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Quantum Entanglement – A Primer

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abstract

The phenomenon of "quantum entanglement" and the related "EPR paradox" has been mystifying physicists ever since Einstein, Podoloky & Rosen published their gedanken experiment in 1935. However, the existence of quantum entanglement is, now, theoretically and experimentally well established. In fact, quantum entanglement is being looked upon as the cardinal resource in quantum communication, quantum computing and other information processing activities to the extent that it could revolutionize the world of information technology. This is despite the fact that an unquestionable physical explanation of entanglement continues to elude all human endeavours. In this review, we present a bird's eye view of all facets of this intriguing phenomenon. Starting from the pre-requisites (Section 2), we introduce "quantum entanglement" in Section 3. We, then, look at the scheme of measurement of the degree of entanglement in terms of "Wootters Concurrence" & "Entanglement of Formation". Using quaternions, we study the geometry of the single and two qubit states of quantum computing. Through the Hopf fibrations, we identify geometric manifestations of the separability and entanglement of two qubit quantum systems. Thereafter, we present the salient features of the EPR experiment and the Bohm's version thereof in terms of particle spins. Possible schemes of resolution of the EPR paradox in terms of Bohr's Complementarity Principle, time retroaction and Bohm's "quantum potential" are also discussed. Anomalies resulting from the von Neumann scheme of quantum measurement are also touched upon. A comprehensive list of references supports this review.

I. INTRODUCTION

It is, now, well acknowledged that quantum entanglement i.e. the existence of nonclassical correlations among quantum states constitutes an invaluable resource for performing various tasks related to quantum information theory e.g. quantum teleportation, quantum dense coding, quantum cryptography etc. In this article we present a primer of this peculiar phenomenon that is intrinsic to quantum mechanics.

II. SOME PREREQUISITES

A. Isolated Systems

Axiomatically, the dynamical evolution of a physical system is represented in a Hilbert space of appropriate dimensions with the so called "quantum state" that describes the system being represented by a vector in the said Hilbert space and the dynamical evolution of the system manifesting itself as the action of a linear operator on the state vector thereby updating the state vector to correspond to the instantaneous state of the system.
In quantum mechanics, the concept of "pure state" and "mixed state" is non trivial and warrants a comprehensive exposition before the main theme of this paper can be addressed. This is because, quantum mechanics exhibits stochastic behavior at various levels which makes it imperative for us to be clear about the various ramifications of such probabilistic behavior.
Let us consider a statistical ensemble $\Pi = \{\pi_1, \pi_2, \pi_3, ..., \pi_n\}$ of a large number of identically prepared quantum systems. Then, the axioms of quantum mechanics postulate that
(a) such an ensemble is completely characterized by a normalized state vector $|\psi\rangle$ in the Hilbert space H of the system;
(b) the measurable properties are represented in the Hilbert spaceHas linear operators $\hat{\Omega}$ that are Hermitian and act on the state vector, thereby evolving it in time;
(c) the Hermitian operator $\hat{\Omega}$ admits a spectral decomposition as

$$\hat{\Omega} = \int_{-\infty}^{\infty} \omega d\lambda_\omega \tag{2.1}$$

where $\omega \in \mathbb{R}$ and λ_ω, the spectral family of $\hat{\Omega}$, is a one parameter family of commuting orthogonal projection operators with the following properties:-
(i) the family of projection operators is monotonically increasing i.e.

$$\lambda_{\omega'} \geq \lambda_\omega \ for \ \omega' > \omega \tag{2.2}$$

(ii) the family is continuous from the right i.e.

$$Lim_{\varepsilon \to +0} \lambda_{\omega+\varepsilon} = \lambda_\omega \tag{2.3}$$

(iii)

$$Lim_{\omega \to -\infty} \lambda_\omega = 0, \ Lim_{\omega \to \infty} \lambda_\omega = 1 \tag{2.4}$$

On performing measurements relating to an observable $\hat{\Omega}$ on the ensemble Π(described by the normalized state vector $|\psi\rangle \in$ H) we obtain a real valued random variable Ω with a cumulative probability distribution function

$$F_\Omega(\omega) = \langle\psi|\lambda_\omega|\psi\rangle \tag{2.5}$$

Using eqs. (2.2) –(2.4), we can easily establish the following:
(i)

$$F_\Omega(\omega_1) \leq F_\Omega(\omega_2) \ for \ \omega_1 < \omega_2 \tag{2.6}$$

(ii)

$$Lim_{\varepsilon \to +0} F_\Omega(\omega + \varepsilon) = F_\Omega(\omega) \tag{2.7}$$

(iii)

$$Lim_{\omega \to -\infty} F_\Omega(\omega) = 0, \ Lim_{\omega \to \infty} F_\Omega(\omega) = 1 \tag{2.8}$$

thereby vindicating that $F_\Omega(\omega)$is, indeed, a cumulative probability distribution function. It also follows from eq. (2.5) that on performing a measurement of the observable $\hat{\Omega}$, the possible outcomes are the values $\omega \in Spec\left(\hat{\Omega}\right)$ for if $\omega \notin Spec\left(\hat{\Omega}\right)$ then λ_ω is constant in a neighborhood of ω, so that $F_\Omega(\omega)$ is constant in this region. This implies that the probability that the random variable Ω falls in this region is zero. Put succinctly, a measurement of the observable represented by the Hermitian operator $\hat{\Omega}$ necessarily returns a real valued random variable Ω that can take values only in $Spec\left(\hat{\Omega}\right)$. Furthermore,

$$E(\Omega) = \int_{-\infty}^{+\infty} \omega dF_\Omega(\omega) = \int_{-\infty}^{+\infty} \omega d\left\langle \psi \left| \lambda_\omega \right| \psi \right\rangle = \langle \psi | \hat{\Omega} | \psi \rangle \tag{2.9}$$

and

$$Var(\Omega) = E\left(\Omega^2\right) - \left[E(\Omega)\right]^2 = \left\langle \psi \left| \hat{\Omega}^2 \right| \psi \right\rangle - \left\langle \psi \left| \hat{\Omega} \right| \psi \right\rangle^2 \tag{2.10}$$

Quantum states characterized as above are called "pure states".
Let us, now, consider a number, say m of ensembles $\Pi_1, \Pi_2, \Pi_3, ..., \Pi_m$, each of which is prepared as above i.e. ensemble $\Pi_i = \{\pi_{i1}, \pi_{i2}, \pi_{i3}, ..., \pi_{in_i}\}$ consists of identically prepared systems $\pi_{i1}, \pi_{i2}, \pi_{i3}, ..., \pi_{in_i}$ so that each of these $\Pi_i = \{\pi_{i1}, \pi_{i2}, \pi_{i3}, ..., \pi_{in_i}\}$ constitutes a "pure state" and hence, is represented by a normalized vector $|\psi_i\rangle \in$ H. Then the ensemble Ξobtained by mixing all the above states with respective weights w_i ($w_i \geq 0$, $\sum_i w_i = 1$) e.g. by mixing l_i states of the ensemble Π_i(so that $w_i = \frac{l_i}{\sum_i l_i}$), produce the so called "mixed quantum state". When a measurement is performed relating to the observable represented by the operator $\hat{\Omega}$ on the state Ξ, the result is a real valued random variable Ω with the cumulative probability distribution

$$F_\Omega(\omega) = \sum_i w_i \langle \psi_i | \lambda_\omega | \psi_i \rangle \tag{2.11}$$

with the mean

$$E(\Omega) = \sum_i w_i \langle \psi_i | \hat{\Omega} | \psi_i \rangle \tag{2.12}$$

In terms of the "density" operator

$$\rho = \sum_i w_i |\psi_i\rangle\langle\psi_i| \tag{2.13}$$

we have

$$F_\Omega(\omega) = Tr\left(\lambda_\omega \rho\right), E(\Omega) = Tr\left(\hat{\Omega}\rho\right), Var(\Omega) = Tr\left(\hat{\Omega}^2\rho\right) - \left[Tr\left(\hat{\Omega}\rho\right)\right]^2 \tag{2.14}$$

so that all the statistical properties of the ensemble Ξ can be obtained from the relevant density matrix. In fact, the density matrix provides a unified description of "pure" as well as "mixed" state dynamics. For pure states, the density matrix takes the form

$$\rho = |\psi\rangle\langle\psi| \tag{2.15}$$

i.e. it is a projection operator onto the state $|\psi\rangle$. In view of the fact that the "density matrix" formulation constitutes the cardinal setup for the study of "entanglement", we state below its relevant properties to facilitate continuity and completeness. We have

(a)

$$\rho^{\dagger} = \rho,\ \rho \geq 0 \ and \ Tr\left(\rho\right) = 1 \tag{2.16}$$

(b) The “density matrix” also admits a spectral decomposition in its eigenbasis. Since $\rho$is hermitian it can be diagonalized in an appropriate basis. Further, since it is positive semidefinite its eigenvalues are also real and non-negative of which a countable number are strictly positive, say p_i that are, at most, finitely degenerate. The point 0 is the only possible limit point of the spectrum. All these properties together imply that ρ admits a spectral decomposition in its eigenbasis as

$$\rho = \sum_i p_i |\varphi_i\rangle\langle\varphi_i| \tag{2.17}$$

with the normalization

$$1 = Tr\left(\rho\right) = \sum_j \langle\varphi_j|\rho|\varphi_j\rangle = \sum_j \langle\varphi_j| \left(\sum_i p_i|\varphi_i\rangle\langle\varphi_i|\right)|\varphi_j\rangle = \sum_i p_i \tag{2.18}$$

(c) We have $E\left(\rho\right) = Tr\left(\rho^2\right) = \sum_j \langle\psi_j|\rho|\psi_j\rangle = \sum_j \langle\psi_j|\left(\sum_i p_i|\varphi_i\rangle\langle\varphi_i|\right)|\psi_j\rangle$

$$= \sum_j \left(\sum_k c^*_{jk}\langle\varphi_k|\right)\left(\sum_i p_i|\varphi_i\rangle\langle\varphi_i|\right)\left(\sum_l c_{jl}|\varphi_l\rangle\right) = \sum_{i,j} c^*_{ji}p_i c_{ji} = \sum_{i,j} \left|c_{ji}\right|^2 p_i \leq 1 \tag{2.19}$$

since $\sum_i p_i = 1$, $\sum_{i,j} \left|c_{ji}\right|^2 = 1$.

(d) The equality would hold in the special case of a “pure state” for which$\rho = |\psi\rangle\langle\psi|$. This is easily seen. Let $\rho = \sum_i p_i|\varphi_i\rangle\langle\varphi_i|$ be its spectral decomposition. Suppose, then, that $\rho^2 = \sum_i p_i^2|\varphi_i\rangle\langle\varphi_i| = \rho$ whence the eigenvalues p_i must satisfy $p_i^2 = p_i \forall i \Rightarrow p_i = 0, 1$. Since $Tr\left(\rho\right) = 1 = \sum_i p_i$, we must have $p_k = 1$ for some k and $p_i = 0 \forall i \neq k$ implying that ρ is the pure state $|\varphi_k\rangle\langle\varphi_k|$. The converse is trivial. Hence, a “pure state” is characterized by $\rho^2 = \rho$.

(e) A pure state can also be identified by $Tr\left(\rho^2\right) = Tr\left(\rho\right)$. Given a pure state, we have $\rho^2 = \rho$ so that $Tr\left(\rho^2\right) = Tr\left(\rho\right)$. Conversely, let $\rho = \sum_i p_i|\varphi_i\rangle\langle\varphi_i|$ be the spectral decomposition of ρ. Then, $\rho^2 = \sum_i p_i^2|\varphi_i\rangle\langle\varphi_i|$ so that $0 = Tr\left(\rho\right) - Tr\left(\rho^2\right) = \sum_i p_i\left(1 - p_i\right)$. Since each summand on the right hand side is non-negative, we must have $p_i = 0, 1$. However, because $Tr\left(\rho\right) = \sum_i p_i = 1$ so that we must have $p_k = 1$ for some k and $p_i = 0 \forall i \neq k$ implying that ρ is the pure state $|\varphi_k\rangle\langle\varphi_k|$.

(f) The set of all density matrices $S\left(\mathrm{H}\right)$ is a convex set i.e.

$$\rho_1, \rho_2 \in S\left(\mathrm{H}\right) \Rightarrow \rho = \lambda\rho_1 + \left(1 - \lambda\right)\rho_2 \in S\left(\mathrm{H}\right) \ \ for\ \lambda \in [0, 1] \tag{2.20}$$

Since a pure state cannot be decomposed into a convex linear combination of other distinguishable pure states, for a pure state, we have $\rho = \rho_1 = \rho_2$ thereby implying that the pure states lie on the boundary of $S\left(\mathrm{H}\right)$.

(g) It needs be emphasized here that the decomposition of the density matrix as a convex combination of normalized (but not necessarily orthogonal) pure states as $\rho = \sum_i w_i|\psi_i\rangle\langle\psi_i|$ is not unique. To examine the issue further, let us introduce $\left|\tilde{\psi}_i\right\rangle = \sqrt{w_i}|\psi_i\rangle$ whence, we can write the density matrix as

$$\rho = \sum_i \left|\tilde{\psi}_i\right\rangle\left\langle\tilde{\psi}_i\right| \tag{2.21}$$

Let us, now assume that two sets of states $\left\{\left|\tilde{\psi}_i\right\rangle\right\}$ and $\left\{\left|\tilde{\zeta}_i\right\rangle\right\}$ generate the same density matrix. We, then, have

$$\rho = \sum_i \left|\tilde{\psi}_i\right\rangle\left\langle\tilde{\psi}_i\right| = \sum_i \left|\tilde{\zeta}_i\right\rangle\left\langle\tilde{\zeta}_i\right| \tag{2.22}$$

where we have appended zero vectors to the set of states having the lesser number of states to make the number of states equal. Eq. (2.22) can hold only if there exists a unitary transformation $\hat{U} \equiv (u_{ij})$ such that

$$\left|\tilde{\psi}_i\right\rangle = \sum_j u_{ij}\left|\tilde{\zeta}_j\right\rangle \tag{2.23}$$

for, by direct substitution, we have

$$\rho = \sum_i \left|\tilde{\psi}_i\right\rangle\left\langle\tilde{\psi}_i\right| = \sum_i \left[\left(\sum_j u_{ij}\left|\tilde{\zeta}_j\right\rangle\right)\left(\sum_k u^*_{ik}\left\langle\tilde{\zeta}_k\right|\right)\right] = \sum_j \left|\tilde{\zeta}_j\right\rangle\left\langle\tilde{\zeta}_j\right| \tag{2.24}$$

because $\sum_i u_{ij}u^*_{ik} = \sum_i u_{ij}u^\dagger_{ki} = \sum_i u^T_{ji}u^{\dagger T}_{ik} = \delta_{jk}$. Conversely, let us assume the spectral decomposition of $\rho = \sum_i p_i|\varphi_i\rangle\langle\varphi_i| = \sum_i |\tilde{\varphi}_i\rangle\langle\tilde{\varphi}_i|$ with $|\tilde{\varphi}_i\rangle = \sqrt{p_i}|\varphi_i\rangle$. Further, let eq. (2.21) also hold and let $|\psi\rangle$ be a state that is orthogonal to the space spanned by the vectors $|\tilde{\varphi}_i\rangle$ so that

$$\langle\psi|\rho|\psi\rangle = \langle\psi|\sum_i |\tilde{\varphi}_i\rangle\langle\tilde{\varphi}_i||\psi\rangle = 0 = \langle\psi|\sum_i \left|\tilde{\psi}_i\right\rangle\left\langle\tilde{\psi}_i\right||\psi\rangle = \sum_i \left|\langle\psi|\left.\tilde{\psi}_i\right\rangle\right|^2 \tag{2.25}$$

implying that $\langle\psi|\left.\tilde{\psi}_i\right\rangle = 0$ for all i and all such $|\psi\rangle$ that are orthogonal to the space spanned by the vectors $|\tilde{\varphi}_i\rangle$. It follows that we can express $\left|\tilde{\psi}_i\right\rangle$ as a linear combination of $|\tilde{\varphi}_i\rangle$ viz.

$$\left|\tilde{\psi}_i\right\rangle = \sum_k c_{ik}|\tilde{\varphi}_k\rangle \tag{2.26}$$

whence

$$\sum_i \left|\tilde{\psi}_i\right\rangle\left\langle\widehat{\psi}_i\right| = \rho = \sum_i |\tilde{\varphi}_i\rangle\langle\tilde{\varphi}_i| = \sum_{i,j}\left(\sum_k c^*_{kj}c_{ki}\right)|\tilde{\varphi}_i\rangle\langle\tilde{\varphi}_j| \Rightarrow \sum_k c^\dagger_{jk}c_{ki} = \delta_{ji} \tag{2.27}$$

In the event that $c = (c_{ki})$ is not a square matrix, we may append zero vectors to the set of states $\{|\tilde{\varphi}_i\rangle\}$ to ensure this. Together with eq. (2.27), this will ensure that $c = (c_{ki})$ is a unitary square matrix. Similarly, we establish that

$$\left|\tilde{\zeta}_i\right\rangle = \sum_k d_{ik}|\tilde{\varphi}_k\rangle \tag{2.28}$$

where d is another unitary matrix whence we may define $u = cd^\dagger$ as the unitary matrix satisfying eq. (2.23).
(h) Given $\Omega = \sum_j \lambda_\omega|\lambda_\omega\rangle\langle\lambda_\omega|$ as the spectral decomposition of an operator Ω, then the probability of a measurement of the observable represented by the operator Ω returning a value $\omega \in Spec\,(\Omega)$ can be expressed in terms of the density operator as $p\,(\omega) = \sum_i p_i\,|\langle\lambda_\omega \mid \psi_i\rangle|^2 = \langle\lambda_\omega|\rho|\lambda_\omega\rangle = Tr\,(P_\omega\rho)$ where $\rho = \sum_i p_i|\psi_i\rangle\langle\psi_i|$ is the density operator expressed in, not necessarily an orthogonal, basis but satisfying $\langle\psi_i \mid \psi_i\rangle = 1$ and $P_\omega = |\lambda_\omega\rangle\langle\lambda_\omega|$ is the relevant projection operator.
At this point it becomes important to distinguish between a "mixed state" that represents a statistical ensemble of a number of "pure states" and the linear superposition $|\psi\rangle = \sum_i \alpha_i|\psi_i\rangle$of pure states $\{|\psi_i\rangle\}$ that yields another pure state.

B. Composite Systems

We consider two quantum systems Q_1, Q_2 represented by state vectors that are linear combinations $|\psi_1\rangle = \sum_i a^i|e_i\rangle \in \mathrm{H}_1$, $|\psi_2\rangle = \sum_j b^j|f_j\rangle \in \mathrm{H}_2$ of the orthonormal basis vectors $\{|e_i\rangle\}, \{|f_j\rangle\}$in the respective Hilbert spaces $\mathrm{H}_1, \mathrm{H}_2$. Then the dynamics of the composite system, Q, evolves in the Hilbert space H that is the tensor product $\mathrm{H} = \mathrm{H}_1 \otimes \mathrm{H}_2$. An arbitrary state $|\Psi\rangle$of the composite system can, accordingly, be written as a linear combination of the tensor product orthonormal basis $\{|e_i\rangle \otimes |f_j\rangle\}$ of H as $|\Psi\rangle = \sum_{i,j} c_{ij}|e_i\rangle \otimes |f_j\rangle$. Given a tensor product operator $\Omega = \Omega_1 \otimes \Omega_2$ with Ω_1 acting in the Hilbert space H_1 and Ω_2 acting in H_2, the action of Ω on the basis vectors $\{|e_i\rangle \otimes |f_j\rangle\}$ is given by

$$(\Omega_1 \otimes \Omega_2)(|e_i\rangle \otimes |f_j\rangle) \equiv (\Omega_1|e_i\rangle) \otimes (\Omega_2|f_j\rangle) \tag{2.29}$$

Furthermore, any operator Ω acting on H can be decomposed as a linear combination of the tensor products

$$\Omega = \sum_j \Omega_{1j} \otimes \Omega_{2j} \tag{2.30}$$

An extension of the postulates of quantum mechanics of isolated systems implies that the observables of system Q_1 can be expressed as the tensor product $\Omega_1 \otimes I_2$ acting in the Hilbert space H while the observables of the system Q_2 can be expressed as the tensor product $I_1 \otimes \Omega_2$ also acting in H.
In the event of Q_1 and Q_2 being uncorrelated, we can factorize the density operator of the composite system into the tensor product of the density operators of the constituent systems i.e.

$$\rho = \rho_1 \otimes \rho_2 \tag{2.31}$$

and consequently, the expectation values of a tensor product operator $\Omega = \Omega_1 \otimes \Omega_2$ also factorises as

$$\langle \Omega_1 \otimes \Omega_2 \rangle \equiv Tr\left\{(\Omega_1 \otimes \Omega_2)\rho\right\} = Tr^{(1)}\left\{\Omega_1\rho_1\right\}.Tr^{(2)}\left\{\Omega_2\rho_2\right\} = \langle \Omega_1 \rangle . \langle \Omega_2 \rangle \tag{2.32}$$

where $Tr^{(1)}, Tr^{(2)}$ denote the partial traces over the Hilbert spaces $\mathrm{H}_1, \mathrm{H}_2$ respectively.
We define the partial transpose $\rho^{pt}$of a bipartite state ρ with respect to the second Hilbert space as

$$\rho_{ij,kl} \rightarrow \rho_{il,kj} \tag{2.33}$$

where $\rho_{ij,kl} = (\langle e_i| \otimes \langle f_j|)\,\rho\,(|e_k\rangle \otimes |f_l\rangle)$. If the state ρ is separable i.e. if $\rho = \sum_i p_i\rho_{1i} \otimes \rho_{2i}$, then

$$\rho^{pt} = \sum_i p_i\rho_{1i} \otimes \rho_{2i}^T \tag{2.34}$$

The partial trace over $\mathrm{H}_2$of an operator Ω acting in the tensor product Hilbert space $\mathrm{H} = \mathrm{H}_1 \otimes \mathrm{H}_2$is an operator acting on H_1 defined by

$$\Omega_1 = Tr^{(2)}\Omega = \sum_k (I \otimes \langle k|)\,\Omega\,(I \otimes |k\rangle) \tag{2.35}$$

We also introduce the concept of the reduced density operator which is defined as

$$\rho^{(1)} = Tr^{(2)}\rho \tag{2.36}$$

The reduced density matrix completely describes the statistical properties of all observables of the subsystem Q_1. The expectation of any operator of the form $\Omega = \Omega_1 \otimes I_2$ is described by

$$\langle\Omega\rangle = Tr^{(1)}\left\{\Omega_1\rho^{(1)}\right\} \tag{2.37}$$

Given a bipartite pure state $|\Psi\rangle_{12} = \sum_{i,\mu} c_{i\mu}|i\rangle_1 \otimes |\mu\rangle_2 \in \mathrm{H}_1 \otimes \mathrm{H}_2$, the reduced density matrix for system Q_1 is given by $\rho^{(1)} = Tr^{(2)}\left(|\Psi\rangle_{12\,12}\langle\Psi|\right) = \sum_\alpha {}_2\langle\alpha|\left(\sum_{i\mu}\sum_{j\nu} c_{i\mu}c^*_{j\nu}|i\rangle_1 \otimes |\mu\rangle_{2\,1}\langle j| \otimes {}_2\langle\nu|\right)|\alpha\rangle_2$
$= \sum_{i,j,\alpha} c_{i\alpha}c^*_{j\alpha}|i\rangle_{1\,1}\langle j|$ where we have used the orthogonalities $\langle\alpha \mid \mu\rangle = \delta_{\alpha\mu}$ and $\langle\nu \mid \alpha\rangle = \delta_{\nu\alpha}$. The fact that $\rho^{(1)}$is a density operator is easily seen. We have ${}_1\langle\phi|\rho^{(1)}|\phi\rangle_1 = \sum_\alpha\sum_i c_{i\alpha\,1}\langle\phi \mid i\rangle_1 \sum_j c^*_{j\alpha\,1}\langle j \mid \phi\rangle_1$ $= \sum_\alpha \left|\sum_i c_{i\alpha\,1}\langle\phi \mid i\rangle_1\right|^2 \geq 0$. Further, $Tr\left(\rho^{(1)}\right) = \sum_{i,j,\alpha} c_{i\alpha}c^*_{j\alpha\,1}\langle j \mid i\rangle_1 = \sum_{i,\alpha} c_{i\alpha}c^*_{i\alpha} = 1$ and the hermiticity of $\rho^{(1)}$ is trivial.

The concept of "state purification" is of immense importance in quantum information processing. Let $\rho_1 = \sum_k p_k|\psi_k\rangle\langle\psi_k|$ be mixed state density matrix of the system Q_1 with Hilbert space H_1. Let us introduce a second Hilbert space H_2 of the same dimension as H_1 and a normalized vector defined by $|\Psi\rangle = \sum_k \sqrt{p_k}|\psi_k\rangle \otimes |\varphi_k\rangle$ where $\{|\varphi_k\rangle\}$ is an orthonormal basis in H_2. Then

$$Tr^{(2)}|\Psi\rangle\langle\Psi| = \sum_{i,j,k}\left(I \otimes \langle\varphi_i|\right)\left[\sqrt{p_j p_k}|\psi_j\rangle|\varphi_j\rangle\langle\psi_k|\langle\varphi_k|\right]\left(I \otimes |\varphi_i\rangle\right) = \sum_k p_k|\psi_k\rangle\langle\psi_k| = \rho_1 \tag{2.38}$$

This process of finding a true state density matrix whose partial trace over an appended Hilbert space yields the given density matrix is called purification. Hence, we can purify a mixed state by tensoring an extra Hilbert space of the same dimension as that of the original Hilbert space. However, this purification is not unique and if $|\Psi\rangle = \sum_k \sqrt{p_k}|\psi_k\rangle \otimes |\varphi_k\rangle$ is a purification of $\rho_1 = \sum_k p_k|\psi_k\rangle\langle\psi_k|$ then so is $|\Psi'\rangle = \sum_k \sqrt{p_k}|\psi_k\rangle \otimes \hat{U}|\varphi_k\rangle$ with $\hat{U}$being an arbitrary unitary operator.

Another concept that finds application in quantum information theory is that of "quantum entropy". Quantum entropy has been defined in a variety of ways e.g. von Neumann entropy, relative entropy, linear entropy etc. We shall restrict ourselves here to a sketch of the various properties of von Neumann entropy. The von Neumann entropy expresses our lack of knowledge about the realization of a particular state $|\varphi_i\rangle$in a statistical mixture obtained by mixing several ensembles of pure states represented by $\{|\varphi_i\rangle\}$with weights p_i. We define von Neumann entropy as

$$S(\rho) \equiv -Tr(\rho \ln \rho) \tag{2.39}$$

with the properties

(a) $S(\rho) \geq 0$

for all density matrices ρ with the equality holding if and only if ρ is a pure state.

(b)

$$S(\rho) \leq \ln\left(Dim\,\mathrm{H}\right) \tag{2.40}$$

for finite dimensional Hilbert space H with the equality holding if and only if ρrepresents a completely mixed state.

(c)

$$S\left(\hat{U}\rho\hat{U}^\dagger\right) = S(\rho) \tag{2.41}$$

for any unitary transformation of the Hilbert space H.

(d)

$$S\left(\sum_i \lambda_i\rho_i\right) \geq \sum_i \lambda_i S(\rho_i) \tag{2.42}$$

for any collection of density matrices ρ_i with $\lambda_i \geq 0$ and $\sum_i \lambda_i = 1$. This property implies that $S(\rho)$ is a concave functional on the space of density matrices. This inequality implies that the uncertainty or lack

of knowledge associated with the state $\sum_i \lambda_i \rho_i$ is not less than the average uncertainty associated with the states $\rho_i$that constitute the composite system.
(e) Finally, let us consider a two component composite system with the density matrix representation ρ for the composite system and the reduced density matrices $\rho^{(1)} = Tr^{(2)}(\rho)$ and $\rho^{(2)} = Tr^{(1)}(\rho)$ of the constituent systems. Then

$$S(\rho) \leq S\left(\rho^{(1)}\right) + S\left(\rho^{(2)}\right) \tag{2.43}$$

The equality holds if the total density operator is factorizable as $\rho = \rho^{(1)} \otimes \rho^{(2)}$ i.e. that the constituent systems are uncorrelated. This inequality implies that the uncertainty associated with the product state $\rho^{(1)} \otimes \rho^{(2)}$ is, in general, greater than the uncertainty associated with the composite system. In other words, by taking partial traces over the composite systems we lose out information on the possible correlations existing between the constituent systems, thereby increasing the entropy.

III. QUANTUM ENTANGLEMENT – THE BASICS

Ever since the seminal work of Einstein, Podolsky & Rosen (EPR) [22], the phenomenon of "quantum entanglement" i.e. the presence of non-classical correlations between distant objects, has been mystifying the entire scientific community. However, entanglement, as an intrinsic feature of the quantum description of physical processes gained acceptance only after John Bell [23] established the futility of explaining "entanglement" as a fallout of the existence of "local hidden variables". The implications of Bell's work have, subsequently, been extensively endorsed [24-41] and the existence of "entanglement" experimentally vindicated, as well [42-56].
We return to our picture of a composite system consisting of two subsystems Q_1, Q_2 represented by state vectors $|\psi_1\rangle, |\psi_2\rangle$ in respective Hilbert spaces$\mathrm{H}_1, \mathrm{H}_2$. The dynamics of the composite system, Q, evolves in the Hilbert space $\mathrm{H} = \mathrm{H}_1 \otimes \mathrm{H}_2$. Thus, an arbitrary state vector of the composite system can be represented in the tensor product basis as $|\Psi\rangle = \sum_{i,j} c_{ij} |e_i\rangle \otimes |f_j\rangle$ thereby showing that the dimension of the tensor product space is given by $Dim(\mathrm{H}) = Dim(\mathrm{H}_1).Dim(\mathrm{H}_2)$. Furthermore, it is obvious that "product states" of the form $|\psi_1\rangle \otimes |\psi_2\rangle$that permit a classical interpretation as "system Q_1 is in state $|\psi_1\rangle$ and Q_2 is in state $|\psi_2\rangle$" are vectors in the space of dimension $Dim(\mathrm{H}_1) + Dim(\mathrm{H}_2)$ [57]. In a nutshell, the so-called "product states' or "separable states" constitute only a subset of the set of possible states of the composite system. These residual states cannot be decomposed as the tensor product of states of the constituent subsystems. Such states exhibit non-classical correlations, violate some form of the Bell's inequality [27] and are called "entangled states" [57]. As an illustration, let us consider the state $|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle \otimes |\uparrow\rangle + |\downarrow\rangle \otimes |\downarrow\rangle)$ of the composite system and attempt to decompose it as $|\Phi^+\rangle = (c_1|\uparrow\rangle + c_2|\downarrow\rangle) \otimes (d_1|\uparrow\rangle + d_2|\downarrow\rangle)$. Comparing coefficients, we get $c_1 d_2 = c_2 d_1 = 0$, $c_1 d_1 = c_2 d_2 = \frac{1}{\sqrt{2}}$ which have no acceptable solution, thereby confirming that $|\Phi^+\rangle$ cannot be represented as a "product state". It can, similarly, be shown that the states $|\Phi^-\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle \otimes |\uparrow\rangle - |\downarrow\rangle \otimes |\downarrow\rangle)$, $|\Psi^+\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle \otimes |\downarrow\rangle + |\downarrow\rangle \otimes |\uparrow\rangle)$, and $|\Psi^-\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle \otimes |\downarrow\rangle - |\downarrow\rangle \otimes |\uparrow\rangle)$ are also entangled states. In fact, these four states constitute the "Bell basis" for the space $\mathbb{C}^2 \otimes \mathbb{C}^2$ of a pair of entangled qubits [57]. The "Bell basis" can be obtained from the logical (binary) basis $\{|\uparrow\rangle \otimes |\uparrow\rangle, |\uparrow\rangle \otimes |\downarrow\rangle, |\downarrow\rangle \otimes |\uparrow\rangle, |\downarrow\rangle \otimes |\downarrow\rangle\}$ by the unitary transformation $\hat{U} \equiv \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 1 & 0 & 0 & -1 \end{pmatrix}$. It is an opportune place here to introduce the so-called "magic basis" [58-60] $\{|\Phi^+\rangle, i|\Phi^-\rangle, i|\Psi^+\rangle, |\Psi^-\rangle\}$ in $\mathbb{C}^2 \otimes \mathbb{C}^2$which is also an orthogonal set of fully entangled states. This basis set has certain pleasing properties viz.
(a) the set of mixtures of the generalized Bell states coincides with the set of states whose density matrices are real when expressed in the "magic basis";
(b) the entanglement of pure states can be expressed in a particularly simple form in the "magic basis";
(c) the set of transformations that act independently on a pair of qubits coincides with the set of real (upto an overall phase factor) unitary transformations when expressed in this "magic basis".

(d) we have, in the logical basis $|0\rangle \equiv (1,0)^T$, $|1\rangle \equiv (0,1)^T$, $\sigma_y|0\rangle = i|1\rangle$, $\sigma_y|1\rangle = -i|0\rangle$ whence $\sigma_y \otimes \sigma_y|00\rangle = \sigma_y|0\rangle \otimes \sigma_y|0\rangle = -|11\rangle$, $\sigma_y \otimes \sigma_y|01\rangle = |10\rangle$, $\sigma_y \otimes \sigma_y|10\rangle = |01\rangle$, $\sigma_y \otimes \sigma_y|11\rangle = -|00\rangle$ so that, in the Bell basis $\sigma_y \otimes \sigma_y|\Phi^+\rangle = -|\Phi^+\rangle$, $\sigma_y \otimes \sigma_y|\Phi^-\rangle = |\Phi^-\rangle$, $\sigma_y \otimes \sigma_y|\Psi^+\rangle = |\Psi^+\rangle$, $\sigma_y \otimes \sigma_y|\Psi^-\rangle = -|\Psi^-\rangle$. Making use of these properties, we can show that the transformation $|\psi\rangle \to \left|\tilde{\psi}\right\rangle = (\sigma_y \otimes \sigma_y)\,|\psi^*\rangle$ in the logical basis corresponds to simply a complex conjugation when expressed in the "magic basis". This is easily seen by writing $|\psi\rangle = a_1|\Phi^+\rangle + a_2 i|\Phi^-\rangle + a_3 i|\Psi^+\rangle + a_4|\Psi^-\rangle$
$= \frac{1}{\sqrt{2}}\left[(a_1 + ia_2)\,|00\rangle + (ia_3 + a_4)\,|01\rangle + (ia_3 - a_4)\,|10\rangle + (a_1 - ia_2)\,|11\rangle\right]$ so that
$|\psi^*\rangle = \frac{1}{\sqrt{2}}\left[(a_1^* - ia_2^*)\,|00\rangle + (-ia_3^* + a_4^*)\,|01\rangle + (-ia_3^* - a_4^*)\,|10\rangle + (a_1^* + ia_2^*)\,|11\rangle\right]$ and

$$\left|\tilde{\psi}\right\rangle = \sigma_y \otimes \sigma_y|\psi^*\rangle = \frac{1}{\sqrt{2}}\left[-\left(a_1^* + ia_2^*\right)|00\rangle - \left(ia_3^* + a_4^*\right)|01\rangle - \left(ia_3^* - a_4^*\right)|10\rangle - \left(a_1^* - ia_2^*\right)|11\rangle\right]$$

$$= -a_1^*\big|\Phi^+\big\rangle - ia_2^*\big|\Phi^-\big\rangle - ia_3^*\big|\Psi^+\big\rangle - a_4^*\big|\Psi^-\big\rangle = a_1^*\left[(\sigma_y \otimes \sigma_y)\big|\Phi^+\big\rangle\right]$$

$$+a_2^*\left\{i^*\left[(\sigma_y \otimes \sigma_y)\big|\Phi^-\big\rangle\right]\right\} + a_3^*\left\{i^*\left[(\sigma_y \otimes \sigma_y)\big|\Psi^+\big\rangle\right]\right\} + a_4^*\left[(\sigma_y \otimes \sigma_y)\big|\Psi^-\big\rangle\right]$$

It is easily seen that

$$\left\langle \psi \,\middle|\, \tilde{\psi} \right\rangle = -\sum_k a_k^{*2} \tag{3.1}$$

The transformation referred to in this section constitutes the "spin flip" transformation, since it reverses the direction of the spin. In fact, for a spin particles it is the "time reversal" transformation.
Another fallout of the above relations is that, given the general state of a pair of qubits represented by the density operator ρ, the corresponding operator for the spin flipped states is obtained as

$$\tilde{\rho} = \sigma_y \otimes \sigma_y \rho^* \sigma_y \otimes \sigma_y \tag{3.2}$$

where we have used the hermiticity of the Pauli matrices and the complex conjugation $\rho^*$is obtained in the logical basis. In analogy with the proof given above, we can easily show that the transformation (3.2) corresponds to a mere "complex conjugation" in the "magic basis". We shall return to these properties again in the sequel.
Particles represented by entangled states show anomalously strong nonlocal correlations even in the absence of present interactions, if they have interacted in the past. A clear distinction exists between entangled and unentangled pure states with the latter being expressible as a tensor product of states of the constituent systems. Entanglement between states cannot be created by local actions on either of the constituent subsystems. The Schmidt number provides an important property for identifying separable states and entangled ones.

IV. SCHMIDT'S DECOMPOSITION:

Every pure state in the composite Hilbert space $\mathrm{H} = \mathrm{H}_1 \otimes \mathrm{H}_2$can be expressed in the form $|\Psi\rangle = \sum_{i=1}^{N} \sqrt{\lambda_i}|e_i\rangle \otimes |f_i\rangle$ where $\{|e_i\rangle\}_{i=1}^{N_1}$ and $\{|f_i\rangle\}_{i=1}^{N_2}$ are respectively orthonormal bases in $\mathrm{H}_1$and H_2respectively, $N \le \min\{N_1, N_2\}$, $\sum_i \lambda_1 = 1$ and $\lambda_i \ge 0 \forall i$. Let us assume $|\Psi\rangle = \sum_{i=1}^{N_1}\sum_{j=1}^{N_2} C_{ij}|e_i'\rangle \otimes \left|f_j'\right\rangle$ in H and introduce $|\phi_i'\rangle = \sum_j C_{ij}\left|f_j'\right\rangle$ (which may not be orthogonal) in terms of which we can write $|\Psi\rangle = \sum_{i=1}^{N_1} |e_i'\rangle \otimes |\phi_i'\rangle$. Let $\rho = |\Psi\rangle\langle\Psi|$ be the density matrix of the system. Taking trace over the degrees of freedom of the second system, we get $\rho^{(1)} = Tr^{(2)}\left(|\Psi\rangle\langle\Psi|\right) = \sum_{i=1}^{N_1}\sum_{j=1}^{N_1}\left\langle \phi_j' \middle| \phi_i' \right\rangle |e_i'\rangle\left\langle e_j'\right|$. Let us, now, perform a unitary transformation to a new basis $\{|e_i\rangle\}$ in H_1in which $\rho^{(1)}$ takes a diagonal form $\rho^{(1)} = \sum_{i=1}^{N_1} \lambda_i |e_i\rangle\langle e_i|$with real and non-negative coefficients λ_i. Finally, we go back and adopt this basis from the beginning. We, then, get $\langle \phi'_j | \phi'_i \rangle = \lambda_i \delta_{ij}$ i.e. we set $|\phi'_i\rangle = \sqrt{\lambda_i}|f_i\rangle$ This completes the proof. We also have $\rho^{(1)} = Tr^{(2)}\rho = CC^\dagger$ and $\rho^{(2)} = Tr^{(1)}\rho = C^T C^*$. The Schmidt decomposition bases are respectively the eigenbasis of the two reduced density matrices corresponding to the constituent systems e.g. $\rho^{(1)} = \sum_{i=1}^{N_1} \lambda_i |e_i\rangle\langle e_i|$ and $\rho^{(2)} = \sum_{i=1}^{N_1} \lambda_i |f_i\rangle\langle f_i|$ respectively. The real numbers λ_i are the Schmidt's coefficients and the number of non-vanishing λ_i is the Schmidt rank of the state $|\Psi\rangle$ and is equal to the rank of the reduced density matrices. For a pure state the number of non zero λ_i must be one. If such number exceeds one, then the state is entangled.

The use of von Neumann entropy as a measure of entanglement of pure states is well accepted [58]. We define the entropy of entanglement in such cases by $E(|\psi\rangle) = S\left(\rho^{(1)}\right) = -Tr\left(\rho^{(1)} \ln \rho^{(1)}\right) = -Tr\left(\rho^{(2)} \ln \rho^{(2)}\right)$. We have already elucidated the generic properties of von Neumann entropy in the previous section. However, the following attributes make $S(\rho)$ a compatible measure of entanglement [58]:

(a) the von Neumann entropy is, obviously, entanglement sensitive. Its value ranges from zero for a product state to $\ln N$for a maximally entangled state of two Nstate particles;

(b) von Neumann entropy for uncorrelated systems is additive;

(c) von Neumann entropy is conserved under the action of any unitary operator that can be factorized as a tensor product $\hat{U} = \hat{U}_1 \otimes \hat{U}_2$ of unitary operators $\hat{U}_1, \hat{U}_2$ acting respectively on the constituent subsystems;

(d) average $S(\rho)$ of a composite system cannot be increased by local non-unitary operations;

(e) $S(\rho)$ can be concentrated or diluted with unit efficiency in the limiting case.

(f) $S(\rho)$ completely parameterizes a "pure state" in the sense that it represents both its "entanglement of formation" defined as "the asymptotic number of standard singlets required to locally prepare the said pure state" as well as its "distillable entanglement" defined as "the asymptotic number of standard singlets that can be prepared from the said pure state by local operations".

V. MIXED STATE QUANTUM ENTANGLEMENT, ENTANGLEMENT OF FORMATION AND WOOTTERS CONCURRENCE

A mixed state is said to be entangled if it cannot be represented as a mixture of factorizable pure states. Such states arise, for instance, when the constituent subsystems of a pure entangled state interact with the environment that causes the pure state to evolve non-unitarily to a mixed state.

The "entanglement of formation" constitutes a universal "entanglement measure" that can be adopted for measuring entanglement of pure states as well as mixed states. We define the "entanglement of formation" as [58-60]

(a) the "entanglement of formation" of a bipartite pure state $|\psi\rangle$ is the von Neumann entropy

$$E(|\psi\rangle) = S\left(Tr^{(1)}|\psi\rangle\langle\psi|\right) = S\left(Tr^{(2)}|\psi\rangle\langle\psi|\right) \tag{5.1}$$

of the reduced density matrix of either of the constituent subsystems. The entropy may be evaluated with respect to either of the constituent subsystems for, we have $S\left(\rho^{(1)}\right) = -Tr\left(\rho^{(1)} \ln \rho^{(1)}\right) = -\sum_i \lambda_i \ln \lambda_i$ where $\{\lambda_i\}$are the eigenvalues of $\rho^{(1)}$ so that $\rho^{(1)} = -\sum_i \lambda_i |i\rangle_1 {}_1\langle i|$ and ${}_1\langle i \mid j\rangle_1 = \delta_{ij}$ is the spectral decomposition of $\rho^{(1)}$. We, then, have $\left(\rho^{(1)}\right)^2 = \sum_{i,j} \lambda_i \lambda_j |i\rangle_1 \langle i \mid j\rangle_1 \langle j| = \sum_i \lambda_i^2 |i\rangle_1 {}_1\langle i|$ whence, by induction $\left(\rho^{(1)}\right)^n = \sum_i \lambda_i^n |i\rangle_1 {}_1\langle i|$. Expanding an arbitrary function $f(\lambda)$ as a power series in λ, we can generalize $f\left(\rho^{(1)}\right) = \sum_i f(\lambda_i) |i\rangle_1 {}_1\langle i|$. Now, by Schmidt's decomposition $|\Psi\rangle = \sum_i \sqrt{p_i}|i\rangle_1 |i\rangle_2$ so that $\rho^{(1)} = \sum_i p_1 |i\rangle_1 {}_1\langle i|$ and$\rho^{(2)} = \sum_i p_1 |i\rangle_2 {}_2\langle i|$ whence the equality follows from the fact that both $\rho^{(1,2)}$ have the same probability weights p_i.

(b) "entanglement of formation" of an ensemble of bipartite pure states $\Pi = \{p_i, |\psi_i\rangle\}$ is given by the ensemble average $E(\Pi) = \sum_i p_i E(|\psi_i\rangle)$ of the entanglement of formation of the constituent pure states of the ensemble.

(c) the "entanglement of formation" of a mixed state is the minimum of $E(\Pi)$ taken over the ensembles $E(\Pi) = \sum_i p_i E(|\psi_i\rangle)$ realizing the mixed state. Thus the entanglement of formation $E(M)$of a mixed state M is the least expected entanglement of any ensemble of pure states realizing M.

To obtain further insights on the "entanglement of formation", we consider a normalized state vector of a pure state of two qubits represented as

$$|\psi\rangle = \sum_{i,j} C_{ij}|i\rangle|j\rangle \tag{5.2}$$

with $\sum_{i,j}|C_{ij}|^2 = 1$. We can write its Schmidt decomposition in the form $|\psi\rangle = \sum_i \sqrt{\mu_i}|e_i\rangle \otimes |f_i\rangle$ where the basis sets $\{|e_i\rangle\}$, $\{|f_i\rangle\}$ are obtained from the bases $\{|i\rangle\}$, $\{|j\rangle\}$ by unitary transformations and hence, retain their orthonormality. It can be easily shown, using the singular value decomposition of the coefficient matrix (C_{ij}) that the μ_i's are eigenvalues of $CC^\dagger$ and they satisfy $\mu_1 + \mu_2 = 1$ as can be seen from the characteristic equation for $CC^\dagger$. The von Neumann entropy is, then, $S(\mu_i) = -\sum_i \mu_i \ln \mu_i$. Further, $(\det C)^2 = \mu_1\mu_2 = \mu_1(1-\mu_1)$ whence

$$\mu_1 = \frac{1}{2}\left(1 - \sqrt{1 - 4(\det C)^2}\right) \tag{5.3}$$

Let us, now, represent the state $|\psi\rangle$ in the "magic basis" as

$$|\psi\rangle = a_1\left|\Phi^+\right\rangle + a_2 i\left|\Phi^-\right\rangle + a_3 i\left|\Psi^+\right\rangle + a_4\left|\Psi^-\right\rangle$$

$$= \frac{1}{\sqrt{2}}\left[(a_1 + ia_2)|00\rangle + (ia_3 + a_4)|01\rangle + (ia_3 - a_4)|10\rangle + (a_1 - ia_2)|11\rangle\right] \tag{5.4}$$

so that

$$C = \frac{1}{\sqrt{2}}\begin{pmatrix} a_1 + ia_2 & ia_3 + a_4 \\ ia_3 - a_4 & a_1 - ia_2 \end{pmatrix} \; and \; \det C = \frac{1}{2}\sum_k a_k^2 \tag{5.5}$$

The quantity

$$\kappa = \left|\sqrt{2\left(1 - Tr\left(\rho^{(1)}\right)^2\right)}\right| = \left|\sqrt{2\left(1 - \sum_j \mu_j^2\right)}\right| = 2\left|\sqrt{\mu_1\mu_2}\right| = 2\left|\det C\right| = \left|\sum_k a_k^2\right| = \left|\left\langle \psi \mid \tilde{\psi}\right\rangle\right| \tag{5.6}$$

is called "Wootter's concurrence" in the literature [58-60], where, in the last step we have used eq. (3.1). κ ranges from 0 to 1 and is monotonically related to the entanglement. It can, therefore, be considered a measure of entanglement. It follows from eq. (5.6) that any real linear combination of the "magic basis" would result in a fully entangled state with unit concurrence. Conversely, any completely entangled state can be written as a linear combination in the "magic basis" with real components, upto an overall phase factor. In fact, these properties are not unique to a state description in the "magic basis" and hold in any other basis that is obtained from the "magic basis" by an orthogonal transformation since orthogonal transformations do not disturb the norm of a state i.e. $\sum_k a_k^2 = \sum_k \acute{a}_k^2$ so that concurrence is not affected by any transformation $O \in SO(4)$ [60].

We, now, obtain an expression for the entanglement of formation of mixed states of a pair of qubits as a function of their density matrix. We follow the following steps [58-60]:

(a) Let $\{|\hat{e}_i\rangle, i = 1, 2, ..., n\}$be an orthonormal eigenbasis in the n dimensional Hilbert space corresponding to the relevant density matrix ρ with respective eigenvalues $\{\lambda_i, i = 1, 2, .., n\}$ of the pair of qubits. Let $\{\lambda_i, i = 1, 2, ..., k\}$ be the set of non-zero eigenvalues and let be the corresponding eigenvectors. We subnormalize these eigenvectors to obtain $\{|e_i\rangle, i = 1, 2, ..., k\}$such that$\langle e_i \mid e_i\rangle = \lambda_i \forall \lambda_i \neq 0$. $\{|e_i\rangle, i = 1, 2, ..., k\}$ constitutes the "eigenensemble" of the density matrixρ. Then the ensemble of states

$$\{|\xi_i\rangle, i = 1, 2, ..., r, r \geq k\} \tag{5.7}$$

where

$$|\xi_i\rangle = \sum_{j=1}^{k} U_{ij}^{*} |e_j\rangle, i = 1, 2, ..., r \tag{5.8}$$

constitutes a general decomposition of the density matrix ρ. In eq. (5.8) U is a $r \times r$ unitary matrix or a $r \times k$ matrix whose columns are orthogonal vectors since we use only k columns in the computation of eq. (5.8) out of the $r$columns. The proof is trivial for

$$\sum_{i=1}^{r} |\xi_i\rangle\langle\xi_i| = \sum_{i=1}^{r} \sum_{l,m=1}^{k} U_{il}^{*} U_{im} |e_l\rangle\langle e_m| = \sum_{i=1}^{r} \sum_{l,m=1}^{k} U_{li}^{\dagger} U_{im} |e_l\rangle\langle e_m| = \sum_{m=1}^{k} |e_m\rangle\langle e_m| = \rho \tag{5.9}$$

We also have $\langle \xi_i \mid \xi_i\rangle = \sum_{l,m=1}^{k} U_{il}^{*} U_{im} \langle e_m \mid e_l\rangle = \sum_{l,m=1}^{k} U_{li}^{\dagger} U_{im} \langle e_m \mid e_l\rangle = \lambda_i$so that the normalization remains unchanged.

(b) The states $\{|e_i\rangle, i = 1, 2, ..., r, r \geq k\}$ are eigenstates of ρ for $\rho|e_k\rangle = \left(\sum_{i=1}^{r} |e_i\rangle\langle e_i|\right)|e_k\rangle = \sum_{i=1}^{r} |e_i\rangle\lambda_k\delta_{ik} = \lambda_k|e_k\rangle$. Similarly, it can be shown that the spin flipped states $|\tilde{e}_k\rangle$ are also eigenstates of the spin flipped density operator$\tilde{\rho}$. Now, let us define the matrix $\Xi_{ij} = \langle e_i \mid \tilde{e}_j\rangle$. Then, this matrix is symmetric because

$$\langle e_i|\sigma_y \otimes \sigma_y|e_j^{*}\rangle = \begin{pmatrix} e_i^{1*} \\ e_i^{2*} \\ e_i^{3*} \\ e_i^{4*} \end{pmatrix}^{T} \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} e_j^{1*} \\ e_j^{2*} \\ e_j^{3*} \\ e_j^{4*} \end{pmatrix} = \begin{pmatrix} -e_i^{1*}e_j^{4*} + e_i^{2*}e_j^{3*} \\ +e_i^{3*}e_j^{2*} - e_i^{4*}e_j^{1*} \end{pmatrix} = \langle e_j|\sigma_y \otimes \sigma_y|e_i^{*}\rangle.$$

Furthermore, the matrices $\rho\tilde{\rho}$ and $\Xi\Xi^{*}$ are cospectral i.e. have the same set of eigenvalues. This is proved as follows:

(i) Since unitarity preserves orthogonality & normalization, without any loss of generality, we can assume the subnormalized eigenbasis as $e_1 \equiv \left(\sqrt{\lambda_1}e^{i\theta_1}, 0, 0, 0\right)^{T}$with similar expressions for $e_j, j = 2, 3, 4$. This gives $\rho = \sum_i |e_i\rangle\langle e_i| = diag\left(\lambda_i, i = 1, 2, 3, 4\right)$ whence $\tilde{\rho} = diag\left(\lambda_4, \lambda_3, \lambda_2, \lambda_1\right)$ and $Tr\left(\rho\tilde{\rho}\right) = 2\lambda_1\lambda_4 + 2\lambda_2\lambda_3$;

(ii) By definition, we have $\langle e_1 \mid \tilde{e}_1\rangle = \langle e_2 \mid \tilde{e}_1\rangle = \langle e_3 \mid \tilde{e}_1\rangle = \langle e_2 \mid \tilde{e}_2\rangle = \langle e_4 \mid \tilde{e}_2\rangle = \langle e_3 \mid \tilde{e}_3\rangle = \langle e_4 \mid \tilde{e}_3\rangle = \langle e_4 \mid \tilde{e}_4\rangle = 0$ and $\langle e_4 \mid \tilde{e}_1\rangle = -\sqrt{\lambda_1\lambda_4}e^{-i(\theta_1+\theta_4)}$, $\langle e_3 \mid \tilde{e}_2\rangle = \sqrt{\lambda_2\lambda_3}e^{-i(\theta_2+\theta_3)}$ so that $\Xi = \begin{pmatrix} 0 & 0 & 0 & -\sqrt{\lambda_1\lambda_4}e^{-i(\theta_1+\theta_4)} \\ 0 & 0 & \sqrt{\lambda_2\lambda_3}e^{-i(\theta_2+\theta_3)} & 0 \\ 0 & \sqrt{\lambda_2\lambda_3}e^{-i(\theta_2+\theta_3)} & 0 & 0 \\ -\sqrt{\lambda_1\lambda_4}e^{-i(\theta_1+\theta_4)} & 0 & 0 & 0 \end{pmatrix}$ whence $Tr\left(\Xi\Xi^{*}\right) = 2\lambda_1\lambda_4 + 2\lambda_2\lambda_3 = Tr\left(\rho\tilde{\rho}\right)$. In fact, we have $Tr\left(\Xi\Xi^{*}\right)^{n} = Tr\left(\rho\tilde{\rho}\right)^{n}$ for all integers n so that $\rho\tilde{\rho}$ and $\Xi\Xi^{*}$have the same characteristic equation and hence, the same spectra.

(c) Now, consider the matrix Ξ. It is a complex symmetric matrix. It follows from the Takagi factorization that there exists a unitary matrix Vsuch that $V\Xi V^{T}$ is diagonal. Now, consider $V\Xi\Xi^{*}V^{\dagger} = \left(V\Xi V^{T}\right)\left(V^{-1T}\Xi^{*}V^{*T}\right) = \left(V\Xi V^{T}\right)\left(V^{*}\Xi^{*}V^{*T}\right) = \left(V\Xi V^{T}\right)\left(V\Xi V^{T}\right)^{*}$ thereby showing that Valso diagonalizes the Hermitian matrix$\Xi\Xi^{*}$.

(d) From the expression $V\Xi\Xi^{*}V^{\dagger} = \left(V\Xi V^{T}\right)\left(V\Xi V^{T}\right)^{*}$, it follows that the diagonal elements (eigenvalues) of $V\Xi V^{T}$(except for a phase factor) are the square roots of the eigenvalues of $\Xi\Xi^{*}$ which, as proved earlier are the eigenvalues of $\rho\tilde{\rho}$. In other words, since Ξ is symmetric, by an appropriate choice of phases of the eigenvectors of$V$, we can make the diagonal elements of $V\Xi V^{T}$equal to the square roots of the eigenvalues of $\Xi\Xi^{*}$.

(e) Since V is a unitary matrix, we can obtain a decomposition of ρas

$$\rho = \sum_i |\zeta_i\rangle\langle\zeta_i| \tag{5.10}$$

where

$$|\zeta_i\rangle = \sum_j V_{ij}^* |e_j\rangle \tag{5.11}$$

We then have

$$\left\langle \zeta_i \,\middle|\, \tilde{\zeta}_j \right\rangle = \sum_{k,m} V_{ik} V_{jm} \langle e_k \mid \tilde{e}_m\rangle = \sum_{k,m} V_{ik} V_{mj}^T \langle e_k \mid \tilde{e}_m\rangle = \sum_{k,m} V_{ik} \Xi_{km} V_{mj}^T = \left(V \Xi V^T\right)_{ij} = \mu_i \delta_{ij} \tag{5.12}$$

where $\mu_i$are the square roots of the eigenvalues of $\rho\tilde{\rho}$ or $\Xi\Xi^*$or the eigenvalues of $V\Xi V^T$. Written out explicitly, we note that the orthogonal matrix $V \equiv \frac{1}{\sqrt{2}} \begin{pmatrix} e^{i\theta_1} & 0 & 0 & e^{i\theta_4} \\ 0 & e^{i\theta_2} & e^{i\theta_3} & 0 \\ 0 & -e^{i\theta_2} & e^{i\theta_3} & 0 \\ -e^{i\theta_1} & 0 & 0 & e^{i\theta_4} \end{pmatrix}$ diagonalizes Ξ in our chosen basis with the eigenvalues $\mu_{1,4} = \mp\sqrt{\lambda_1\lambda_4}$, $\mu_{2,3} = \pm\sqrt{\lambda_2\lambda_3}$ respectively. We also have $|\zeta_1\rangle = \sum_j V_{1j}^* |e_j\rangle = 2^{-1/2}\left(\sqrt{\lambda_1}, 0, 0, \sqrt{\lambda_4}\right)^T$, $|\zeta_2\rangle = 2^{-1/2}\left(0, \sqrt{\lambda_2}, \sqrt{\lambda_3}, 0\right)^T$, $|\zeta_3\rangle = 2^{-1/2}\left(0, -\sqrt{\lambda_2}, \sqrt{\lambda_3}, 0\right)^T$, $|\zeta_4\rangle = 2^{-1/2}\left(-\sqrt{\lambda_1}, 0, 0, \sqrt{\lambda_4}\right)^T$ whence, for example, $\left|\tilde{\zeta}_1\right\rangle = 2^{-1/2}\left(-\sqrt{\lambda_4}, 0, 0, -\sqrt{\lambda_1}\right)^T$ whence $\left\langle \zeta_1 \,\middle|\, \tilde{\zeta}_1\right\rangle = -\sqrt{\lambda_1\lambda_4} = \mu_1$.

(f) For mixed states, we define Wootters Concurrence as $\kappa(\rho) = \max\{0, \mu_1 - \mu_2 - \mu_3 - \mu_4\}$ where μ_i are the square roots of the eigenvalues of the product matrix$\rho\tilde{\rho}$. We, first consider the case when $\mu_1 - \mu_2 - \mu_3 - \mu_4 \geq 0$. Let us introduce the set of states$|\psi_1\rangle = |\zeta_1\rangle$, $|\psi_2\rangle = i|\zeta_2\rangle$, $|\psi_3\rangle = i|\zeta_3\rangle$ and $|\psi_4\rangle = i|\zeta_4\rangle$. It is easily seen that $\left\langle \psi_1 \,\middle|\, \tilde{\psi}_1\right\rangle = \langle\zeta_1 \mid \zeta_1\rangle = \mu_1$, $\left\langle \psi_{2,3,4} \,\middle|\, \tilde{\psi}_{2,3,4}\right\rangle = -\langle\zeta_{2,3,4} \mid \zeta_{2,3,4}\rangle = -\mu_{2,3,4}$ so that if we define the matrix $\Psi = (\Psi_{ij}) = \left(\left\langle \psi_i \,\middle|\, \tilde{\psi}_j\right\rangle\right)$, then $Tr\Psi = \mu_1 - \mu_2 - \mu_3 - \mu_4 = \kappa(\rho)$.

(g) The final decomposition that we obtain is one in which the concurrence of each state is $\kappa(\rho)$. In this decomposition, then, the average entanglement shall be equal to the entanglement of any state. In this context, we note that the concurrence remains unchanged under a unitary transformation of the states, for if $|\phi_i\rangle = \sum_j W_{ij}|\psi_j\rangle$ then $\kappa(\rho)$ is unaffected under this transformation. This is easily seen. We have$\sum_i \left\langle \phi_i \,\middle|\, \tilde{\phi}_i\right\rangle = \sum_i \left(W\Psi W^T\right)_{ii} = Tr\left(W\Psi W^T\right)$. Now, $\Psi \equiv \left\langle \psi_i \,\middle|\, \tilde{\psi}_j\right\rangle$ is a real symmetric matrix. Hence, we can take the unitary W to be real unitary i.e. orthogonal whence $W^{-1} = W^T$ and the above trace is unaffected. Let us assume that the transformation effected by the orthogonal matrix W results in a decomposition such that the concurrence of each state and hence, the average concurrence is $\kappa(\rho)$. Our problem is, now, to establish the existence of an orthogonal matrix W. This can be done by the following scheme:

(i) Calculate the "preconcurrence" defined as $\kappa_{pre}(|\psi_i\rangle) = \frac{\langle\psi_i|\tilde{\psi}_i\rangle}{\langle\psi_i|\psi_i\rangle}$ corresponding to each of the above states $|\psi_i\rangle$.

(ii) Select the two states with the largest and smallest preconcurrences, say $|\psi_\alpha\rangle, |\psi_\beta\rangle$. There exists a set of positive determinant orthogonal transformations $\{W\}$ that acts only on these two extreme states $|\psi_\alpha\rangle, |\psi_\beta\rangle$and leaves the other ones $\{|\psi_i\rangle, i \neq \alpha, \beta\}$unchanged. Further, there also exists an element of this set that would operate to interchange the two extreme states $|\psi_\alpha\rangle, |\psi_\beta\rangle$ with each other (and hence interchanging their preconcurrences, as well) while leaving the other states $\{|\psi_i\rangle, i \neq \alpha, \beta\}$unchanged. By continuity considerations, then, there must exist a $W \in \{W\}$ that would operate on $|\psi_\alpha\rangle, |\psi_\beta\rangle$and transforming $|\psi_\alpha\rangle$ (say) such that its preconcurrence equals $\kappa(\rho)$. We perform this transformation. The result is that one element of the set of states has now attained the desired concurrence.

(iii) We iterate this process again and again, thereby making the concurrence of the remaining states also equal to $\kappa(\rho)$. The transformed set of states, then, constitutes a decomposition of the density matrix ρ that has the property that each constituent pure state of the decomposition has the same concurrence $\kappa(\rho)$ which is also, obviously, the average concurrence of the ensemble. Let us call this decomposition $\{|\phi_i\rangle\}$.

(h) We have defined the entanglement of formation of a bipartite mixed state as the minimum of the entanglement of formation of the ensembles of pure states that realize the desired mixed state. In this context, the entanglement of formation of the ensembles of pure states is the average of the entanglement of formation of the set of pure states constituting the ensemble. Hence, we need also to establish that that no decomposition of ρ other than $\{|\phi_i\rangle\}$ has a lower average entanglement. We proceed to establish this.

As has been mentioned earlier, unitary transformation of states does not change the average concurrence. Hence, we can write the average concurrence of the decomposition $\{|\phi_i\rangle\}$ as $\langle\kappa(\rho)\rangle = \sum_i |(W\Psi W^t)_{ii}| = \sum_i \left|\sum_j (W_{ij})^2 \Psi_{jj}\right|$. We note that $\sum_i \left|(W_{ij})^2\right| = 1$. Hence, we need to show that $\sum_i \left|\sum_j (W_{ij})^2 \Psi_{jj}\right| \geq \mu_1 - \mu_2 - \mu_3 - \mu_4$. To prove this, let us assume that $(W_{i1})^2$ for each$i$to be real and positive and adjust the phase of the remaining $(W_{ij})^2, j \neq 1$ appropriately. Then $\sum_i \left|\sum_i (W_{ij})^2 \Psi_{ji}\right| \geq \left|\sum_{i,j} (W_{ij})^2 \Psi_{jj}\right| = \left|\mu_1 - \sum_{j=2}\left(\sum_i (W_{ij})^2\right)\mu_j\right|$ $\geq \mu_1 - \mu_2 - \mu_3 - \mu_4$ which establishes our result that $\{|\phi_i\rangle\}$ is the optimal decomposition of ρ each of whose constituent states have a concurrence of $\kappa(\rho) = \mu_1 - \mu_2 - \mu_3 - \mu_4$. It follows from the convexity of the dependence of the entanglement of formation on the average concurrence that the optimality of the decomposition in terms of concurrence implies optimality in terms of the entanglement of formation.
(i) We, now, address the case of $\mu_1 - \mu_2 - \mu_3 - \mu_4 < 0$. In this case, we can find four phases $\omega_i$such that $\sum_{j=1}^4 e^{-2i\omega_j}\mu_j = 0$. This is possible because when $\mu_1 - \mu_2 - \mu_3 - \mu_4 < 0$, we can constitute a closed quadrangle whose sides are respectively μ_j (It may be noted that such a closed polygon cannot be constructed when $\mu_1 - \mu_2 - \mu_3 - \mu_4 \geq 0$ for $\mu_1 \leq \mu_1 + \mu_2 + \mu_3$). Using these phase factors, we can define four pure states as

$$|\phi_1\rangle = \frac{1}{2}\left(e^{i\omega_1}|\zeta_1\rangle + e^{i\omega_2}|\zeta_2\rangle + e^{i\omega_3}|\zeta_3\rangle + e^{i\omega_4}|\zeta_4\rangle\right) \tag{5.13}$$

$$|\phi_2\rangle = \frac{1}{2}\left(e^{i\omega_1}|\zeta_1\rangle + e^{i\omega_2}|\zeta_2\rangle - e^{i\omega_3}|\zeta_3\rangle - e^{i\omega_4}|\zeta_4\rangle\right) \tag{5.14}$$

$$|\phi_3\rangle = \frac{1}{2}\left(e^{i\omega_1}|\zeta_1\rangle - e^{i\omega_2}|\zeta_2\rangle + e^{i\omega_3}|\zeta_3\rangle - e^{i\omega_4}|\zeta_4\rangle\right) \tag{5.15}$$

$$|\phi_4\rangle = \frac{1}{2}\left(e^{i\omega_1}|\zeta_1\rangle - e^{i\omega_2}|\zeta_2\rangle - e^{i\omega_3}|\zeta_3\rangle + e^{i\omega_4}|\zeta_4\rangle\right) \tag{5.16}$$

It is easily seen that $\sum_i |\phi_i\rangle\langle\phi_i| = \sum_i |\zeta_i\rangle\langle\zeta_i| = \rho$ so that the states (5.13) – (5.16) constitute a decomposition of ρ. Furthermore, we also have $\left\langle \phi_i \,\middle|\, \tilde{\phi}_i \right\rangle = 0$ for $i = 1, 2, 3, 4$ since, as already proved $\left\langle \zeta_i \,\middle|\, \tilde{\zeta}_j \right\rangle = \mu_i\delta_{ij}$ and $\sum_{j=1}^4 e^{-2i\omega_j}\mu_j = 0$. We illustrate the calculations by explicitly working our $\left\langle \phi_1 \,\middle|\, \tilde{\phi}_1 \right\rangle$. We have, from above, $|\zeta_1\rangle = 2^{-1/2}\left(\sqrt{\lambda_1}, 0, 0, \sqrt{\lambda_4}\right)^T$, $|\zeta_2\rangle = 2^{-1/2}\left(0, \sqrt{\lambda_2}, \sqrt{\lambda_3}, 0\right)^T$, $|\zeta_3\rangle = 2^{-1/2}\left(0, -\sqrt{\lambda_2}, \sqrt{\lambda_3}, 0\right)^T$, $|\zeta_4\rangle = 2^{-1/2}\left(-\sqrt{\lambda_1}, 0, 0, \sqrt{\lambda_4}\right)^T$. Therefore,
$|\phi_1\rangle = \frac{1}{2\sqrt{2}}\begin{pmatrix} \sqrt{\lambda_1}e^{i\omega_1} - \sqrt{\lambda_1}e^{i\omega_4}, \sqrt{\lambda_2}e^{i\omega_2} - \sqrt{\lambda_2}e^{i\omega_3}, \\ \sqrt{\lambda_3}e^{i\omega_2} + \sqrt{\lambda_3}e^{i\omega_3}, \sqrt{\lambda_4}e^{i\omega_1} + \sqrt{\lambda_4}e^{i\omega_4} \end{pmatrix}^T$ whence
$\left|\tilde{\phi}_1\right\rangle = \frac{1}{2\sqrt{2}}\begin{pmatrix} -\sqrt{\lambda_4}e^{i\omega_1} - \sqrt{\lambda_4}e^{i\omega_4}, \sqrt{\lambda_3}e^{i\omega_2} + \sqrt{\lambda_3}e^{i\omega_3}, \\ \sqrt{\lambda_2}e^{i\omega_2} - \sqrt{\lambda_2}e^{i\omega_3}, -\sqrt{\lambda_1}e^{i\omega_1} + \sqrt{\lambda_1}e^{i\omega_4} \end{pmatrix}^T$ so that
$\left\langle \phi_1 \,\middle|\, \tilde{\phi}_1 \right\rangle = \frac{1}{4}\left(-\sqrt{\lambda_1\lambda_4}e^{-2i\omega_1} + \sqrt{\lambda_1\lambda_4}e^{-2i\omega_4} + \sqrt{\lambda_2\lambda_3}e^{-2i\omega_2} - \sqrt{\lambda_2\lambda_3}e^{-2i\omega_3}\right) = \frac{1}{4}\sum_j e^{-2i\omega_j}\mu_j = 0$. It follows that each pure state $|\phi_i\rangle$ has zero concurrence and is separable implying that ρ is also separable.

VI. THE GEOMETRY OF QUANTUM ENTANGLEMENT

A. The Geometry of a Single Qubit

The "quantum bit" or "qubit" plays the role of a "bit" in quantum computing [19,57,99-104] and constitutes a unit of quantum information [19,57,99-104]. It is represented by a state vector of a two-level quantum system. The representation space is, therefore, a two dimensional Hilbert space of the complex numbers and the basis vectors are usually chosen as $|0\rangle \equiv \begin{pmatrix} 1 & 0 \end{pmatrix}^T$ and $|1\rangle \equiv \begin{pmatrix} 0 & 1 \end{pmatrix}^T$, being the eigenvectors of the "spin" operator σ_3 in the direction of the $z$axis.

The fundamental difference between the "classical bit" and the "qubit" is that the former can have only two possible values viz. 0,1. The "qubit", on the other hand, can occur in an infinite number of states being the superposition of the "pure states" represented by the basis vectors. We can, therefore, express a qubit as a linear combination of the two basis states as $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$. $\alpha, \beta \in \mathbb{C}$ are the probability amplitudes whose squares provide a measure of the probability of the qubit being in state $|0\rangle$ and state $|1\rangle$ respectively. We must, therefore, have $|\alpha|^2 + |\beta|^2 = 1$

The state space of a single qubit quantum register admits a geometrical representation as a Bloch sphere [105]. This is established as follows:-

The state space of a two level quantum system is conventionally taken as the Hilbert space $\mathrm{H} \equiv \mathbb{C} \otimes \mathbb{C}$ [106]. Now, if two physical states $|\psi\rangle, |\phi\rangle$differ merely by a phase i.e. a complex number of unit magnitude i.e. $|\psi\rangle = e^{i\omega}|\phi\rangle$, then they represent the same physical state. It follows, therefore, that the proper space for a two level quantum system is the above Hilbert space $\mathrm{H} \equiv \mathbb{C}\otimes\mathbb{C}$quotiented by the equivalence relation $|\psi\rangle \sim |\phi\rangle$ iff $|\psi\rangle = e^{i\omega}|\phi\rangle$. It will, thus, be the projective Hilbert space created by this equivalence relation and may be defined as $\Pi(\mathrm{H}) = \mathrm{H}/\sim$. Sets of points in Hdiffering only in phase (i.e. the same quantum ray) will be mapped onto the same point in $\Pi(\mathrm{H})$. Thus, $\psi \mapsto \Pi(\psi) =: \frac{|\psi\rangle\langle\psi|}{\langle\psi|\psi\rangle}$. Now, the complex space $\mathbb{C}^2$ can be identified with the algebra of quaternions $\mathbb{Q}$ through the symplectic decomposition of an arbitrary quaternion $q \in \mathbb{Q}$ as $q = (q_0 + q_1 i) + j(q_2 - q_3 i) = q_\alpha + j\bar{q}_\beta$ $q_\alpha = (q_0 + q_1 i)$, $q_\beta = (q_2 + q_3 i) \in \mathbb{C}$. The set of normalized quaternions i.e. quaternions with unit modulus get mapped into a sphere S^3 embedded in $\mathbb{R}^4$. It, therefore, follows that normalized state vectors in $\mathbb{C}^2$ can also be canonically identified with the sphere S^3 embedded in $\mathbb{R}^4$. Quotienting $\mathbb{C}^2$by the equivalence relation $|\psi\rangle \sim |\phi\rangle$ iff $|\psi\rangle = e^{i\omega}|\phi\rangle$ to get the projective Hilbert space$\Pi(\mathrm{H}) = \mathrm{H}/\sim$, amounts to constructing the complex projective space $CP(1)$ i.e. $S^3/U(1)$ which yields the sphere S^2usually referred to in the literature as the Bloch sphere. In other words, the geometry of the two level quantum system (qubits) can be conveniently represented by the Bloch sphere.

B. The Hopf map

The identification of S^3in $\mathbb{R}^4$ with the Bloch sphere (S^2) is done through the well studied Hopf map. As a by product of the Hopf analysis, one also recovers the association between the geometry of qubits [59, 107-109] and quaternions [110-116]. To construct the Hopf map, we recall that the sphere S^3 is the group manifold of the special unitary group of matrices $SU(2)$i.e. matrices with unit determinant that is isomorphic to the symplectic group $Sp(1)$ of transformations that preserve the quaternionic form. Elements on S^3 can be expressed in terms of quaternions $q \equiv (z_\alpha, z_\beta)$ through the symplectic decomposition $q = z_\alpha + j\bar{z}_\beta$, $z_\alpha, z_\beta \in \mathbb{C}$ or equivalently by matrices $q_m = \begin{pmatrix} z_\alpha & z_\beta \\ -\bar{z}_\beta & \bar{z}_\alpha \end{pmatrix}$with $z_\alpha\bar{z}_\alpha + z_\beta\bar{z}_\beta = 1$ for, writing $z_\alpha = q_0 + iq_1$, $z_\beta = q_2 + iq_3$, we obtain $q_0^2 + q_1^2 + q_2^2 + q_3^2 = 1$. confirming that $q \equiv (z_\alpha, z_\beta)$ lies on the sphere S^3.

To obtain explicit expressions for the Hopf map, we make use of the canonical representation of the quaternion units by the well known Pauli matrices $\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ as $i \equiv -i\sigma_1$, $j \equiv -i\sigma_2$,$i \equiv -i\sigma_3$. In terms of these matrices, acting as the basis, the Hopf mapping is defined by $x = \pi(q) = \begin{pmatrix} \bar{z}_\alpha & \bar{z}_\beta \end{pmatrix} \sigma \begin{pmatrix} z_\alpha & z_\beta \end{pmatrix}^T$yielding $x = \left(\bar{z}_\beta z_\alpha + \bar{z}_\alpha z_\beta, i(\bar{z}_\beta z_\alpha - z_\beta\bar{z}_\alpha), |z_\alpha|^2 - |z_\beta|^2\right)$

$$= \left(2(q_0q_2 + q_1q_3), 2(q_0q_3 - q_1q_2), q_0^2 + q_1^2 - q_2^2 - q_3^2\right).$$

Let us take an element of the unitary group $U(1)$, say, $\varphi = \begin{pmatrix} \eta & 0 \\ 0 & \bar{\eta} \end{pmatrix} = \lambda I + \mu\sigma_3$. We, then, have $\pi(q\varphi) = (q\varphi)^\dagger \sigma q\varphi = \varphi^\dagger x\varphi = x$ confirming, thereby that $\pi(q) = \pi(q\varphi)$ for $\varphi \in U(1)$ and hence, establishing the projective nature of the Hopf map taking all elements of S^3 connected through a unitary transformation to a single image. The image set is confirmed to be S^2since $x^2 = 1$as can be easily verified.

Thus, the Hopf map creates a principal bundle structure for S^3 with the base manifold being S^2 and the fibres being circles S^1 (members of the unitary group $U(1)$.
To obtain the local charts and the transition functions for the Hopf map, we parameterize the sphere S^3 by the stereographic projection coordinates. Let (X, Y)be the stereographic projection coordinates of a point in the southern hemisphere $U_S$of S^2 from the North Pole. Consider a complex plane that contains the equator of S^2. Then, $Z = X + iY$ lies within the circle of unit radius on the plane. Further, from the standard expressions for stereographic coordinates, we have $Z = \frac{x_1+ix_2}{1-x_3} = \frac{q_0-iq_1}{q_2-iq_3} = \frac{\bar{z}_\alpha}{\bar{z}_\beta}$.
The projective nature of the Hopf map again manifests itself here as the invariance of Zunder the transformation $(z_\alpha, z_\beta) \to (\lambda z_\alpha, \lambda z_\beta)$ for $|\lambda| = 1$. Similarly, the stereographic coordinates (U, V) of a point in the northern hemisphere $U_N$with respect to the South Pole will be given by $W = U + iV = \frac{\bar{z}_\beta}{\bar{z}_\alpha}$.
We can, now, define the fibre bundle structure of the Hopf map. The local trivializations in the northern and southern hemisphere are respectively given by:-
(i) $\phi_N^{-1} : \pi^{-1}(U_N) \to U_N \times U(1)$ by $(z_\alpha, z_\beta) \mapsto \left(\frac{\bar{z}_\beta}{\bar{z}_\alpha}, \frac{z_\alpha}{|z_\alpha|}\right)$
(ii) $\phi_S^{-1} : \pi^{-1}(U_S) \to U_S \times U(1)$ by $(z_\alpha, z_\beta) \mapsto \left(\frac{\bar{z}_\alpha}{\bar{z}_\beta}, \frac{z_\beta}{|z_\beta|}\right)$
(Both these trivializations are well defined on the respective charts for, in the northern hemisphere $z_\alpha \neq 0$ and in the southern hemisphere $z_\beta \neq 0$).
(iii) On the equator, $x_3 = 0$ so that $|z_\alpha| = |z_\beta| = 2^{-1/2}$, whence, on the equator, the local trivializations become $\phi_N^{-1} : (z_\alpha, z_\beta) \mapsto \left(\frac{\bar{z}_\beta}{\bar{z}_\alpha}, \sqrt{2}z_\alpha\right)$ and $\phi_N^{-1} : (z_\alpha, z_\beta) \mapsto \left(\frac{\bar{z}_\alpha}{\bar{z}_\beta}, \sqrt{2}z_\beta\right)$ leading to the equatorial transition function $t_{NS} = \frac{z_\alpha}{z_\beta}$.

C. The Geometry of Two Qubit States & Quantum Entanglement

The Hopf map described above can easily be generalized to $\pi : S^7 \to S^4$. This motivates us to examine the geometry of a two qubit quantum state using the formalism of the Hopf map. However, when addressing multiple qubit states, one needs to carefully consider the issue of quantum entanglement. The "quaternions" again come in handy in studying the two qubit state.
The Hilbert space for the compound system Hwill be the tensor product of the individual Hilbert spaces H_A, H_B of the two qubits and the basis vectors will be the direct product of the bases of the two spaces. We can, therefore, write a pure state of a two qubit system as $|\Phi\rangle = \alpha|00\rangle + \beta|01\rangle + \chi|10\rangle + \delta|11\rangle$ where $|ij\rangle \equiv |i\rangle \otimes |j\rangle$, $|i\rangle \in H_A, |j\rangle \in H_B$, $\alpha, \beta, \chi, \delta \in \mathbb{C}$, $\alpha = \alpha_{Re} + i\alpha_{Im}$, $\beta = \beta_{Re} + i\beta_{Im}$, and $\delta = \delta_{Re} + i\delta_{Im}$, $|\alpha|^2 + |\beta|^2 + |\chi|^2 + |\delta|^2 = 1$. This normalization condition translates to a sphere S^7 embedded in $\mathbb{R}^8$. Now, if the two qubit state is a composition of two one qubit states, then it should be possible to write the composite state as the tensor product of the two single qubit states. Writing $|\phi\rangle_A = a_1|0\rangle_A + a_2|1\rangle_A$, $|\phi\rangle_B = b_1|0\rangle_B + b_2|1\rangle_B$, we have, for separable states $|\Phi\rangle = |\phi\rangle_A \otimes |\phi\rangle_B = a_1b_1|00\rangle + a_1b_2|01\rangle + a_2b_1|10\rangle + a_2b_2|11\rangle$ whence, the separability condition can be inferred as $\alpha\delta - \beta\chi = 0$.
To introduce the Hopf fibration $\pi : S^7 \to S^4$through the quaternions, we write the probability amplitudes $\alpha, \beta, \chi, \delta \in \mathbb{C}$ in the form of two quaternions using the symplectic decomposition as $q_1 = \alpha_{Re} + \alpha_{Im}i + \beta_{Re}j + \beta_{Im}k$ and $q_2 = \chi_{Re} + \chi_{Im}i + \delta_{Re}j + \delta_{Im}k$. Obviously, the normalization condition implies that $|q_1|^2 + |q_2|^2 = 1$. Parametrizing the sphere S^4as $\sum_{l=1}^{5} \xi_l^2 = 1$, we obtain the Hopf map $\pi : S^7 \to S^4$ by the mapping $\xi_1 = Q_0$, $\xi_2 = Q_1$,$\xi_3 = Q_2$, $\xi_4 = Q_3$ and $\xi_5 = \sqrt{\left(1 - |Q|^2\right)}$ where$\pi(q_1, q_2) = Q = Q_0 + Q_1 i + Q_2 j + Q_3 k = 2\,\overline{(q_1\bar{q}_2)}$. Explicit computation using the values of the quaternions q_1 and q_2 yield

$$\xi_1 = 2(\alpha_{Re}\chi_{Re} + \beta_{Re}\delta_{Re} + \alpha_{Im}\chi_{Im} + \beta_{Im}\chi_{Im})$$

$$\xi_2 = 2(\alpha_{Re}\chi_{Im} - \alpha_{Im}\chi_{Re} + \beta_{Re}\delta_{Im} - \beta_{Im}\delta_{Re})$$

$$\xi_3 = 2(\alpha_{Re}\delta_{Re} - \alpha_{Im}\delta_{Im} - \beta_{Re}\chi_{Re} + \beta_{Im}\chi_{Im})$$

$$\xi_4 = 2\left(\alpha_{Re}\delta_{Im} + \alpha_{Im}\delta_{Re} - \beta_{Re}\chi_{Im} - \beta_{Im}\chi_{Re}\right)$$

$$\xi_5 = 1 - 2\left|q_1 q_2\right|$$

The Hopf map $\pi : S^7 \to S^4$is equivalent to the mapping of S^7onto a fibre bundle with the base space being the unit sphere $S^4$and the fibres being spheres S^3(this is evidenced by the invariance of this map under the transformation $(q_1, q_2) \mapsto (\lambda q_1, \lambda q_2)$,$|\lambda| = 1$)

A perusal of the above expressions reveals an intriguing feature of the Hopf map. If the two qubit states are separable i.e. $\alpha\delta - \beta\chi = 0$, then $\xi_3 = \xi_4 = 0$and the base space reduces to S^2which is the Bloch sphere discussed in the earlier section of this manuscript. This Bloch sphere (the base space) constitutes the state space of one of the qubits of the two qubit separable system. The obvious question to be posed, then is – What about the state space of the other qubit of this separable system? A possible solution is to introduce a second Hopf map that fibres out the fibrings of the first Hopf map. As mentioned earlier the fibres of the map $\pi : S^7 \to S^4$ consist of spheres S^3attached to the base space S^4. By means of another Hopf map $\pi' : S^3 \to S^2$ we can further, fibrate the fibres of the first map into a base space (the two sphere S^2) and fibres (being the one dimensional sphere). This creates another Bloch sphere that can be considered as the state space of the second qubit in the two qubit separable composite system. It needs be emphasized here that such a construction is not permissible in an entangled system because of the non vanishing of the coordinates ξ_3, ξ_4.

VII. THE PHYSICS OF QUANTUM ENTANGLEMENT & EPR PARADOX

We present below, what can, at best, be called a "status report" on the present state of knowledge in regard to the "EPR paradox" [22] since the issue at hand is, perhaps, as much resolved as it was in 1935 when Einstein, Podolsky & Rosen published their seminal work in the Physical Review [22]. Although quantum entanglement is, now, well accepted among contemporary physicists as a physical phenomenon having unquestioned existence, a universally acceptable and non controversial explanation thereof continues to defy human endeavours. The issue is extremely deep rooted and goes to question the very foundations of quantum theory. In fact, the EPR thought experiment [22] continues to haunt the votaries of quantum mechanics even to this day, almost eight decades after its entre. In a nutshell, EPR contemplates to question the "completeness" of the quantum theory based on their objective criterion of physical realism stated thus [22]

"If, without in any way disturbing a system, we can predict with certainty (i.e. with probability equal to unity) the value of a physical quantity, then there exists an element of physical reality corresponding to this physical quantity".

They further claim that [22]

"The elements of physical reality cannot be determined by a priori philosophical considerations, but must be found by an appeal to results of experiments and measurements".

Applying this criterion of physical realism to two particles with spatial separation and represented by an entangled wavefunction, they conclude that the description of a quantum system by the wavefunction is incomplete [22].

Postulates of quantum mechanics mandate that a physical property of a system can be represented by an operator that acts on a quantum state (that is a vector in Hilbert space) thereby returning an eigenvalue of the operator when a measurement on the system is performed with a corresponding collapse of the wavefunction to the state returned by the measurement process. Furthermore, physical properties represented by non-commuting operators cannot be measured simultaneously and a measurement performed on the system to determine one of them disturbs the state of the system so as to destroy knowledge of the other [20].

On the basis of the above, EPR infer that either [22]

(a) "*the quantum mechanical description of reality given by the wavefunction is not complete; or*
(b) *when the operators corresponding to two physical quantities do not commute the two quantities cannot have simultaneous reality. For, if both of them had simultaneous reality-and thus definite values - these values would enter into the complete description, according to the condition of completeness. If, then, the wavefunction provided such a complete description of reality, it would contain these values; these would then be predictable. This not being the case, we are left with the alternatives stated.*"
Performing the following thought experiment, they, then, concluded that physical quantities represented by non commuting operators can have simultaneous reality thereby contradicting (b) so that "the wavefunction does not lend to a complete description of the quantum system" [22].
For the purpose, they consider a composite system consisting of two parts whose initial states are fully known and which interact for a finite period and thereafter move apart to attain a spatial separation to the extent that no classical interaction remains viable. We designate the wavefunction of the composite system byΨ that can be expanded in the eigenbasis$\{u_i\left(x_1\right)\}$ with respective eigenvalues $\{a_i\}$of an operator $\hat{A}$ representing a physical quantity$A$ as

$$\Psi\left(x_1, x_2\right) = \sum_{n=1}^{\infty} \psi_n\left(x_2\right) u_n\left(x_1\right)$$

where $\psi_n\left(x_2\right)$are respectively the expansion coefficients of the eigenexpansion of Ψ. If a measurement of A returns the value a_k then the wavefunction of the composite system $\Psi$is deemed to collapse to the value $\psi_k\left(x_2\right) u_k\left(x_1\right)$. This is so because, sequel to the measurement returning the value a_k the first system must necessarily update itself to the state $u_k\left(x_1\right)$ whence there is no choice left for the composite system but to take the value $\psi_k\left(x_2\right) u_k\left(x_1\right)$ so that the second system must attain $\psi_k\left(x_2\right)$. It is emphasized that the eigenvectors $\{u_i\left(x_1\right)\}$ are eigenvectors of the operator $\hat{A}$, that represents the property A so that if we now choose to measure a different property B with the operator representation $\hat{B}$(that does not commute with $\hat{A}$), we shall obtain a different set of eigenvectors $\{v_i\left(x_1\right)\}$ with eigenvalues $\{b_i\}$ and a corresponding eigenexpansion of Ψas

$$\Psi\left(x_1, x_2\right) = \sum_{n=1}^{\infty} \varphi_n\left(x_2\right) v_n\left(x_1\right)$$

so that if a measurement of B on the composite system returns $b_r$then the second system must necessarily collapse to the state $\varphi_r\left(x_2\right)$. Thus, as a consequence of the measurement of two different properties of the first system constituting the composite system, the second system relates to two different eigenfunctions viz. $\psi_k\left(x_2\right)$ and $\varphi_r\left(x_2\right)$. The important point is that, at the time of measurement the two systems are spatially separated with no interaction between them and [22] "*no real change can take place in the second system in consequence of anything that is done to the first system*". Thus, [22] "*it is possible to assign two different wavefunctions ($\psi_k\left(x_2\right)$ and $\varphi_r\left(x_2\right)$) to the same reality (the second system after the interaction with the first).*"
Now, it is possible that the two wavefunctions $\psi_k\left(x_2\right)$ and $\varphi_r\left(x_2\right)$ are the eigenfunctions of two non-commuting operators that relate to the physical quantities P, Q with corresponding eigenvalues p_k, q_rrespectively. It follows, then, that "*by measuring either A or Bwe are in a position to predict with certainty, and without in any way disturbing the second system, either the value of the quantity P(that is p_k) or the value of the quantity Q(that is q_r)*" so that "in the first case,Pmust be considered an element of reality and, in the second case Q, obviously contradicting the inference of the preceding paragraph that $\psi_k\left(x_2\right)$and$\varphi_r\left(x_2\right)$relate to the same reality (the second system after the interaction with the first). EPR thereby conclude that [22]
"*Previously we proved that either* (2.1) *the quantum-mechanical description of reality given by the wave function is not complete or* (2.2) *when the operators corresponding to two physical quantities do not commute the two quantities cannot have simultaneous reality. Starting, then with the assumption that the wave function does give a complete description of the physical reality we arrived at the conclusion that two physical quantities with noncommuting operators, can have simultaneous reality. Thus, the negation of (1) leads to the negation of the only other alternative* (2.2). *We are thus forced to conclude that the quantum-mechanical description of physical reality given by wave functions is not complete.*"

The EPR criterion of "physical reality" cited above (verbatim) warrants some special remarks. We have [117]:
(a) The element of "physical reality" is believed to exist before the measurement process is initiated i.e. the "physical reality" is not a consequence of the process of measurement. "Physical reality" exists if one can predict with certainty the outcome of the process of measurement.
(b) It is well accepted in the quantum theory that the measuring apparatus interacts with the measured system in the process of measurement. However, EPR mandate that the element of "physical reality" belongs to the measured system and not to the measuring apparatus.
(c) A very important point here is that EPR "physical reality" envisages prediction with "certainty" of the outcome of a measurement. This a very fundamental issue for its application implies that the physical system must necessarily be in an eigenstate of some Hermitian operator for only in such a case can one make predictions with certainty. This would, in turn, mean that states that are superpositions of eigenstates do not correspond to "physical reality". There is another related issue. The Wigner-Araki-Yanase theorem mandates the necessary existence of imperfect measurement if quantum theory is to explain all possible interactions.
(d) The EPR "physical reality" strongly emphasizes causality. In fact, the "element of reality" constitutes the cause of the outcome of measurement that is precisely predictable.
(e) The element of "physical reality" may not necessarily coincide with the measurement outcome. What is necessary is that the predictable (with certainty) outcome of measurement must necessarily relate to an underlying physical cause. However, such cause need not necessarily be known.
(f) The EPR criterion envisions the existence of an "arrow of time" directed from the past to the future so that a future act of measurement cannot retroactively affect the "physical reality" as existing in the past.
(g) Separability is key to the EPR criterion and essentially implies that the two constituent systems should be so far spatially apart that either one "is unable to create physical reality in the other system". An interesting conjecture is that objects that may be separated in three dimensional space could well be contiguous and, hence capable of mutual influence in $3N$dimensional "configuration space" of N interacting particles.
(h) Votaries of "counterfactual definiteness" claim that by associating elements of reality with observables that cannot be simultaneously measured, the criterion of "physical reality" assigns well defined values to such observables.
An analysis of the EPR experiment [22] leaves us with a situation in which at least one of the following must necessarily be false [117-118]:
(a) the completeness of statistical quantum mechanics;
(b) locality
(c) the independent reality of distant things.
Escape routes out of each of these possible falsehoods are [118]:-
(a) the postulation of the existence of "hidden variables" that account for the non-local correlations;
(b) a non-local interpretation of Quantum Mechanics e.g. Bohr's Copenhagen interpretation [119-120] that assumes that the position and momentum of a particle cannot be measured simultaneously with exactness for they are complementary properties (that cannot be measured by the same apparatus). In fact, Bohr and Heisenberg believed that the measuring apparatus interacted with the system under measurement in the process of measurement so that the properties of the system did not exist independently of the measuring apparatus. Non- locality becomes inherent to this interpretation, for the detection of a particle at one point in space would instantaneously collapse the wavefunction over all space in order that another observer at an arbitrarily distant spatial point may *not* observe the object at the second location. Put succinctly, the Copenhagen interpretation envisaged that (i) the measuring apparatus interacted with the system under measurement to the extent that measurement outcomes obtained by an experiment were created by the apparatus; (ii) the wavefunction ψrelated itself to the probabilities of possible measurement outcomes; (iii) upon measurements the wavefunction collapsed to reflect the measured value as returned by the measuring apparatus. Non-locality, then, manifests itself in step (iii) insofar as upon measurement the particle wavefunction (that may be spread over all space) undergoes instantaneous collapse so that a distant measurement of the same property may not yield a contradictory outcome;
(c) the absence of "physical reality" of distant objects i.e. that the "real world" exists only within one's own "past light cone". In fact, it is, now, believed that Einstein had visualized the possible existence of such an escape route in 1946 by stating that [118]

"*One can escape from this conclusion (that statistical quantum theory is incomplete) only either by assuming that the measurement of S_1(telepathically) changes the real situation of S_2 or by denying independent real situations as such to things which are spatially separated from each other. Both alternatives appear to me to be equally unacceptable*".

It is important to point out here that the non-locality intrinsic to the Copenhagen interpretation emanates causally from the *real* existence of two observers A for system $S_1$and $B$for system S_2. An intriguing issue (that was visualized by Einstein and in context whereof the preceding quote was made) is, what happens if there is only one real observer i.e. if we interpret the wavefunction as A's knowledge about the quantum system so that when a measurement is performed, it is A's perception of the wavefunction that collapses to the returned value of the measurement? Then, because it is only a change in A's knowledge of the system under consideration, his measurement and the consequential change in his perception of the wavefunction cannot affect the measurement outcome obtained by $B$and consequentially B's perception of the wavefunction. But such a scheme of things provides no defence against the possibility of the two observers returning different measurement outcomes for the same property at the same instant thereby making measurements observer dependent, and hence, questioning the very premises of "measurement". The only way out is, then, to deny the *real* existence of one of the observers, say B and to interpret that the wavefunction represents an individual's knowledge of the quantum state and *not* collective knowledge. Sequel to the seminal work of EPR, numerous attempts were made to absorb "Einsteinian localism" within the frame work of quantum mechanics. A seemingly discreet attempt to explain the existence of non-classical correlations among spatially removed particles was made by postulating the existence of "hidden variables". However, the theory fell flat in its infancy with experimental evidence vindicating the violation of Bell's inequalities (that are mathematical expressions derived on the premises that there exist "hidden variables" that result in the creation of entanglement). The escape route (a) is, therefore, lost and we are left with the choices of living in a world (a) which is intrinsically non-local or (b) in which there exists no independent reality of spatially separated events [121].

VIII. EPR & SPIN STATES (BOHM'S VERSION OF EPR EXPERIMENT)

Let us consider two independent systems Q_1& Q_2 represented respectively by the wavefunctions $\Psi_1(x_1)$ and $\Psi_2(x_2)$so that the composite system is expressible by the product wavefunctions $\Psi(x_1, x_2) = \Psi_1(x_1)\Psi_2(x_2)$. This factorizability of the composite wavefunction can be traced to the "assumed" independence of the constituent systems for, in such a case, from elementary probability, we must have $P(Q_1 = \alpha, Q_2 = \beta) = P(Q_1 = \alpha) P(Q_2 = \beta)$ whence $|\Psi(x_1, x_2)|^2 = |\Psi_1(x_1)|^2 |\Psi_2(x_2)|^2$. Introducing spin states, let the spin components along the three axes of Q_1 be $\left(s_x^1, s_y^1, s_z^1\right)$ and those of Q_2 be $\left(s_x^2, s_y^2, s_z^2\right)$respectively. Further let $\hat{S}_z^1|\psi_{z+}^1\rangle = \sigma_z|\psi_{z+}^1\rangle = +|\psi_{z+}^1\rangle$, $\hat{S}_z^1|\psi_{z-}^1\rangle = -|\psi_{z-}^1\rangle$, $\hat{S}_z^2|\psi_{z+}^2\rangle = \tau_z|\psi_{z+}^2\rangle = +|\psi_{z+}^2\rangle$, $\hat{S}_z^2|\psi_{z-}^2\rangle = -|\psi_{z-}^2\rangle$.
We can, then, write $|\psi_{z+}^1\rangle = (1,0)_1^T$, $|\psi_{z-}^1\rangle = (0,1)_1^T$, $|\psi_{z+}^2\rangle = (1,0)_2^T$, $|\psi_{z-}^2\rangle = (0,1)_2^T$. Using these eigenstates of $\hat{S}_z^1$, $\hat{S}_z^2$, we can construct the four factorizable states of the composite system as $|\psi_{z\pm}^1\rangle \otimes |\psi_{z\pm}^2\rangle$ and $|\psi_{z\pm}^1\rangle \otimes |\psi_{z\mp}^2\rangle$. However, as mentioned earlier, this set of four states does not exhaust all possible states of a composite system. As testimony of this, we consider the singlet state $|\Psi^-\rangle = \frac{1}{\sqrt{2}}\left(|\psi_{z+}^1\rangle \otimes |\psi_{z-}^2\rangle - |\psi_{z-}^1\rangle \otimes |\psi_{z+}^2\rangle\right)$ which can easily be shown to be non-factorizable in the form $|\phi^1\rangle \otimes |\phi^2\rangle$ in terms of the constituent system states $|\phi^1\rangle$ and $|\phi^2\rangle$ respectively. Furthermore, $\left(\hat{S}_z^1 + \hat{S}_z^2\right)|\Psi^-\rangle = 0$ showing that $|\Psi^-\rangle$ gives opposite results for measurements of the $z$component of the spins of Q_1& Q_2. Similarly,
$\Sigma^2|\Psi^-\rangle = \left[(\sigma_x + \tau_x)^2 + (\sigma_x + \tau_x)^2 + (\sigma_x + \tau_x)^2\right]|\Psi^-\rangle = (6 + 2\sigma.\tau)|\Psi^-\rangle = 0$ whence a measurement of the squared total spin of Q_1& Q_2 yields zero with certainty. Besides, the form of $|\Psi^-\rangle$ remains invariant under rotations for if $\sigma.n|\psi_{n\pm}^1\rangle = \pm|\psi_{n\pm}^1\rangle$ and $\tau.n|\psi_{n\pm}^2\rangle = \pm|\psi_{n\pm}^2\rangle$, then $|\Psi_n^-\rangle = \frac{1}{\sqrt{2}}\left(|\psi_{n+}^1\rangle \otimes |\psi_{n-}^2\rangle - |\psi_{n-}^1\rangle \otimes |\psi_{n+}^2\rangle\right)$.
Let us, now consider the state $|\Psi^+\rangle = \frac{1}{\sqrt{2}}\left(|\psi_{z+}^1\rangle \otimes |\psi_{z-}^2\rangle + |\psi_{z-}^1\rangle \otimes |\psi_{z+}^2\rangle\right)$ called the triplet state. This state has the properties (i) it is not factorizable; (ii) $\left(\hat{S}_z^1 + \hat{S}_z^2\right)|\Psi^+\rangle = 0$; (iii) $\Sigma^2|\Psi^+\rangle = 2\hbar^2|\Psi^+\rangle$; (iv) it is not rotationally invariant.

Now, we can write $\left|\psi^1_{z+}\right\rangle \otimes \left|\psi^2_{z-}\right\rangle = \frac{1}{\sqrt{2}}\left(\left|\Psi^+\right\rangle + \left|\Psi^-\right\rangle\right)$ and $\left|\psi^1_{z-}\right\rangle \otimes \left|\psi^2_{z+}\right\rangle = \frac{1}{\sqrt{2}}\left(\left|\Psi^+\right\rangle - \left|\Psi^-\right\rangle\right)$ from which it follows that a measurement of the squared total spin of particle pairs that are described by the states $\left|\psi^1_{z+}\right\rangle \otimes \left|\psi^2_{z-}\right\rangle$ shall produce the results zero and $2\hbar^2$ with equal probabilities. Similar is the case for pair of particles represented by $\left|\psi^1_{z-}\right\rangle \otimes \left|\psi^2_{z+}\right\rangle$.

It follows from the above that if we take a large ensemble of particles prepared in the state $\left|\Psi^-\right\rangle$ and perform measurement of the squared total spin on each of them, we shall invariably get the outcome zero (i.e. with certainty). On the contrary, if take an ensemble of particles consisting of equal numbers in respective states $\left|\psi^1_{z+}\right\rangle \otimes \left|\psi^2_{z-}\right\rangle$ and $\left|\psi^1_{z-}\right\rangle \otimes \left|\psi^2_{z+}\right\rangle$ and perform like measurements of squared total spin on each pair, we get the results zero and one with equal frequency thereby leading to the EPR paradox.

In view of the foregoing i.e. the EPR paradox and violation of Bell inequalities by composite systems we are left with little choice but to accept the existence of quantum entanglement as a physical phenomenon intrinsic to the quantum description of matter. Yet an explanation of the physics behind quantum entanglement continues to elude the human mind. The options available to the scientific community to wade out of this quagmire (on the basis of the present state of knowledge) have been succinctly presented by Selleri [117] as:

(a) A straightforward solution to the EPR paradox is to reject the EPR "reality criterion" on the premises that there exists no physical reality that is independent of its observation & measurement. This line of reasoning was put forth by Bohr [120] in his response to the EPR publication [22]. However, the EPR team had already envisioned such a stance while presenting the "thought" experiment as they write [22]

"One could object to this conclusion on the grounds that our criterion of reality is not sufficiently restrictive. Indeed, one would not arrive at our conclusion if one insisted that two or more physical quantities can be regarded as simultaneous elements of reality only when they can be simultaneously measured or predicted. On this point of view, since either one or the other, but not both simultaneously, of the quantities P and Q can be predicted, they are not simultaneously real. This makes the reality of P and Q depend upon the process of measurement carried out on the first system, which does not disturb the second system in any way. No reasonable definition of reality could be expected to permit this."

Niels Bohr [120], however, questioned this premise. It was his belief that there exists no reality "that is independent of measurement". In other words, things are real only when they are measured. In a sense, Bohr did not contradict the conclusions drawn by EPR on the basis of the premises assumed by them – it were the premises that, he believed, to be wrong. Specifically, he found the definition of "physical reality" as contemplated by EPR to be fundamentally flawed on account of the use of *"without in any way disturbing the system"*. It was his belief "that the measuring apparatus invariably interacted with the system under observation during the process of measurement" thereby disturbing the latter and, as such, the notion of "a physical measurement" without disturbing the observed system's condition was untenable. He attributed this "invariable disturbance" of the observed system during the measurement process due to its interaction with the measuring apparatus to the "very existence of the quantum of interaction". Indeed, he writes [120]

"Indeed the finite interaction between object and measuring agencies conditioned by the very existence of the quantum of action entails – because of the impossibility of controlling the reaction of the object on the measuring instrument if these are to serve their purpose- the necessity of the final renunciation of the classical ideal of causality and a radical revision of our attitude towards the problem of physical reality".

Explaining further, Bohr observes [120]

"From our point of view, we now see that the wording of the above mentioned criterion of physical reality proposed by Einstein, Podolsky and Rosen contains an ambiguity as regards the meaning of the expression "without in any way disturbing the system". Of course there is in a case like that just considered no question of a mechanical disturbance of the system under investigation during the last critical stage of the measuring procedure. But even at this stage there is essentially the question of an influence on the very conditions which define the possible types of predictions regarding the future behaviour of the system. Since these conditions constitute an inherent element of the description of any phenomenon to which the term physical reality can be properly attached we see that the argumentation of the mentioned authors does not justify their conclusion that quantum-mechanical description is essentially incomplete On the contrary this description as appears from the preceding discussion may be characterized as a rational utilization of all possibilities of unambiguous interpretation of measurement compatible with the finite and uncontrollable interaction between the objects and the measuring instruments in the field of quantum theory. In fact, it is only the mutual exclusion of any two experimental procedures, permitting the unambiguous definition of complementary physical quantities, which provides room for new physical laws, the coexistence of which

might at first sight appear irreconcilable with the basic principles of science. It is just this entirely new situation as regards the description of physical phenomena, that the notion of complementarity aims at characterizing".

Based on Bohr's line of reasoning, the EPR paradox automatically resolves itself for, now "an element of reality is necessarily associated with a precise act of measurement and there is no reality that exists independent of the process of measurement" and so there cannot simultaneously exist elements of physical reality corresponding to position and momentum for, one can never perform simultaneous measurement of position and momentum and, if there is no measurement, there exists no element of physical reality.

To bring home the point explicitly, we consider the entangled wavefunction $\psi\left(x_1 - x_2, p_1 + p_2, x_1, x_2\right)$and introduce apparatus Q_1 & P_1(Q_2 & P_2) capable of measuring the position and momentum of the two constituent systems 1 & 2 respectively. Then, Bohr's argument mandates that we may perform simultaneous measurements using Q_1&Q_2 to obtain precise positions of the systems 1 & 2 or use simultaneously the apparatus P_1&P_2 to obtain precise values of the momenta of 1 & 2. We may, however, never use Q_1&P_1 together simultaneously since these apparatus are mutually incompatible so that we can never measure the corresponding physical quantities simultaneously and there exists no physical reality corresponding to these physical quantities. Similarly we cannot useQ_2&P_2 together simultaneously [120].

(b) A second approach to the resolution of the EPR paradox is to acknowledge "retroaction in time" as a possible and acceptable physical phenomenon. The premise for such an approach, in the words of Rietdijk is [122]

"If we measure a conserved quantity that was known beforehand, no compensating change in the same quantity takes place in the measuring apparatus. Since this holds for all possible definite values of the relevant quantity, it is equally true for that quantity if it was unknown beforehand and becomes definite at the measurement".

Applying the above prescription to the spin of a particle, Rietdijk [122] claims that if the z spin component $s_z$of a particle $P$is known before its actual measurement to be $+\frac{1}{2}\hbar$ or $-\frac{1}{2}\hbar$, then a subsequent measurement thereof shall not entail any exchange of z angular momentum between P and the measuring apparatus, say A. Now, if we use the same apparatus to measure (in random order) the$s_z$of an ensemble of equal number of particles with known $s_z = +\frac{1}{2}\hbar$ and $s_z = -\frac{1}{2}\hbar$ respectively, the situation would be no different and no exchange of z angular momentum between the mixture and A would take place. Extrapolating further, if the$y$component of the spins (instead of the$z$component) of the particles constituting the ensemble are known to be $s_y = +\frac{1}{2}\hbar$ and $s_y = -\frac{1}{2}\hbar$ respectively or, for that matter, in any other direction, there would still be no compensating exchange of z angular momentum between the mixture and A when measurements are performed by A for s_z on the particles constituting the ensemble. The fallout of the above reasoning together with the mandated conservation of angular momentum is that (since there occurs no transfer of momentum between the measured particle and the apparatus) the measured particle's s_z would not have changed as a consequence of the process of measurement i.e. the particle's $s_z$must have had the value $s_z = \pm\frac{\hbar}{2}$ before it reached the measuring instrument. However, if, instead of measuring s_z, we chose to measure s_x or, for that matter, the spin in an arbitrary direction ni.e. s_n we would still have obtained $s_x = \pm\frac{\hbar}{2}$ or $s_n = \pm\frac{\hbar}{2}$, as the case may be. Now, it is cardinally true that the spin of a particle can have a definite value $\pm\frac{\hbar}{2}$ in at most one direction. It, therefore, must necessarily be true that the spin component in the direction to be measured would have adjusted itself beforehand i.e. before (at least) entering the measuring apparatus (or earlier) to the value $\pm\frac{\hbar}{2}$ (that is reported on measurement). Thus, there is a process that occurs retroactively in time informing the particle of the direction in which the measurement is going to take place thereby enabling it to orient its spin accordingly.

Building on the above premises, Rietdijk attempts to explain the EPR paradox as follows [122]:

1. We consider two diametrically oppositely moving spin $\frac{1}{2}$ particles P & Q that are emitted from a common source E. Let E'corresponding to the event Ebe the origin of a four dimensional coordinate system and points A' and B' be the points corresponding to the measuring event A and B in this coordinate system respectively so that $E'A'$ and $E'B'$ are respectively the world lines of particles P & Q.

2. Working in the inertial frame O of the apparatus, let us assume that the measurement event B' has occurred yielding the result $s_{zQ} = \frac{\hbar}{2}$. It follows that simultaneously with this discovery, we

infer that$s_{zP} = -\frac{\hbar}{2}$ with certainty. Hence, if the point A_1 on $E'A'$ is simultaneous with B', then, between the points A_1 and A' on $E'A'$, we must have $s_{zP} = -\frac{\hbar}{2}$ with certainty.

3. Let us, now, consider another inertial frame O'. Let C be a point in O' lying along $E'A'$ between A' and A_1. Let C in O' be simultaneous with C_1 in O' lying along $E'B'$ between B' and E'. Applying conservation of angular momentum in O', we obtain $s_{zC} + s_{zC_1} = 0$ where s_{zC} is P's $s_z$in O' and s_{zC_1} is Q's. Now, since C lies between A' and A_1 on $E'A'$and since by (i) $s_{zP} = -\frac{\hbar}{2}$ with certainty between A_1 and A' on $E'A'$, we must have $s_{zC} = -\frac{\hbar}{2}$ with certainty whence $s_{zC_1} = +\frac{\hbar}{2}$with certainty.

4. Proceeding similarly, we can identify another inertial frame $O''$and identify point C_2 on $E'A'$ in O'' that is simultaneous with C_1 (with reference to O''). Hence, we must have, by retracing the above logic, $s_{zC_2} = -\frac{\hbar}{2}$ in O''.

5. Proceeding iteratively as above, we can conclude that the values of s_z of P & Qmeasured at A' & B' were enshrined in them at the point of their very emission i.e. E'. Since, the z direction was chosen arbitrarily, isotropy of space leads to the inference that the two particles P & Q had the spin components $\pm\frac{\hbar}{2}$ in any arbitrarily chosen direction. However, as mentioned earlier, maxims of quantum mechanics mandate that it is impossible for spin components to have $\pm\frac{\hbar}{2}$ in all directions simultaneously. We, therefore, reach the same conclusion as elucidated in the previous paragraphs viz. that there exists a retroactive effect in time that enables the particles to orient their spin commensurate with the directions in which they are to be measured by the apparatus.

Further evidence of "time retroactivity" is provided by Rietdijk in [123-124]. Other approaches to explain the EPR paradox on similar lines using retroaction in time were conjectured by Costa de Beauregard [125-128] & Rayski [129].
(c) An extensive analysis of the quantum measurement process and its relationship to the EPR paradox and Bohr's Complementarity Principle was performed by Furry [130-131]. For the sake of completeness of this article we reproduce herein the salient features thereof. Adapting the von Neumann formalism, Furry defines the expectation value of an arbitrary observable $F$for a quantum system in a pure state represented by the wavefunction φas $\bar{F} = (\varphi, F\varphi) = \int^{d}$. If the system is in a pure state that is the linear combination of pure states φ_i i.e. $\varphi = \sum_i (w_i)^{1/2} \varphi_i$ then $\bar{F} = \left(\sum_i (w_i)^{1/2} \varphi_i, F \sum_i (w_i)^{1/2} \varphi_i\right)$. Furthermore, if the system is in a stateφ_i, which is a constituent of a statistical ensemble of quantum states consisting of states φ_i with respective probabilities w_i, then $\bar{F} = \sum_i w_i (\varphi_i, F\varphi_i)$. The measurement of observable F in relation to our quantum system returns one of the eigenvalues of F, say λ and the probability of this measurement outcome is $|(\varphi, \chi_\lambda)|^2$ if the system is in pure state φ, where $\chi_\lambda$is the eigenfunction of F corresponding to the eigenvalue λ. If the system is in the superposed pure state $\varphi = \sum_i (w_i)^{1/2} \varphi_i$, then the corresponding probability is $\left|\left(\sum_i (w_i)^{1/2} \varphi_i, \chi_\lambda\right)\right|^2$ and if it is in the mixed state, then the probability is $\sum_i w_i |(\varphi_i, \chi_\lambda)|^2$. Using the above relations, Furry could establish incompatibility between the entangled states and mixtures. His line of argument runs as follows:-
(i) An arbitrary state vector of a composite system consisting of two subsystems I and II can be written as $|\Psi\rangle = \sum_{i,j} c_{ij}|\phi\rangle|\varphi\rangle$. In a suitably chosen orthonormal basis, through Schmidt's decomposition, we can also write this state as $|\Psi\rangle = \sum_i d_i|e_i\rangle|f_i\rangle$ with a single nonzero coefficient d_i implying separable states and more than one nonzero d_i implying entangled states. Let the two basis sets $\{|e_i\rangle\}$ and $\{|f_i\rangle\}$ be eigensets of two Hermitian operators A & B (observables of the systems I and IIrespectively) so that $A|e_i\rangle = a_i|e_i\rangle$ and $B|f_i\rangle = b_i|f_i\rangle$. Now, if a measurement of the observable A made on the subsystem I of the composite system at time t_I returns the eigenvalue a_k, it necessary follows that the wavefunction of the composite system collapses to the eigenstate that corresponds to the eigenvalue a_k of system I of observable A i.e. to the state $|\Psi_k\rangle = d_k|e_k\rangle|f_k\rangle$ in the Schmidt basis so that any subsequent measurement performed on the system II at time $t > t_I$ of the observable B in the Schmidt basis must necessarily return b_k (the index k is the same). This implies that for all time $t > t_I$, the measurement outcome of B on II is certain and is equal to b_k.
(ii) Now, the subsystems I and II can be sufficiently spatially separated to the extent that no perturbation of the state of II could have occurred consequent to the measurement performed on I. This together with the fact that, as per (i) above, for all time $t > t_I$, the measurement outcome of B on II is certain

and is equal to b_k must imply that the system II must necessarily have been in the state corresponding to eigenvalue b_k of Bbefore the measurement of A on I was performed so that the state of II must have been $|f_k\rangle$ before the time instant t_I. This implies that the state vector of I even before time instant t_I would be $|e_k\rangle$ so that the state of the composite system even before t_I was$|e_k\rangle|f_k\rangle$. Hence, if we consider a statistical ensemble of of the composite system $I + II$ and perform the above measurement process on each constituent of the ensemble, we should obtain the state vectors of the composite system as $|e_1\rangle|f_1\rangle$ with probability $|d_1|^2$, $|e_2\rangle|f_2\rangle$ with probability $|d_2|^2$ and, in general, $|e_i\rangle|f_i\rangle$ with probability $|d_i|^2$. However, this goes contrary to our assumption that the state of the composite system is $|\Psi\rangle = \sum_i d_i|e_i\rangle|f_i\rangle = \sum_i d_i|\Psi_i\rangle$.

1. To unambiguously confirm that the two mathematical expressions above do not yield equivalent expressions for the same quantum system. We consider two new observables A'& B' in relation to the systems I and II respectively that are not compatible with A & Brespectively i.e. $[A, A'] \neq 0 \neq [B, B']$ so that they cannot be simultaneously assigned definite values by any measurement process. As in the previous case, we introduce complete orthonormal bases $\{|e'_i\rangle\}$, $\{|f'_i\rangle\}$ consisting of the eigenfunctions of A'& B' respectively so that $A'|e'_i\rangle = a'_i|e'_i\rangle$ and $B'|f'_i\rangle = b'_i|f'_i\rangle$. Then, the probability of the composite measurement A'on I and B' on II returning the eigenvalues a'_m and b'_nrespectively is given by $P_1(a'_m, b'_n) = |\langle\Psi \mid e'_m f'_n\rangle|^2 = |\sum_k d^*_k \langle\Psi_k \mid e'_m f'_n\rangle|^2$if we assume that the ensemble is described by $|\Psi\rangle = \sum_i d_i|e_i\rangle|f_i\rangle$. However, if we describe the system by the mixture of states $|e_i\rangle|f_i\rangle$ with probability $|d_i|^2$, the same probability is given by $P_2(a'_m, b'_n) = \sum_k |d_k|^2 |\langle\Psi_k \mid e'_m f'_n\rangle|^2$. It is very obvious that $P_1(a'_m, b'_n) \neq P_2(a'_m, b'_n)$ due to the presence of the interference terms $\sum_{k\neq k'} d^*_k d_{k'} \langle\Psi_k \mid e'_m f'_n\rangle\langle\Psi_{k'} \mid e'_m f'_n\rangle^*$thereby establishing the incompatibility of quantum mechanics with the proposition that *"it is possible to assign separate physical reality to a composite entangled quantum system"*.

2. An identical conclusion was obtained by Fortunata [132-133] by calculating the expectation value of the Hermitian projection operator $\Pi_\Psi = |\Psi\rangle\langle\Psi|$ in the two frameworks. Using $|\Psi\rangle = \sum_i d_i|e_i\rangle|f_i\rangle$, we easily obtain $\langle\Pi_\Psi\rangle = \langle\Psi|\Pi_\Psi|\Psi\rangle = 1$ since the orthogonal basis vectors are normalized. On the other hand, if we describe the system by a mixed state consisting of the ensemble of states $|e_i\rangle|f_i\rangle$ with probabilities $|d_i|^2$, then $\langle\Pi_\Psi\rangle = \sum_k |d_k|^2 \langle e_k f_k|\Pi_\Psi|e_k f_k\rangle = \sum_k |d_k|^2 \langle e_k f_k \mid \Psi\rangle\langle\Psi \mid e_k f_k\rangle$ since we must have $\sum_k |d_k|^2 = 1$ with the equality holding only when only one of the $d_k \neq 0$, which is, precisely the characterization of a pure state. In other words, for all mixed states, we must have $\langle\Pi_\Psi\rangle < 1$ thereby vindicating the incompatibility of the two frameworks once again.

(d) No discourse on the EPR experiment can be complete without reference to the "quantum potential" based causal interpretation of David Bohm. The authors, Bohm & Hiley [134], introduce the concept through a review of the single particle dynamics and, thereafter, extend the formalism to the many body problem. They consider an electron with well defined coordinates Xthat are functions of time t. Additionally, in line with the causal interpretation (that attributes a "wave-particle duality in interaction" to all fundamental constituents of matter), they also assign a wave field $\psi(X, t) = R\exp\left(\frac{iS}{\hbar}\right)$ with real R, S to the electron. The Schrodinger equation then becomes $\frac{\partial P}{\partial t} + \mathrm{div}\left(P\frac{\nabla S}{m}\right) = 0$ where $P = R^2 = \psi^*\psi$ and $\frac{\partial S}{\partial t} + \frac{(\nabla S)^2}{2m} + V + Q = 0$ where $Q = -\frac{\hbar^2}{2m}\frac{\nabla^2 R}{R}$. In the above analysis, V represents the usual classical potential whereas Q has been introduced by the authors as a new type of "potential" named "quantum potential". It is easy to see that the first equation of this set is a manifestation of conservation of probability density $P = \psi^*\psi$ with the probability current $j = \frac{P\nabla S}{m}$. However, the second equation needs some explanation. In the limit $\hbar \to 0$, it reduces to the Hamilton Jacobi equation for the motion of a particle under a classical potential V. The authors [134] follow this rationale for imparting a "potential" like characteristic to Q and term it a "quantum potential" that assumes significance at the microscopic level. An intuitive explanation of two key quantum processes viz. quantum interference and quantum barrier penetration is provided on the basis of this quantum potential, the former being explained as occurring at points where this potential assumes infinite values (thereby repelling the particles from points at which $R = |\psi| = 0$) and the latter in terms of the fluctuations of this potential so that there exist situations where it becomes negative enough to supercede the barrier potential V, thereby permitting penetration occasionally. It is pertinent to mention here that Bohm's framework for the one body case completely reproduces the classical dynamics in the limit $\hbar \to 0$. The many body problem adduces some more intriguing non-trivial features of the analysis. We consider a N body system consisting of particles of equal mass m. Writing the configuration space wavefunction $\Psi\left(X_1,...,X_n,t\right)$ as $\Psi = R\exp\left(\frac{iS}{\hbar}\right)$, we obtain from the Nparticle Schrdinger equation, $\frac{\partial P}{\partial t} + \sum_k \nabla_k.\left(P\frac{\nabla_k S}{m}\right) = 0$ where $P = \Psi^*\Psi$ and $\frac{\partial S}{\partial t} + \sum_k \frac{(\nabla_k S)^2}{m} + \sum_{i<j} V\left(x_i, x_j\right) + Q = 0$ with $Q = -\frac{\hbar^2}{2m}\sum_k \frac{\nabla_k^2 R}{R}$. The first of these equations is, again, probability conservation in the configuration space of the$N$particles while the second again reduces to Hamilton Jacobi in the classical limit. It is instructive, however, to explore the character of the quantum potential. The said potential is a function of the configuration space coordinates $Q \equiv Q\left(X_1,...,X_N\right)$ and is not localized to pairs of particles in the sense that the interaction between any pair depends on all the other particles. Another important feature is that $Q\left(X_1,...,X_N\right)$ does not, in general, result in a vanishing interaction between any pair i,j as $|X_i - X_j| \to \infty$ so that particles separated by large distances continue to show interactive behaviour. A third feature of this "quantum potential" is that it depends on $\Psi\left(X_1,...,X_n,t\right)$ i.e. on the "quantum state" of the composite system. On the basis of these postulates, Bohm & Hiley are able to explain away the EPR paradox. To do so, they consider a system of two particles and replace the EPR quantum state of the system $\Psi = \exp\left[\frac{ip(X_1+X_2)}{2}\right]\delta\left(X_1 - X_2 - a\right)$ by $\Psi = \exp\left[\frac{ip(X_1+X_2)}{2}\right] f\left(X_1 - X_2\right)$ where $f\left(X_1 - X_2\right)$is a real function that has a sharp peak at$X_1 - X_2 = a$. The former wavefunction can, obviously be approximated (to the desired accuracy) by the latter by enhancing the sharpness of the peak. The quantum potential corresponding to this wavefunction is $Q = -\frac{\hbar^2}{2m}\frac{\nabla^2 f(X_1-X_2)}{f(X_1-X_2)}$ which is apparently, non vanishing and which, in fact, does not vanish even in the limit $|X_1 - X_2| \to \infty$ thereby, leading to the conclusion that two particles that are spatially separated to the extent of forbidding any classical interaction do still have quantal relationships that depend on the quantum state of the system. Thus, the measurement of any observable on the first system results in a consequential transformation on the second system that is brought about by the non-vanishing quantum potential so that the EPR paradox stands resolved.

It needs to be emphasized strongly at this point that the approaches outlined above for the resolution of the EPR paradox are purely illustrative and, by no means, exhaustive. They have been chosen for inclusion in this review primarily on the grounds of their novelty, historical significance and their importance in providing direction for more refined attempts [135-156] to resolve the paradox.

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