



## MPZ : Minimum Propagation Zone for a Tridimensional Superconductor

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abstract

The implementation of a group of thermic stability criterion of a superconductor presents a certain interest. Within this framework, many work are elaborated [2], [6], studying the unidimensional case. It aims , in fact , at controlling a critic energy which is associated to the solution of the intermediate equilibrium, called instable normal state. The intended energy presents a sort of threshold relative to thermic disturbances. In this paper , we will present:

- A generalization of this critic energy which present a good approximation of the quantity of energy able to transit the superconductor to a normal state
- Some control estimations of this energy.

**Key Words** : Nonlinear problem, Critic enregy, Thermic stability, normal state, Superconductor.

### I. POSITION OF THE PROBLEM

#### A. Study framework

Thermic balance state of a great number of superconductors, is managed by the following mathematic problem :

$$-\sum_{i,j=1}^N \partial_{x_i}(k(u)\partial_{x_j}) = \lambda F(u) \quad \text{on } \Omega \quad (1)$$

$$u \equiv 0 \quad \text{on } \Gamma = \partial\Omega \quad (2)$$

In this model:

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- k: indicates thermal conductivity. It is a continuous function, strictly positive, bounded, and there is a constant  $k_0 > 0$  such as:

$$k(u) \leq k_0 \text{ for all } u \geq 0 \tag{3}$$

- F: The term of energetic competition between a power dissipated by Joule effect G, and that absorbed by the cryogenic bath Q:

$$F(u) = G(u) - Q(u)$$

The diversity of the techniques carrying out the superconductive state and maintaining its thermal stability, generates a whole of acceptable classes  $U_{ad}^G$  for the term F. These classes are characterized primarily, but not only, by the existence of an element  $u_1 > 0$  such as:

$$F(u_1) = 0$$

Let us note, moreover, that all elements of  $U_{ad}^G$  are  $C^2$  and null for the negative values ( $u \leq 0$ ). These assumptions represent the noted hypothesis  $H_0$ . In a quasi-general way, there are three techniques or systems of cooling ( Helium I, Helium II and the temperature control by the field edges). Each system is modeled by the hypothesis  $H_0$  and an only hypothesis  $H_m$  with  $m \in \{1, 2, 3\}$  such as:

–  $H_1$  :

there exists  $u_2 > u_1$  such as  $F(u_2) = 0$  and  $F'(u) \leq 0$  for all  $u \geq u_2$

–  $H_2$  :

$$\lim_{u \rightarrow +\infty} F(u) = F_\infty$$

–  $H_3$  :

$$\exists \gamma_1 > 0 \text{ such as } \lim_{u \rightarrow \infty} \left( \frac{F(u)}{u} \right) \leq \gamma_1 \text{ and } \lim_{u \rightarrow \infty} F(u) = +\infty$$

Then, we define classes of this term as follows:

$$U_{ad}^m = \{F \text{ checking } H_0 \text{ and } H_m\} \text{ with } 1 \leq m \leq 3$$

Thus, the general class  $U_{ad}^G$  is defined by:

$$U_{ad}^G = \bigsqcup_{m=1}^3 U_{ad}^m$$

The whole of these assumptions id deduced from nature of experimental curves, given by the various cooling systems [14],[15].

**Remark. 1.**

By reasons of necessity, we need specific heat. Therefore, we recall that model (Pe) represents the stationary case of the dynamic model:

$$\begin{cases} c(u)\partial_t u - \sum_{i,j=1}^N \partial_{x_i}(k(u)\partial_{x_j}) = \lambda F(u) & \text{on } \Omega \\ u \equiv 0 & \text{on } \Gamma = \partial\Omega \end{cases}$$

where c is the specific heat considered as a strictly positive function in  $C^1$  checking: There is a constant  $B > 0$  such as:

$$|c'| \leq B$$

**B. Consequences of the canonical transformations**

**Definition.1** We define Kirchoff's transformation Y associated with the function k by:

$$Y(u) = \int_0^u k(u)du \tag{4}$$

**Definition.2** We define enthalpic's transformation Z associated with the function c by:

$$Z(\omega) = \int_0^\omega c(s)ds \tag{5}$$

**Theorem. 1** [9] Let  $F \in U_{ad}^G$ . Then we have:

$$F \in U_{ad}^m \implies \tilde{F} = F \circ Y^{-1} \in U_{ad}^m \text{ for } 1 \leq m \leq 3$$

Where  $Y^{-1}$  is the opposite transformation of Y.

**Proposition .1** Problem (Pe) is equivalent to the problem ( $\tilde{P}e$ ) according to :

$$-\Delta y = \lambda \tilde{F}(y) \text{ for all } x \text{ in } \Omega \tag{6}$$

$$y = 0 \text{ on } \Gamma \tag{7}$$

**Proof**

As k is continuous and strictly positive function, then Y is continuously derivable, bi-univocal, and applies the variation compact  $\bar{\Omega}$  (closed bounded of  $IR^n$ ) of u on  $\bar{\Omega}$  of variation of y. Since  $u \equiv 0$ , on  $\Gamma$  is equivalent with  $y \equiv 0$  on  $\Gamma$ , then formulate (4) makes it possible to deduce:

$$k(u)\partial_{x_i}u = \partial_{x_i}y$$

and problem (Pe) can be replaced by problem ( $\tilde{P}e$ ) according to:

$$-\Delta y = \lambda \tilde{F}(y) \text{ for all } x \text{ in } \Omega \tag{8}$$

**Remark. 2.**

1. The problem ( $\tilde{P}e$ ) is a semi linear problem simpler to study than (Pe).
2. Theorem (1) shows classes invariance by the Kirchoff's transformation.
3. Any solution y of ( $\tilde{P}e$ ) gives in a bi-univocal way a solution u of problem (Pe) such as:

$$u = Y^{-1}(y)$$

**Proposition .2** For all  $F \in U_{ad}^m$ , function  $\tilde{F}$  checks the following assertion:

There exists  $\tilde{F}_0 > 0$  and  $q_m \geq 0$  such as:

$$| \tilde{F}(y) | \leq \tilde{F}_0 | y |^{q_m} \tag{9}$$

**Idea of proof**

As F is  $C^2$  in  $IR^+$ , and for  $u \geq u_2$  behaves like a polynomial at least of dismantles 2 for the class  $U_{ad}^2$  and in a way under linear or finished for the other classes, this makes it possible to conclude that there exists  $q_m$  and  $F_0$  such as:

$$| F(y) | \leq F_0 | y |^{q_m} \tag{10}$$

While exploiting, on the one hand, the inequality (3), and on the other hand the inequality (9) and the definition (1) we deduce that:

$$| \tilde{F}(y) | \leq \tilde{F}_0 | y |^{q_m} \tag{11}$$

with  $\tilde{F}_0 = F_0 k_0^{-q_m}$

**C. Recall of existence conditions related to normal states**

By the modeling given in [1], [5], [9] and [14], the superconductive state represents the single balance state ( $F(0) = 0$ ), then any resistive state not null is regarded as a parasitic one. Thus, it will be interesting to determine conditions giving the unicity. Proofs of results of this paragraph are given in [9].

**Definition.3**

1. We call fundamental or superconductive state, the trivial solution  $y = 0$ .
2. We call normal or resistive state, all state  $y$  not null positive solution, in the energy space  $H_0^1(\Omega)$  of the problem:

$$(\tilde{P}e) \quad \Delta y = \lambda \tilde{F}(y) \quad \text{in } \Omega$$

The following theorem gives result for the general case ( unspecified dimension).

**Theorem.2** For any  $\tilde{F}$  element of  $U_{ad}$  checking:

$$\tilde{F}(y) \leq \frac{\lambda_1}{\lambda} \quad \text{for all } y \geq 0 \tag{12}$$

Problem  $(\tilde{P}e)$  admits the fundamental state as a unique solution.

**Corollary.1** Let us pose :

$$E(\omega) = \int_0^\omega \tilde{F}(\omega) d\omega$$

Problem  $(\tilde{P}e)$  admits a unique solution if we have one of the following situations:

- a)  $\tilde{F} \in U_{ad}^1$  and  $E(y_2) \leq 0$  where  $y_2 = Y(u_2)$ .
- b)  $\tilde{F} \in U_{ad}^2$  and  $\lambda \leq \lambda_c^2$  where  $\tilde{F}_m = \sup_{y \geq 0} \tilde{F}(y)$  and  $\lambda_c^2 = \frac{\lambda_1}{\sup(1, \tilde{F}_m)}$
- c)  $\tilde{F} \in U_{ad}^3$  and  $\lambda \leq \lambda_c^3$   
where

$$\lambda_c^3 = \lambda_1 \sup\left(\frac{1}{\gamma_1}, \frac{1}{\gamma_3}\right), \quad \gamma_3 = \sup_{y_3 \geq y \geq y_1} \tilde{F}(y)$$

and  $y_3$  is such as :  $\tilde{F}(y) \leq \gamma_1 y$  for all  $y \geq y_3$

Proofs of theorem and corollary exist in [9].

Fundamental state is the only one guaranteeing [5], [9], [14], stability of superconductor. So it 's interesting to find the conditions supporting the existence of other normal states.

Let  $\psi(r)$  be the function defined on the field  $D = ]0, \sqrt[N]{2}[$  by:

$$\psi(r) = \left(\frac{1+r}{r}\right)^2 \left(\frac{(1+r)^N - 1}{2 - (1+r)^N}\right) \tag{13}$$

With  $\psi(r_0) = \min_{r \in D} \psi(r)$  and  $r_{max}$  the ray of the largest ball contained in  $\Omega$ .

**Theorem.3** [9] Let  $\tilde{F}$  be an element of  $U_{ad}^G$ . If the following conditions are checked:

a) There exists  $\omega_0 > 0$  such as:

$$\int_0^{\omega_0} \tilde{F}(s)ds \geq \left(\frac{\omega_0}{r_{max}\sqrt{2}}\right)\psi(r_0) \tag{14}$$

b) There exists  $\omega_1 > 0$  such as:

$$\tilde{F}(y) \leq \gamma_0 y \quad \text{for all } y \geq \omega_1 \tag{15}$$

Then the problem  $(\tilde{P}e)$  admits two completely ordered normal states

$$(y(x) \leq \tilde{y}(x), \text{ for } x \text{ in } \Omega)$$

## II. GENERALIZATION OF MPZ CONCEPT

### A. Recall of MPZ notion for the unidimensional case

In this part, we will give stability results of the superconductive state, which generalize the approaches described by [2], [5], [11], [13], [14] and [16]. This generalization is characterized by its compatibility with all types of cooling and its dependence in temperature, only, of the thermophysical characteristics (c and k), when the fundamental state is not preserved overall in the field.

To our knowledge, the three-dimensionnel case is not studied yet. For this reason, we propose ( by extension of the unidimensional case) a general defintion of critic energy, and we give some results related to this energy.

#### Seol and Chu defintion of critic energy for unidimensional case:

For the unidimensionnel case, critic energy is given by the following [9]:

$$\Theta_{dim=1} = A \int_0^L \int_0^{T(X)} C(\omega)d\omega dX \tag{16}$$

Where A, L and  $C(T)$  indicate, respectively, the surface of the cross-section of the superconductor considered as filforme, its length and its specific heat. Function  $T(X)$  represents thermic balance corresponding, by a modeling, to the first normal solution  $y$ .

The interest carried to this energy comes from various works existing in literature [13], [14] and [16]. These works are centered on the comprehension of the evolution phenomenon of superconductors. They show that energy  $\Theta_{dim=1}$  presents a good approximation of the suceptible quantity of energy to make forward the superconductor.

### B. Quantification of critic energy in unspecified dimension

According to the quoted work, we generalize the concept of critic energy for a three-dimensional superconductors. By extension of (16), we define:

$$\Theta_N = \int_{\Omega_s} \int_0^{T(X)} C(s)ds dX \tag{17}$$

Where  $\Omega_s$  is the physical field. C and T indicate, respectively, specific heat and thermal field associated to the first stationary solution  $y$  of the balance problem  $(Pe)$ , whose existence is ensured by theorem 3.

**Proposition. 3** Let  $J$  be the functional defined by:

$$J(y(x)) = ZoY^{-1}(y(x)) \tag{18}$$

Critic energy is written in the form:

$$\Theta_N = \sigma_N \int_{\Omega} J(y(x)) dx \quad (19)$$

Where  $\sigma_N$  is a constant which depends only on the physical parameters.

**Proof.** Dimensional study given in [8] and [9], gives rise to a number of parameters whose natures can describe the geometry of the field  $\Omega$ , or the operation data of the cryogenic system. Let  $T_c - T_b$ , be the difference between critical temperature and that of cryogenic bath [9], and  $(L_i)_{1 \leq i \leq 3}$  maximum lengths of the field  $\Omega$  following the directions  $(e_i)_{i=1,2,3}$ .

If we pose:

$$u = \frac{T - T_b}{T_c - T_b} \text{ adimensional field, and } x = BX$$

With B the matrix defined by:

$$B = \begin{pmatrix} \frac{1}{L_1} & 0 & 0 \\ 0 & \frac{1}{L_2} & 0 \\ 0 & 0 & \frac{1}{L_3} \end{pmatrix}$$

and

$$\tilde{c} = c[(T_c - T_b)u + T_b], \quad \bar{c} = \int_{\Omega} \tilde{c}(s) ds \quad \text{and} \quad c(u) = \frac{\tilde{c}(u)}{\bar{c}(u)} \quad (20)$$

While writing  $\Theta_N$  according to dimensional sizes, it comes that:

$$\Theta_N = (T_c - T_b) \bar{c} \prod_{i=1}^{i=3} L_i \int_{\Omega} \int_0^{u(x)} c(s) ds dx$$

Then we find :

$$\Theta_N = \sigma_N \int_{\Omega} J(y(x)) dx$$

With :

$$\sigma_N = (T_c - T_b) \bar{c} \prod_{i=1}^{i=3} L_i$$

**Remark.3**

1. Let us note that energy expression thus defined, is independent of the energy competition, since it utilizes only the thermophysical characteristics of the field c and k, and its geometrical magnitudes.
2. A simple dimensional analysis of the constant  $\sigma_N$  shows that it has the dimension of an energy.

### III. ESTIMATES RELATED TO THE NORMAL STATES

The whole of the estimates [1],[3],[4],[11] and [15] starting from the normal states, has a double interest. Indeed, they make it possible to obtain the treshold control of the critic energy determining the minimal propagation zone (MPZ), and help to maintain thermic stability of the domain thanks to this control.

**A. Useful inequalities**

The interest of these inequalities is that they allow to determine a critic energy threshold.

Let  $p^*$  be the Sobolev exponents such as:

$$p^* = \frac{2N}{N-2} \quad (N \geq 3)$$

There exists a constant  $C(N) > 0$  [3] such as:

$$\left( \int_{\Omega} |y|^{p^*} dx \right)^{\frac{2}{p^*}} \leq C(N) \|\nabla y\|_{L^2(\Omega)}^2 \tag{21}$$

If we consider the norm  $\|y\|_{0,p}$  defined on  $W^{0,p}$  then:

$$\|y\|_{0,p} \leq C(N, p, \Omega) \|\nabla y\|_{L^2(\Omega)}^2 \tag{22}$$

**B. Minimal value of the norm related to the solution modeling a normal state**

**Theorem. 4**

Let  $y$  be a normal solution of the problem  $(\tilde{P}e)$ , and  $q_m$  the constant data in proposition 1 . For all  $q_m \in ]0, \frac{N+2}{N-2}]$ , there exists a constant  $c_m^1 > 0$  such as:

$$\|y\|_{L^{q_m+1}(\Omega)} \geq c_m^1$$

**Proof**

As  $y$  is an element of  $H_0^1(\Omega)$ , then from (22) we have

$$\|y\|_{0,q_m}^2 \leq C(N, q_m, \Omega) \|\nabla y\|_{L^2(\Omega)}^2 \tag{23}$$

Let us multiply by  $y$  equation (6). An integration by parts gives:

$$\|\nabla y\|_{L^2(\Omega)}^2 = \lambda \int_{\Omega} y(x) \tilde{F}(y(x)) dx \tag{24}$$

Inequalities (9) and (21) give:

$$\frac{1}{C(N, q_m, \Omega)} \|y\|_{0,q_m+1}^2 \leq \|\nabla y\|_{L^2(\Omega)}^2 \leq \lambda \tilde{F}_0 \|y\|_{q_m+1}^{q_m-1} \tag{25}$$

The following choice:

$$c_m = \frac{1}{C(N, q_m + 1, \Omega)} \quad \text{and} \quad c_m^1 = \sqrt[q_m-1]{c_m}$$

allows to obtain

$$c_m^1 \leq \|y\|_{L^{q_m+1}(\Omega)}$$

This inequality is true for  $q_m + 1 \in ]1, p^*]$ , then  $q_m \in ]1, \frac{N+2}{N-2}]$ .

**Fundamental remark. 4**

1. We do not know if the constant  $c_m^1$  is optimal.
2. Since the physical field  $\Omega_s$  is three-dimensional, then  $N = 3$  and  $q_m \in ]0, 5[$ , which largely covers the three classes of F.

**Corollary. 2** If  $q_m = 1$  then a necessary condition so that the problem  $(\tilde{P}e)$  admits a non null solution, is that:

$$\lambda \geq \lambda_c^1 = \lambda_1 k_0 F_0^{-1} \tag{26}$$

where  $F_0$  is the constant of the growth condition [see proposition 1]

**Proof**

If  $q_m = 1$  then  $\|y\|_{q_m+1}^{q_m-1} = 1$  with:

$$\frac{1}{\lambda \tilde{F}_0 C(N, 2, \Omega)} \leq 1 \tag{27}$$

However

$$C(N, 2, \Omega) = \frac{1}{\lambda_1}$$

where  $\lambda_1$  is the first eigenvalue given by:

$$\lambda_1 = \inf \left\{ \int_{\Omega} |\nabla y|^2 dx, y \in H_0^1(\Omega) \text{ and } \|y\|_{L^2(\Omega)}^2 = 1 \right\} \tag{28}$$

Finally, we obtain the result of the corollary.

### C. Maximum value of the solution modeling a normal state

**Theorem. 5** For all  $F \in U_{ad}^G$  the following assertions are true:

1. If  $q_m \in ]1, \frac{N+2}{N-2}]$  then for any normal solution  $y$ , there exists a constant  $c_m^* > 0$  such as:

$$\sup_{x \in \Omega} |y(x)| \geq c_m^* \tag{29}$$

2. If  $q_m \in ]0, 1]$ , then for any normal solution  $y$ , there exists a constant  $\tilde{c}_m^* > 0$  such as:

$$\sup_{x \in \Omega} |y(x)| \leq \tilde{c}_m^* \tag{30}$$

**Proof**

1. For the first point of the theorem, if  $q_m \in ]1, \frac{N+2}{N-2}]$  we apply the inequality (25) which gives :

$$\|y\|_{0, q_m+1} \geq (c_m)^{\frac{1}{q_m-1}} \text{ avec } c_m = \frac{1}{\lambda \tilde{F}_0 C(N, q_m + 1, \Omega)} \tag{31}$$

As  $\Omega$  is bounded, thus:

$$\|y\|_{0, q_m+1} \leq |\Omega|^{\frac{1}{q_m+1}} \sup_{x \in \Omega} |y(x)| \tag{32}$$

From inequalities (31) and (32) we have:

$$\sup_{x \in \Omega} |y(x)| \geq c_m^{\frac{1}{q_m-1}} \left( \frac{1}{|\Omega|} \right)^{\frac{1}{q_m+1}} = c_m^* \tag{33}$$

2. For the second point, the inequalities (9) and (23) give:

$$|\tilde{F}(y)| \leq \tilde{F}_0 |y|^{q_m} \text{ for all } \tilde{F} \in U_{ad}^m$$

Since  $q_m \in ]0, 1]$ , thus:

$$\lim_{|y| \rightarrow \infty} |\tilde{F}(y)| = 0 \text{ or } 0 \leq \tilde{F}(y) \leq 1$$

From this, there is a constant  $M$ , (who depends on  $q_m$ ), such as:

$$|\tilde{F}(y)| \leq M_{q_m} \tag{34}$$

with:

$$\|y\|_{0, q_m+1} \leq (\lambda \tilde{F}_0 C(N, q_m + 1, \Omega))^{\frac{1}{1-q_m}} \tag{35}$$

Condition (9) and inequality (34) gives that:

$$\|\tilde{F}(y)\|_{0, \frac{1}{q_m}+1} \leq \tilde{F}_0 \|y\|_{0, q_m+1}^{q_m} \tag{36}$$

An integration of the equation (6), and (35), imply that:

$$\|\nabla y\|_{L^2(\Omega)}^2 = \lambda \int_{\Omega} y(x) \tilde{F}(y(x)) dx \leq M_{q_m} \lambda^{\frac{1}{1-q_m}} \tag{37}$$

Setting

$$p_m^0 = 1 + \frac{1}{q_m} \tag{38}$$

$$p_m^{j+1} = \frac{N p_m^j}{N - 2 p_m^j} \geq \left(\frac{N}{N - 2}\right) p_m^j \tag{39}$$

and iterating the arguments in equalities (36), (37) and (38) yields (39) by the usual "boot strap" process [4] since  $W^{2,p} \subset C^0(\Omega)$  with continues injection, for  $p$  sufficiently large.  $p_{jm}$  is positive and strictly increasing. We obtain the result.

**Remark. 5** If we pose:

$$\sup_{x \in \Omega} |y(x)| = \beta$$

By holding account of proposition (3) and according to corollary 2 and 5, for  $q_m = 1$  we have :

$$\Theta_N \leq |\Omega| \bar{c}(T_c - T_b) \prod_{i=1}^3 L_i Z(\beta)$$

#### IV. CONTROL OF CRITIC ENERGY FROM THE CANONICAL TRANSFORMATIONS

##### A. Growth condition of the functional related to critic energy

We can write, from proposition 2, the critic energy as:

$$\Theta_N = \sigma_N \int_{\Omega} \int_0^{y(x)} j(\omega) d\omega \tag{40}$$

Where  $j$  is derivative of  $J$ .

**Proposition 4** Functional  $j$ , null in 0, check the growth condition :

$$|j(\omega)| \leq a(x) + b |\omega|^{\frac{p}{q}} \tag{41}$$

Where  $a(x) \geq 0$ .

**Proof**

Since:

$$j(\omega) = \left(\frac{\partial J}{\partial \omega}\right)(\omega) = \left(\frac{\partial Z \circ Y^{-1}}{\partial \omega}\right)(\omega)$$

this makes it possible to write:

$$j(\omega) = c(Y^{-1}(\omega)) \frac{\partial Y^{-1}}{\partial \omega}(\omega)$$

As  $Y^{-1}(0) = 0$ , then we have:

$$j(0) = c(0) = 0$$

For the growth condition, owing to the fact that thermic conductivity  $k$  is strictly positive, increasing and bounded, then the Kirchoff's transformation, defined from  $IR^+$  in  $IR^+$  increasing and is bounded. Since  $c$  is positive, bounded and increasing, there is  $B$  such as (remark 1):

$$c(u) \leq Bu$$

We deduce that:

$$j(\omega) = \frac{B}{2} \frac{d}{d\omega} \left[ ((Y^{-1})^2)(\omega) \right]$$

By exploiting the fact that  $Y^{-1}$  is increasing bounded, there is  $p$  and  $q$  such as:

$$\frac{d}{d\omega} \left[ ((Y^{-1})^2)(\omega) \right] \leq |\omega|^{\frac{p}{q}}$$

It's enough to choose :

$$a(x) \geq 0, \quad \text{and} \quad b = \frac{B}{2}$$

**B. Control of critic energy from the growth condition**

**Theorem. 6** For each normal state  $y$ , there exists two positive numbers  $\tilde{a}$  and  $\tilde{b}$  such as:

$$\Theta_N(y) \leq \tilde{a} \|y\|_{L^1} + \tilde{b} \|y\|_{L^{\frac{p}{q}+1}}^{\frac{p}{q}+1} \tag{42}$$

**Proof**

Growth condition for  $j$  gives:

$$j(\omega) \leq a(x) + \frac{B}{2} |\omega|^{\frac{p}{q}} \quad \text{with} \quad a(x) \geq 0$$

However:

$$\Theta_N(y) = \sigma_N \int_{\Omega} J(y(x)) dx = \sigma_N \int_{\Omega} \left( \int_0^{y(x)} j(\omega) d\omega \right) dx$$

Which makes it possible to write:

$$\Theta_N(y) \leq \sigma_N \int_{\Omega} \left( \int_0^{y(x)} a(x) + \frac{B}{2} |\omega|^{\frac{p}{q}} d\omega \right) dx$$

$$\begin{aligned} &\leq \sigma_N \sup_{x \in \Omega} a(x) \int_{\Omega} y(x) dx + \sigma_N \frac{B}{2} \int_{\Omega} \frac{y(x)^{\frac{p}{q}}}{\frac{p}{q} + 1} dx \\ &\leq \sigma_N \| a \|_{L^\infty(\Omega)} \| y \|_{L^1(\Omega)} + \sigma_N \frac{qB}{2(p+q)} \| y \|_{L^{\frac{p}{q}+1}}^{\frac{p}{q}+1} \end{aligned}$$

It's enough to take:

$$\tilde{a} = \sigma_N \| a \|_{L^\infty(\Omega)}$$

$$\tilde{b} = \sigma_N \frac{qB}{2(p+q)}$$

**Corollary. 3** For  $a(x) = 0$ , we have the following increases:

1. If  $p = q$ :

$$\Theta_N \leq \sigma_N \frac{B}{4} \| y \|_{L^2}^2$$

2. If  $p < q$ :

$$\Theta_N \leq \sigma_N \frac{qB}{2(p+q)} \| y \|_{L^\infty(\Omega)} | \Omega |$$

3. If  $p > q$ :

$$\Theta_N \leq \sigma_N \frac{B}{4} \| y \|_{L^{\frac{p}{q}+1}}^{\frac{p}{q}+1}$$

**Proof**

1. For  $p = q$ , it is enough to apply the preceding theorem.

2. If  $p < q$ , then according to theorem (5) we have:

$$\sup_{x \in \Omega} | y(x) | = \| y \|_{L^\infty(\Omega)} \leq \tilde{c}_m^*$$

which gives the second point.

3.  $p + q \geq 2$ .

**V. CONCLUSION**

In this paper, we gave a three-dimensional formulation, generalizing the expression of the critic energy introduced for the unidimensional case. Thanks to this expression, we have given a whole of control results relating to the thermic stability of the superconductor. Justifications related to these results, are based on the analytical properties corresponding to the source term. In a forthcoming work, we will present estimations by behaviors of the specific heat c.

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