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Generalized Symmetries and Some new Solution of the Fokker-Planck Equation

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abstract

In this paper, the third and fourth generalized symmetries of the Fokker-Planck equation are directly computed. We show how to get higher order generalized symmetries by using of the recursion operators, and we derive some exact solutions.

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I. INTRODUCTION

Symmetries play an important role in the mathematical analysis of differential equations, first with the work of Lie, symmetry groupe method and their recent generalizations have prouved useful in understanding conservation lows[1−4], in constructing exact solutions[5−7] and in etablishing integrability of certain systems of differential equations.

The aim of this paper is to determine third and fourth generalized symmetries of the Fokker-Planck equation[8,9]

$$u_t = u_{xx} + xu_x + u, \tag{1.1}$$

and using of recursion operators, higher order generalized symmetries are derived with some polynomial solutions. The Fokker-Planck equation also known as the Kolmogorov Forward equation describes the time evolution of the probability density function of position and velocity of a particle and it can be generalized to any other observable, too.

This paper is organized as follows. In section two, Invariance criterion of an evolutionary equation is recalled. The third section is devoted to determine the third generalized symmetries admitted by equation (1) and how these symmetries can be obtained from Lie point symmetries. Finally in section four, we compute the fourth generalized symmetries, we use recursion operators to construct higher order generalized symmetries and we give some exact solutions.

II. GENERALIZED SYMMETRIES OF THE EQUATION (1.1)

Lie point symmetries group transformations are characterized to act on the space of independent and dependent variables. Here we extend the group transformations to act not only on space of independent and dependent variables, but also on derivatives of dependent variables. The associated vector fields of such transformations is known as generalized symmetry (Lie-Bäcklund symmetries). To compute generalized symmetries First we begin by definition :

Definition A generalized vector field

$$V = \xi[u]\frac{\partial}{\partial x} + \eta[u]\frac{\partial}{\partial t} + \varphi[u]\frac{\partial}{\partial u}, \tag{2.1}$$

where the square brackets will serve to remind us that ξ, η and φ depend on x, t, u and derivatives of u, is a generalized symmetry of an evolutionary equation

$$u_t = P[u], \tag{2.2}$$

if and only if

$$PrV(u_t - P[u])|_{u_t - P[u]=0} = 0. \tag{2.3}$$

where P depends in x, t, u and derivatives of u with respect x, and $Pr(V)$ denotes the extension of V.

For mathematical convenience, to search for generalized symmetries, we adopt to do this with V not in the form (2.1) but in it's canonical form (also called evolutionary form) given by

$$V_Q = Q[u]\frac{\partial}{\partial u}, \tag{2.4}$$

where

$$Q = \varphi - \xi u_x - \eta u_t. \tag{2.5}$$

The above choice is based on the following theorem.

Theorem[2] A generalized vector field V is a symmetry of a system of differential equations if and only if it's canonical form is V_Q.

III. THIRD GENERALIZED SYMMETRIES

Determination for three order generalized symmetries admitted by the Fokker-Planck equation, needs that any time derivatives in the characteristic Q is ignored, because it can be replaced by exploiting the right-hand side of equation (1.1) itself. For example, u_t is replaced by $u_{xx} + xu_x + u$, u_{tx} with $u_{xxx} + xu_{xx} + 2u_x$ and so on.

Now if we put

$$Q = Q(x, t, u, u_x, u_{xx}, u_{xxx}), \tag{3.1}$$

the condition of invariance (2.3) becomes :

$$D_tQ = D_{xx}Q + xD_xQ + Q, \tag{3.2}$$

which must hold for all solutions of the equation (1). The operator D_a, denotes the total derivation given by

$$D_a = \frac{\partial}{\partial a} + u_a\frac{\partial}{\partial u} + u_{ax}\frac{\partial}{\partial u_x} + ... \tag{3.3}$$

After substituting for time derivatives their expressions in the equation (3.2), equating coefficient of various powers of derivatives of u starting with higher order, and analyzing the equation (3.2), the coefficients of terms involving the fourth derivative u_{xxxx}, lead to

$$Q = M(t)u_{xxx} + N, \tag{3.4}$$

with M is a function depending only in the time, and N is a function depending on x, t, u, u_x and u_{xx}.

Combining equations (3.4) and (3.2), the coefficient of terms $u_{xxx}^2, u_{xxx}u_{xx}$ and $u_{xxx}u_x$ give

$$Q = M(t)u_{xxx} + N_1 u_{xx} + H, \tag{3.5}$$

with N_1 and H are functions depending on (x, t) and (x, t, u, u_x) respectively. The remaining term involving the third derivative u_{xxx} yields

$$M^{'}(t) + 3M(t) - 2\frac{\partial N_1}{\partial x} = 0, \tag{3.6}$$

substituting equation (3.5) in equation (3.2), and equating coefficient of u_{xx}^2 and $u_{xx}u_x$ to zero, then (3.5) becomes

$$Q = M(t)u_{xxx} + N_1(t, x)u_{xx} + N_2 u_x + H_1, \tag{3.7}$$

where N_2 and H_1 are functions depending on (t, x) and (t, x, u) respectively. We introduce the above equation in (3.2), the coefficient of u_{xx} yields

$$\frac{\partial^2 N_1}{\partial x^2} + 2\frac{\partial N_2}{\partial x} + x\frac{\partial N_1}{\partial x} - 2N_1 - \frac{\partial N_1}{\partial t} = 0, \tag{3.8}$$

equation (3.7) in (3.2), and equating the coefficient of u_x^2 to zero, we conclude that H_1 is linear in u, thus expression (3.7) of Q becomes

$$Q = M(t)u_{xxx} + N_1(t, x)u_{xx} + N_2(t, x,)u_x + N_3 u + N_4(t, x), \tag{3.9}$$

for some functions N_4 and N_3 depending on the space x and time t. Substituting (3.9) in (3.2) this immediately allows us to write

$$\frac{\partial^2 N_2}{\partial x^2} + x\frac{\partial N_2}{\partial x} - N_2 - \frac{\partial N_2}{\partial t} + 2\frac{\partial N_3}{\partial x} = 0, \tag{3.10}$$

$$\frac{\partial^2 N_3}{\partial x^2} + x\frac{\partial N_3}{\partial x} - \frac{\partial N_3}{\partial t} = 0, \tag{3.11}$$

$$\frac{\partial^2 N_4}{\partial x^2} + x\frac{\partial N_4}{\partial x} - \frac{\partial N_4}{\partial t} + N_4 = 0. \tag{3.12}$$

On solving equations (3.6), (3.8), (3.10), (3.11) and (3.12), we obtain

$$M(t) = c_5 e^t + c_8 e^{-t} + c_1 e^{3t} + c_{10} e^{-3t}, \tag{3.13}$$

$$N_1(t, x) = (2c_5 e^t + 3c_1 e^{3t} + c_8 e^{-t})x + c_2 e^{2t} + c_9 e^{-2t} + c_6, \tag{3.14}$$

$$\begin{aligned} N_2(t, x) &= (3c_1 e^{3t} + c_5 e^t)x^2 + (2c_2 e^{2t} + c_6)x \\ &\quad + 3c_1 e^{3t} + (c_5 + c_3)e^t + c_7 e^{-t}, \end{aligned} \tag{3.15}$$

$$N_3(t, x) = c_1 e^{3t} x^3 + c_2 e^{2t} x^2 + (3c_1 e^{3t} + c_3 e^t)x + c_2 e^{2t} + c_4, \tag{3.16}$$

where $c_1, ..., c_{10}$ are arbitrary constants.

Consequently, all third order generalized symmetry of Fokker-Planck equation (1.1) has it's characteristic as a linear constant coefficient combination of characteristics

$$Q_1 = e^{3t}\{u_{xxx} + 3xu_{xx} + 3(x^2 + 1)u_x + x(x^2 + 3)u\}, \tag{3.17}$$

$$Q_2 = e^{2t}\{u_{xx} + 2xu_x + (x^2 + 1)u\}, \tag{3.18}$$

$$Q_3 = e^t(u_x + xu), \tag{3.19}$$

$$Q_4 = u, \tag{3.20}$$

$$Q_5 = e^t\{u_{xxx} + 2xu_{xx} + (x^2 + 1)u_x\}, \tag{3.21}$$

$$Q_6 = u_{xx} + xu_x, \tag{3.22}$$

$$Q_7 = e^{-t}u_x, \tag{3.23}$$

$$Q_8 = e^{-t}\{u_{xxx} + xu_{xx}\}, \tag{3.24}$$

$$Q_9 = e^{-2t}u_{xx}, \tag{3.25}$$

$$Q_{10} = e^{-3t}u_{xxx}, \tag{3.26}$$

and an infinite family of characteristics

$$Q_\beta = \beta, \tag{3.27}$$

with β is an arbitrary solution of the Fokker-Planck equation (1.1).

Note that all linear combination of $Q_2, Q_3, Q_4, Q_6, Q_7, Q_9$, and Q_β can be obtained directly as constant linear combination of characteristics arising from canonical form of generators of the Lie point symmetries obtained in[10]:

$$\begin{array}{ll} V_1 = -e^t \frac{\partial}{\partial x} + e^t xu \frac{\partial}{\partial u} & V_2 = e^{2t} x \frac{\partial}{\partial x} + e^{2t} \frac{\partial}{\partial t} - e^{2t} x^2 u \frac{\partial}{\partial u}, \\ V_3 = -e^{-2t} x \frac{\partial}{\partial x} + e^{-2t} \frac{\partial}{\partial t} + e^{-2t} u \frac{\partial}{\partial u} & V_4 = \frac{\partial}{\partial t}, \\ V_5 = u \frac{\partial}{\partial u} & V_6 = e^{-t} \frac{\partial}{\partial x}, \end{array}$$

and an infinite generators with the form:

$$V_\beta = \beta(x,t) \frac{\partial}{\partial u}, \tag{3.28}$$

where $\beta(x,t)$ is an arbitrary solution of the equation (1.1) .

For example, if we consider

$$V_2 = e^{2t} x \frac{\partial}{\partial x} + e^{2t} \frac{\partial}{\partial t} - e^{2t} x^2 u \frac{\partial}{\partial u},$$

then it's canonical form is

$$(-e^{2t} x^2 u - e^{2t} x u_x - e^{2t} u_t) \frac{\partial}{\partial u},$$

on substituting u_t by $u_{xx} + xu_x + u$, we obtain that the characteristic of V_2 is $-Q_2$. If we do the same of all V_i, we remark that Lie point symmetries are included in the third generalized symmetries via their canonical form.

IV. FOURTH GENERALIZED SYMMETRIES

In this case, to obtain fourth generalized symmetries, we include the fourth derivative u_{xxxx} in the characteristics Q, that is to consider

$$V = Q(x, t, u, u_x, u_{xx}, u_{xxx}, u_{xxxx}) \frac{\partial}{\partial u}. \tag{4.1}$$

As in the previous case, the invariance condition is

$$D_t Q = D_{xx} Q + x D_x Q + Q,$$

and the only distinction isn't more than in this case it will be appear more of terms than in the last case.

For example:

$$D_t Q = \frac{\partial Q}{\partial t} + u_t \frac{\partial Q}{\partial u} + u_{tx} \frac{\partial Q}{\partial u_x} + ... + u_{txxxx} \frac{\partial Q}{\partial u_{xxxx}},$$

we remark in this case that it appear u_{txxxx}, but in the previous case only we have u_{txxx}.

From invariance condition, coefficients of all terms containing the five derivatives u_{xxxxx}, allow us to write

$$Q = g(t)u_{xxxx} + G_1(x, t, u, u_x, u_{xx}, u_{xxx}), \tag{4.2}$$

for some functions g and G_1. Combining (4.2) and (3.2), we see that coefficients of terms involving $u^2_{xxxx}, u_{xxxx}u_{xxx}, u_{xxxx}u_{xx}$, and $u_{xxxx}u_x$ lead to

$$G_1 = f(t, x)u_{xxx} + G_2, \tag{4.3}$$

where G_2 and f are functions depending on (x, t, u, u_x, u_{xx}) and (t, x) respectively. Then (4.2) becomes

$$Q = g(t)u_{xxxx} + f(t, x)u_{xxx} + G_2. \tag{4.4}$$

Examining the coefficient of u_{xxxx}, we conclude that

$$g^{'}(t) + 4g(t) = 2\frac{\partial f}{\partial x}. \tag{4.5}$$

Now, using the obtained equation (4.4) in the invariance condition (3.2), and from coefficient of terms $u^2_{xxx}, u_{xxx}u_{xx}$, and $u_{xxx}u_x$ we get

$$G_2 = h(x, t)u_{xx} + G_3, \tag{4.6}$$

for some functions G_3 and h depending on (x, t, u, u_x) and (x, t) respectively. Hence, Q becomes

$$Q = g(t)u_{xxxx} + f(t, x)u_{xxx} + h(x, t)u_{xx} + G_3. \tag{4.7}$$

The coefficient of the remaining third derivative u_{xxx} implies that

$$\frac{\partial f}{\partial t} + 3f = \frac{\partial^2 f}{\partial x^2} + 2\frac{\partial h}{\partial x} + x\frac{\partial f}{\partial x}. \tag{4.8}$$

Substituting the above expression in the equation (3.2), we conclude from the coefficients of terms u^2_{xx} and $u_{xx}u_x$ that

$$G_3 = \theta(x, t)u_x + G_4, \tag{4.9}$$

where θ and G_4 are functions depending on (x, t) and (x, t, u) respectively. Consequently, Q becomes

$$Q = g(t)u_{xxxx} + f(t, x)u_{xxx} + h(x, t)u_{xx} + \theta(x, t)u_x + G_4. \tag{4.10}$$

Analyzing the coefficient of the last term involving the second derivative, we get

$$\frac{\partial h}{\partial t} + 2h = \frac{\partial^2 h}{\partial x^2} + x\frac{\partial h}{\partial x} + 2\frac{\partial \theta}{\partial x}, \tag{4.11}$$

equation (4.10) in (3.2), it remains

$$\begin{aligned}\frac{\partial \theta}{\partial t}u_x + \frac{\partial G_4}{\partial t} + u\frac{\partial G_4}{\partial u} + 2\theta u_x = {} & \frac{\partial^2 \theta}{\partial x^2}u_x + \frac{\partial^2 G_4}{\partial x^2} + 2\frac{\partial^2 G_4}{\partial x \partial u}u_x + u_x^2\frac{\partial^2 G_4}{\partial u^2} \\ & + x\frac{\partial \theta}{\partial x}u_x + x\frac{\partial G_4}{\partial x} + \theta u_x + G_4,\end{aligned} \tag{4.12}$$

we see that the coefficient of u_x^2, leads to

$$G_4 = \alpha(t, x)u + \beta(t, x), \tag{4.13}$$

for some functions α and β. If we observe the coefficient of u_x, we find that

$$\frac{\partial \theta}{\partial t} + \theta = \frac{\partial^2 \theta}{\partial x^2} + x\frac{\partial \theta}{\partial x} + 2\frac{\partial \alpha}{\partial x}, \tag{4.14}$$

from terms involving zero derivatives, we derive

$$\frac{\partial\alpha}{\partial t} = \frac{\partial^2\alpha}{\partial x^2} + x\frac{\partial\alpha}{\partial x}, \tag{4.15}$$

$$\frac{\partial\beta}{\partial t} = \frac{\partial^2\beta}{\partial x^2} + x\frac{\partial\beta}{\partial x} + \beta. \tag{4.16}$$

We see that from equations (4.5) and (4.8), we must have

$$f(t,x) = f_1(t)x + f_2(t), \tag{4.17}$$

$$h(t,x) = h_1(t)x^2 + h_2(t)x + h_3(t), \tag{4.18}$$

where f_1, f_2, h_1, h_2 and h_3 are functions depending on time t. Then equations (4.17), (4.18) and (4.8) allow to

$$f_1^{'}(t) + 2f_1(t) = 4h_1(t), \tag{4.19}$$

$$f_2^{'}(t) + 3f_2(t) = 2h_2(t), \tag{4.20}$$

equations (4.11) and (4.18), lead to

$$\theta = \theta_1 x^3 + \theta_2 x^2 + \theta_3 x + \theta_4, \tag{4.21}$$

the above equation and (4.14) yield to

$$\alpha = \alpha_1 x^4 + \alpha_2 x^3 + \alpha_3 x^2 + \alpha_4 x + \alpha_5, \tag{4.22}$$

where $\theta_1, ..., \theta_4$ and $\alpha_1, ..., \alpha_5$ are functions depending on time t only. Now substituting (4.22) in (4.15), we get

$$\alpha_1 = k_1 e^{4t}, \tag{4.23}$$

$$\alpha_2 = k_2 e^{3t}, \tag{4.24}$$

$$\alpha_3 = k_3 e^{2t} + 6k_1 e^{4t}, \tag{4.25}$$

$$\alpha_4 = k_4 e^{t} + 3k_2 e^{3t}, \tag{4.26}$$

$$\alpha_5 = k_3 e^{2t} + 3k_1 e^{4t} + k_5, \tag{4.27}$$

where $k_1, ..., k_5$ are arbitrary constants. Equations (4.21) and (4.14) allow us to

$$\theta_1 = 4k_1 e^{4t} + k_6 e^{2t}, \tag{4.28}$$

$$\theta_2 = 3k_2 e^{3t} + k_7 e^{t}, \tag{4.29}$$

$$\theta_3 = 12k_1 e^{4t} + (3k_6 + 2k_3)e^{2t} + k_8, \tag{4.30}$$

$$\theta_4 = k_9 e^{-t} + 3k_2 e^{3t} + (k_7 + k_4)e^{t}, \tag{4.31}$$

with $k_6, ..., k_9$ are arbitrary constants. Using (4.18), the above expressions and equation (4.11) we obtain

$$h_1 = 6k_1 e^{4t} + 3k_6 e^{2t} + k_{10}, \tag{4.32}$$

$$h_2 = k_{11} e^{-t} + 3k_2 e^{3t} + 2k_7 e^{t}, \tag{4.33}$$

$$h_3 = k_{12} e^{-2t} + 6k_1 e^{4t} + (3k_6 + k_3)e^{2t} + k_8 + k_{10}, \tag{4.34}$$

with k_{10}, k_{11} and k_{12} are arbitrary constants. Hence, substituting the above equations in (4.19) and (4.20), we get

$$f_1 = 4k_1 e^{4t} + 3k_6 e^{2t} + k_{13} e^{-2t} + 2k_{10}, \tag{4.35}$$

$$f_2 = k_{14} e^{-3t} + k_7 e^{t} + k_2 e^{3t} + k_{11} e^{-t}, \tag{4.36}$$

with k_{12}, k_{13} and k_{14} are arbitrary constants. From equation (4.5) and taking in mind the above results, we obtain that

$$g(t) = k_1 e^{4t} + k_6 e^{2t} + k_{13} e^{-2t} + k_{15} e^{-4t} + k_{10}, \tag{4.37}$$

where k_{15} is an arbitrary constant. Consequently, all fourth generalized symmetry has as it's characteristic, a linear constant coefficient combination of characteristics, $Q_1, ..., Q_{10}, Q_\beta$ and the following five characteristics :

$$\begin{aligned} Q_{11} &= e^{4t}\{u_{xxxx} + 4xu_{xxx} + 6(x^2+1)u_{xx} + 4x(x^2+3)u_x \\ &\quad +(x^4+6x^2+3)u\}, && (4.38)\\ Q_{12} &= e^{2t}\{u_{xxxx} + 3xu_{xxx} + 3(x^2+1)u_{xx} + x(x^2+3)u_x\}, && (4.39)\\ Q_{13} &= u_{xxxx} + 2xu_{xxx} + (x^2+1)u_{xx}, && (4.40)\\ Q_{14} &= e^{-2t}\{u_{xxxx} + xu_{xxx}\}, && (4.41)\\ Q_{15} &= e^{-4t}u_{xxxx}. && (4.42) \end{aligned}$$

Note that, to look for generalized symmetries, the order is predetermined, so to compute fifth generalized symmetries needs to include u_{xxxxx} in Q and it will be computationally intensive.

Searching higher-order generalized symmetries, can be done directly from low-order symmetries without having to solve all equations encountered in previous subsections, that need an extensive algebraic manipulations. However, one can use what are called recursion operators. We define a recursion operator as an operator which maps any symmetry of a given equation into a symmetry of the same equation. As equation (1.1) is a linear partial differential equation, higher-order generalized symmetries can be generated from Lie point symmetries in their canonical form by using recursion operators:

$$\begin{aligned} \mathcal{R}_1 &= e^{-t}D_x, && (4.43)\\ \mathcal{R}_2 &= D_x^2 + xD_x, && (4.44)\\ \mathcal{R}_3 &= e^tD_x^2 + e^tx, && (4.45)\\ \mathcal{R}_4 &= e^{2t}D_x^2 + 2e^{2t}xD_x + e^{2t}(x^2+1). && (4.46) \end{aligned}$$

Now let us show how characteristics of third and fourth generalized symmetries can be obtained by using of above operators :

$$\begin{aligned} Q_1 &= \mathcal{R}_4\mathcal{R}_3[u], & Q_{15} &= \mathcal{R}_1^4[u], && (4.47)\\ Q_5 &= \mathcal{R}_4\mathcal{R}_1[u], & Q_{14} &= \mathcal{R}_2\mathcal{R}_1^2[u], && (4.48)\\ Q_8 &= \mathcal{R}_2\mathcal{R}_1[u], & Q_{13} &= \mathcal{R}_4\mathcal{R}_1^2[u], && (4.49)\\ Q_{10} &= \mathcal{R}_1\mathcal{R}_1^2[u], & Q_{12} &= \mathcal{R}_4\mathcal{R}_2[u], && (4.50)\\ Q_{11} &= \mathcal{R}_4\mathcal{R}_3^2[u]. & & && (4.51) \end{aligned}$$

As $\mathcal{R}_i^j, i = 1, ..., 4$ and $j = 0, 1, 2, ...$ are characteristics of generalized symmetries admitted by the equation (1.1), thus we can determine invariant solutions corresponding to these generators.

Example.1 $i = 1, j = 3$

In this case the symmetry generator is

$$e^{-t}u_{xxx}\frac{\partial}{\partial u},$$

then it's similarity solution will have the form

$$u(x,t) = p_1(t)x^2 + p_2(t)x + p_3(t), \tag{4.52}$$

substituting the above equation in (1) we get the exact solution

$$u(x,t) = ae^{3t}x^2 + be^{2t}x + ae^{3t} + ce^{2t}, \tag{4.53}$$

where a, b and c are arbitrary constants.

Example.2 $i = 1, j = 4$

The generator corresponding $j = 4$ is

$$e^{-t} u_{xxxx} \frac{\partial}{\partial u},$$

so the similarity solution of the above generator will be a third order polynomial function in the variable x

$$u(x,t) = s_1(t)x^3 + s_2(t)x^2 + s_3(t)x + s_4, \tag{4.54}$$

on substituting this in equation (1.1) we obtain the solution

$$u(x,t) = de^{4t}x^3 + ee^{3t}x^2 + (3de^{4t} + me^{2t})x + ee^{3t} + ke^t, \tag{4.55}$$

where d, e, m and k are arbitrary constants. Consequently, we can repeat the procedure to derive higher-order polynomial solutions of the equation (1.1). An additional solution of the Fokker-Planck equation can be obtain from the generalized symmetry

$$V_{Q_5} = e^t\{u_{xxx} + 2xu_{xx} + (x^2 + 1)u_x\} \frac{\partial}{\partial u},$$

which is

$$u(x,t) = e^{-\frac{x^2}{2}}(l_1 + l_2 e^{-t} erf(\frac{x}{\sqrt{2}})), \tag{4.56}$$

where l_1, l_2 are arbitrary constants and erf is the error function.

V. CONCLUSION

The higher order generalized symmetries play an important role in the construction of some exact solutions, to find them needs in general an extensive calculus. Using a recursion operator was of great interest to generate higher order generalized symmetries from canonical form of Lie point symmetries admitted by the Fokker-Planck equation. Employing higher order generalized symmetries generating by recursion operator, so the higher order polynomial solutions can be obtained.

REFERENCES

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[1] Bergurs J.M.; *The Nonlinear Diffusion Equation: Asymptotic Solutions and Statistical Problems*(Dordrecht:Reidel) 1974.

[2] Olver P.J.; *Application of Lie groups to Differential Equation.* Springer, New York 1986.

[3] Bluman G.W and Kumei S.; *Symmetries and Differentials Equations*, Springer, New York 1989.

[4] Bluman G.W and Cole.J.D.; *Similarity Methods for Differential Equations, in Applied Mathematical Sciences, No.13 Equations*, Springer, New York 1974.

[5] Ibragimov N.H.; (ed), *CRC Handbook of Lie Group Anaysis Of differential Equations, Vol.I : Symmetries, Exact Solutions, and Conservations Laws*, CRC Press, Boca Raton, 1994.

[6] Ibragimov N.H.; *Elementry Lie Group Analysis and Ordinary Differential Equations*, Wiley, New York, 1999.

[7] Hydon P.E.; *Symmetry Methods for Differential Equations: A Beginner's Guide*, Cambridge University Press, Cambridge, 2000.

[8] A.Fokker, *Ann.Phys.*, Leipzig **43**, 312(1914).

[9] M.Plank, Sitzungsber.Preuss. Acad.Wiss.Phys. Math. Kl. 324 (1917)

[10] A. Ouhadan , E. H. El Kinani, M. Rahmoune and A. Awane *"Symétries ponctuelles et potentielles de l'équation de Fokker-Planck "*African Journal of mathematical physics Vol 5(2007)33-41.