'} + T\right) + \Gamma_4 \left(2Q^{''} + W\right) \\
\gamma_5 &= \Gamma_0 + \Gamma_2 J + \Gamma_3 \left(J^{'} - Q\right) + \Gamma_4 \left(T + 2J^{''}\right) \\
\gamma_6 &= \Gamma_1 + \Gamma_3 J + \Gamma_4 Q \\
\gamma_7 &= \Gamma_2 + \Gamma_4 J \\
\gamma_8 &= \Gamma_3 \\
\gamma_9 &= \Gamma_4
\end{aligned}
\tag{IV.51}
$$

with $\Delta(\gamma_i) = \frac{5-i}{2}$. On the other hand the explicit expressions of $L.(P.L)_+$ is determined by similar calculations. We find:

$$
(P.L)_+ = \sum_{i=0}^{4} \lambda_i D^i \tag{IV.52}
$$

where λ_i are superfields of spin $\Delta(\lambda_i) = \frac{-i}{2}$ given by:

$$
\lambda_0 = -\left(\frac{\delta f}{\delta W}\right)^{(4)} + \frac{\delta f}{\delta W}\left(J^{''} - Q^{'} - T\right) - \left(\frac{\delta f}{\delta W}\right)^{'} Q + \left(\frac{\delta f}{\delta W}\right)^{''} J \tag{IV.53}
$$

$$
+\frac{\delta f}{\delta T}.Q - \frac{\delta f}{\delta Q}.J + 2\left(\frac{\delta f}{\delta Q}\right)^{''} + X_5 \tag{IV.54}
$$

$$
\lambda_1 = \left(\frac{\delta f}{\delta W}\right)^{'''} - \left(\frac{\delta f}{\delta T}\right)^{''} - \left(\frac{\delta f}{\delta W}.J\right)^{'} - \left(\frac{\delta f}{\delta Q}\right)^{'} + \frac{\delta f}{\delta W}.Q + \frac{\delta f}{\delta T}.J + \frac{\delta f}{\delta J} \tag{IV.55}
$$

$$
\lambda_2 = \left(\frac{\delta f}{\delta W}\right)^{''} - \frac{\delta f}{\delta W}.J - \frac{\delta f}{\delta Q} \tag{IV.56}
$$

$$
\lambda_3 = -\left(\frac{\delta f}{\delta W}\right)^{'} + \frac{\delta f}{\delta T} \tag{IV.57}
$$

$$
\lambda_4 = -\frac{\delta f}{\delta W} \tag{IV.58}
$$

One can then easily check that $L.(P.L)_+$ is of the form

$$
L.(P.L)_+ = \sum_{k=0}^{9} \left(\sum_{j=0}^{5} \Lambda_{j,k-j}\right) D^k \tag{IV.59}
$$

$$
= \sum_{k=0}^{9} \beta_k D^k \tag{IV.60}
$$

where

$$
\Lambda_{j,k} = \begin{cases} 0 & \text{if } j > 5 \text{ or } k < 0 \text{ or } k > 4 \\ \neq 0 & \text{otherwise} \end{cases} \tag{IV.61}
$$

The non vanishing values of the superfields $\Lambda_{j,k}$ are

$$\Lambda_{0,i} = \lambda_i^{(5)} + J\lambda_i^{'''} + Q\lambda_i^{''} + T\lambda_i^{'} + W\lambda_i \tag{IV.62}$$
$$\Lambda_{1,i} = (-)^i \left[\lambda_i^{(4)} + J\lambda_i^{''} + T\lambda_i\right] \tag{IV.63}$$
$$\Lambda_{2,i} = 2\lambda_i^{'''} + J\lambda_i^{'} + Q\lambda_i \tag{IV.64}$$
$$\Lambda_{3,i} = (-)^i \left[2\lambda_i^{''} + J\lambda_i\right] \tag{IV.65}$$
$$\Lambda_{4,i} = \lambda_i^{'} \tag{IV.66}$$
$$\Lambda_{5,i} = (-)^i \lambda_i \tag{IV.67}$$

Therefore we have

$$\beta_0 = \Lambda_{0,0} \tag{IV.68}$$
$$\beta_1 = \Lambda_{0,1} + \Lambda_{1,0} \tag{IV.69}$$
$$\beta_2 = \Lambda_{0,2} + \Lambda_{1,1} + \Lambda_{2,0} \tag{IV.70}$$
$$\beta_3 = \Lambda_{0,3} + \Lambda_{1,2} + \Lambda_{2,1} + \Lambda_{3,0} \tag{IV.71}$$
$$\beta_4 = \Lambda_{0,4} + \Lambda_{1,3} + \Lambda_{2,2} + \Lambda_{3,1} + \Lambda_{4,0} \tag{IV.72}$$
$$\beta_5 = \Lambda_{1,4} + \Lambda_{2,3} + \Lambda_{3,2} + \Lambda_{4,1} + \Lambda_{5,0} \tag{IV.73}$$
$$\beta_6 = \Lambda_{2,4} + \Lambda_{3,3} + \Lambda_{4,2} + \Lambda_{5,1} \tag{IV.74}$$
$$\beta_7 = \Lambda_{3,4} + \Lambda_{4,3} + \Lambda_{5,2} \tag{IV.75}$$
$$\beta_8 = \Lambda_{4,4} + \Lambda_{5,3} \tag{IV.76}$$
$$\beta_9 = \Lambda_{5,4} \tag{IV.77}$$

The differential operator $V_P(L)$ then reads

$$V_P(L) = \sum_{i=0}^{n} (\beta_k - \gamma_k)\, D^k \tag{IV.78}$$

This is easily seen, because

$$\beta_9 = \gamma_9 \tag{IV.79}$$
$$\beta_8 = \gamma_8 \tag{IV.80}$$
$$\beta_7 = \gamma_7 \tag{IV.81}$$
$$\beta_6 = \gamma_6 \tag{IV.82}$$
$$\beta_5 = \gamma_5 \tag{IV.83}$$

To define an $A\,(2 \mid 2)^{(1)}$ affine structure on $V_P(L)$, one must require the vanishing of the super trace, which is equivalent to set

$$Sres\,\{L, P\} = 0 \tag{IV.84}$$

or simply

$$\beta_4 - \gamma_4 = 0 \tag{IV.85}$$

We find

$$X_5^{'}(\hat{z}) = -X_1^{(5)} - X_2^{(4)} + 2\,(X_3)''' + (X_1 J)''' + 2\,(X_4)'' + (X_2 J)'' - (X_1 Q)'' - (X_1 T)' - (X_3 J)' \tag{IV.86}$$

or equivalently:

$$X_5(\hat{z}) = \left(\tfrac{\delta f}{\delta W}\right)^{(4)} - \left(\tfrac{\delta f}{\delta T}\right)^{(3)} - 2\left(\tfrac{\delta f}{\delta Q}\right)'' - \left(\tfrac{\delta f}{\delta W} J\right)'' + 2\left(\tfrac{\delta f}{\delta J}\right)' + \left(\tfrac{\delta f}{\delta T} J\right)' + \left(\tfrac{\delta f}{\delta W} Q\right)' + \left(\tfrac{\delta f}{\delta W} T\right) + \left(\tfrac{\delta f}{\delta Q} J\right) \tag{IV.87}$$

Putting the constraint equation Eq.(4.86) into the expressions of the non vanishing values of $(\beta_k - \gamma_k)$ one can show that $V_P(L)$ is a differential operator of degrees $(0,3)$:

$$V_P(L) = A_3 D^3 + A_2 D^2 + A_1 D + A_0 \tag{IV.88}$$

where $A_i = \beta_i - \gamma_i$ are superfields of dimension $\Delta(A_i) = \frac{5-i}{2}$. We give here below the explicit form of the terms A_i needed in the derivation of the $N = 2$ supersymmetric W_3-algebra.

$$\begin{aligned} A_3 &= -2X_1^{(6)} - 3X_2^{(5)} + 3X_3^{(4)} + 2\left(X_1 J\right)^{(4)} + 6X_4^{(3)} + 3\left(X_2 J\right)^{(3)} \\ &\quad -2\left(X_1 Q\right)^{(3)} + \left(X_2 Q\right)'' - \left(X_3 J\right)'' - 2\left(X_1 T\right)'' - \left(X_4 J\right)' - \left(X_2 T\right)' \\ &\quad +JX_4' + QX_3' + TX_2' + WX_1' + 2QX_4 + 2WX_2 \end{aligned} \tag{IV.89}$$

$$\begin{aligned} A_2 &= -X_2^{(6)} + 3X_4^{(4)} + \left(X_2 J\right)^{(4)} + \left(X_2 Q\right)^{(3)} + JX_4'' + TX_2'' - WX_1'' \\ &\quad - \left(X_3 Q\right)'' - \left(X_4 Q\right)' - \left(X_2 W\right)' + \left(X_2 T\right)'' - 2\left(X_1 W\right)'' - QX_3'' \end{aligned} \tag{IV.90}$$

$$\begin{aligned} A_1 &= -X_1^{(8)} - 2X_2^{(7)} + X_3^{(6)} - JX_1^{(6)} + \left(X_1 J\right)^{(6)} + 3X_4^{(5)} - 2JX_2^{(5)} \\ &\quad - \left(X_1 Q\right)^{(5)} - QX_1^{(5)} + 2\left(X_2 J\right)^{(5)} + \left(X_2 Q\right)^{(4)} + JX_3^{(4)} - QX_2^{(4)} \\ &\quad +J\left(X_1 J\right)^{(4)} - \left(X_1 T\right)^{(4)} + 2J\left(X_2 J\right)^{(3)} + 3JX_4^{(3)} + Q\left(X_1 J\right)^{(3)} \\ &\quad +QX_3^{(3)} - TX_2^{(3)} - J\left(X_1 Q\right)^{(3)} - WX_1^{(3)} - \left(X_2 T\right)^{(3)} + J\left(X_2 Q\right)'' \\ &\quad -Q\left(X_1 Q\right)'' + Q\left(X_2 J\right)'' + QX_4'' - WX_2'' - J\left(X_1 T\right)'' - \left(X_3 T\right)'' - X_3'' T \\ &\quad - \left(X_2 W\right)'' + T\left(X_2 J\right)' + TX_4' + W\left(X_1 J\right)' - J\left(X_2 T\right)' + WX_3' \\ &\quad - \left(X_4 T\right)' - Q\left(X_1 T\right)' + 2T\left(X_2 Q\right) - 2WX_1 Q + 2WX_2 R + 2WX_4 \end{aligned} \tag{IV.91}$$

$$\begin{aligned} A_0 &= -X_2^{(8)} + 2X_4^{(6)} + \left(X_2 R\right)^{(6)} - RX_2^{(6)} + \left(X_2 Q\right)^{(5)} - QX_2^{(5)} + 2RX_4^{(4)} \\ &\quad +R\left(X_2 R\right)^{(4)} - TX_2^{(4)} - \left(X_1 W\right)^{(4)} + X_1^{(4)} W + R\left(X_2 Q\right)^{(3)} + 2QX_4^{(3)} \\ &\quad - \left(X_2 W\right)^{(3)} + Q\left(X_2 R\right)^{(3)} + Q\left(X_2 Q\right)'' + 2TX_4'' - \left(X_3 W\right)'' - 2X_3'' W \\ &\quad +T\left(X_2 R\right)'' - R\left(X_1 W\right)'' - \left(X_1 R\right)'' W - \left(X_4 W\right)' - R\left(X_2 W\right)' \\ &\quad -Q\left(X_1 W\right)' + \left(X_1 Q\right)' W + T\left(X_2 Q\right)' + 2WX_2 Q \end{aligned} \tag{IV.92}$$

An important step towards deriving the supersymmetric Gelfand Dickey Poisson brackets associated to the affine Lie algebra $A\left(2 \mid 2\right)^{(1)}$ generated by the superfields $(J, Q, T, W) \equiv \left(U_1, U_{\frac{3}{2}}, U_2, U_{\frac{5}{2}}\right)$ is towards the superconformal transformation of the super Lax operator $L^{(0,5)}$ namely:

$$\begin{aligned} \tilde{L} &= \tilde{D}^5 + \tilde{J}\tilde{D}^3 + \tilde{Q}\tilde{D}^2 + \tilde{T}\tilde{D} + \tilde{W} \\ &= \left(D\tilde{\theta}\right)^{-3} \left[D^5 + JD^3 + QD^2 + TD + W\right] \left(D\tilde{\theta}\right)^{-2} \end{aligned} \tag{IV.93}$$

Identifying both sides of this equation, one obtain:

$$\begin{aligned}
J &= \left(D\tilde{\theta}\right)^2 \tilde{J} \\
Q &= \left(D\tilde{\theta}\right)^3 \tilde{Q} + \tilde{J}\left(D\tilde{\theta}\right)\left(D^2\tilde{\theta}\right) + 3S\left(z,\tilde{z}\right) \\
T &= \left(D\tilde{\theta}\right)^4 \tilde{T} - \left(D\tilde{\theta}\right)^2 \left(D^2\tilde{\theta}\right)\tilde{Q} + \left(D\tilde{\theta}\right)\left(D^3\tilde{\theta}\right)\tilde{J} \\
&\quad + \left(DS\left(z,\tilde{z}\right)\right) \\
W &= \tilde{W}\left(D\tilde{\theta}\right)^5 + 2\tilde{T}\left(D\tilde{\theta}\right)^3\left(D^2\tilde{\theta}\right) + \left(D\tilde{\theta}\right)^2\left(D^3\tilde{\theta}\right)\tilde{Q} \\
&\quad + 2\tilde{J}\left[\left(D\tilde{\theta}\right)\left(D^4\tilde{\theta}\right) - D^2\tilde{\theta}D^3\tilde{\theta}\right] + 2\left(D^2S\left(z,\tilde{z}\right)\right)
\end{aligned} \tag{IV.94}$$

The $N = 2$ super-W_3 algebra is generated by the superfields $(J, Q, T, W) \equiv \left(U_1, U_{\frac{3}{2}}, U_2, U_{\frac{5}{2}}\right)$ and gives rise to the supermultiplet $\left(1, \left(\frac{3}{2}\right)^2, 2, \left(\frac{5}{2}\right)^2, 3\right)$

The computation of the super GD bracket generating the $N = 2$ version of the w_3-algebra contains long and complicated expressions that we are not putting in the manuscript. However, we note that we have present a consistent algebraic analysis and several important properties as well as the crucial steps necessary to derive the super GD bracket of the $N = 2$ super w_3-algebra. The principal key in this context is given by the hamiltonian operator $V_P(L)$ that we have compute completely.

V. CONCLUDING REMARKS

We use in this work a consistent and systematic analysis developed in previous occasions[7] to study an important problem namely the supersymmetric version of N=2 Gelfand Dickey algebra. The conformal algebra and its supersymmetric extensions have played a central role in the study of string dynamics, statistical models of critical phenomena, and more generally in two dimensional conformal field theories (CFT)[2,3]. These are symmetries generated by conformal spin s currents with $s \leq 2$. The extension à la Zamolodchikov incorporates also currents of higher conformal spin 3 and 5/2 and gives then the super w-algebra involving besides the usual spin-2 energy momentum tensor, a conformal spin-3 conserved current[8,9]. The w-symmetry, which initially was identified as the symmetry of the critical three states Potts model, has also been realized as the gauge symmetry of the so-called w_3 gravity. In relation to integrable systems, these symmetries are shown to play a pioneering role as their existence gives, in some sense, a guarantee of integrability.
All these physical ideas are behind our initiative to renew the interest in the supersymmetric version of the Gelfand-Dickey algebra although the underlying computations are very tedious and complicated. We focuss in nearest occasion to go beyond these extensions and apply our analysis to integrable systems in non trivial symmetries and geometries.

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