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On The Memory Of Non-Locally Damped Harmonic Oscillator

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abstract

We investigate the equation of motion for damped oscillator with arbitrary time memory. We show that the classical dynamics which breaks down the local composition law still preserved the basic uncertainty relation. The propagator and the wavefunction of the system is also evaluated.

I. INTRODUCTION

Approximation methods such as perturbation theory have been used to evaluate solution time dependent Schrödinger equation [1-9]. Khandekar and Lawande [10] had evaluated the exact quantum theory of a classical force oscillator with a time-dependent frequency and a velocity-dependent damping term using path integral approach. However, the damped harmonic oscillator is a system displaying energy dissipation. There are different approach in describing these quantum dissipative system such as Caldirola-Kanai formation [11-12] and the canonical formation of classical mechanics to dissipative system through the non-Hamiltonian system using the double number of degrees of freedom proposed by Bateman [13.14]. Recently, the investigated the time-dependent Harmonic oscillator via a path integral approach with a modified Caldirola-Kanai Hamiltanian [15].

In the article, our aim will be to analyzed the equation of motion of this modified equation with an arbitrary time memory to [16]. Unlike the result of D. Chruseinski and J. Jurkowski [16] which corresponds to the asymptotic regime γt $\to\infty$, our analysis is of the short time regime γt $\to$0 and it corresponds to the notion of switch on the interaction adiabatically. In many body systems, one specified the Hamiltonian describing this system as [17].

$$\hat{H} = \hat{H}_0 + e^{-\gamma t}\hat{H}_1, \qquad (1)$$

At the regime γt $\to$0, the Hamiltonian $\hat{H}$ in Eq (1) becomes the full Hamiltonian of the interaction system and when γt $\to$0, the system is switch on and off adiabatically.

The modified Lagrangian of our model is [15].

$$L = e^{\sin\gamma t}\left[\frac{1}{2}m\dot{q}^2 - \frac{1}{2}m\omega^2(t)q^2\right] \qquad (2)$$

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where ω(t) is the time dependent frequency and γ is a damping factor. The equation of motion of the particle defined by Eq. (2) is

$$\ddot{q}(t) + \beta\gamma \cos\gamma(t - t_0)\dot{q} + \omega^2 q = 0 \qquad (3)$$

where t0 is an arbitrary constant and corresponds to the effective memory in the time past in addition to the local time t in the future. The difference between our modified model Eq. 3 with the standard damp oscillator of Caldirola-Kanai is the arbitrary initial moment of time responsible for the effective memory. As was observed by [16] changing the time past t0 will inevitably change Eq. (3) and hence the corresponding time evolution of the system. D. Kochan [18-19] constructed the quantum mechanical system with an arbitrary time t0 and shows that the damped quantum is governed by the non local Hamiltonian and that the standard composition of the transition amplitude (propagator) given as

$$K\left(q, t, q_0, t_0\right) = \int q_1 \left\langle q, t \right| \left. q_1, t_1 \right\rangle \left\langle q_1, t_1 \mid q_0, t_0 \right\rangle$$

is violated for a damped system of Eq. (3). We follow the approach [16] to investigate Eq. (3) which violated the standard composition relation Eq. (4) and compare our results with that of the standard Caldirola-Kanai system [11-12].

II. HAMILTONIAN AND THE CLASSICAL TRAJECTORY

The Hamiltonian of the system defined for the modified Lagrangian is defined as

$$H = \frac{\partial L}{\partial q}\dot{q} - L\left(q, \dot{q}, t\right)$$

$$= \frac{1}{2m} e^{-\sin\gamma(t-t_0)} p^2 + \frac{1}{2} m\omega^2 q^2 e^{\sin\gamma(t-t_0)} \qquad (4)$$

Equation (4) reduces to time-independent Hamiltonian in the limits γt→ 0 and Eq. (4) satisfies in a compact form the time-dependent Schrödinger equation

$$i\hbar \frac{\partial \psi}{\partial t}(q, t) = \hat{H}(t)\psi\left(q, t\right), \qquad (5)$$

where $\hat{H}(t)$is the time-dependent quantum Hamiltonian operator. The solution of the time-dependent Schrödinger equation of Eq. (5) is given by

$$\psi\left(q, t\right) = \sum_k A_k e^{-iE_k(t-t_0)} \phi(q, t_0) \qquad (6)$$

with Ak being the time-dependent expansion coefficient defined as

$$A_k(t) = e^{iE_k(t-t_0)} \langle \phi_k(t_0) \mid \psi(q, t) \rangle \qquad (7)$$

The path traversed by the particle defined by the equation of motion Eq. (3) is

$$q(t) = e^{-\cos\gamma(t-t_0)} \left[A e^{i\Omega t} + B e^{-i\Omega t}\right] \qquad (8)$$

where we have approximated the argument in Eq. (8) as

$$\sqrt{\gamma^2\cos^2\gamma t-4\omega^2}\simeq\sqrt{\gamma^2-4\omega^2}=2i\omega\sqrt{1-\frac{\gamma^2}{4\omega^2}}$$

and we set $\cos^2\gamma t\to 1$ and $\Omega=\omega\sqrt{1-\frac{\gamma^2}{4\omega^2}}$. The standard Caldirola-Kanai oscillator with variable mass $m(t)=me^{\frac{\gamma t}{m}}$ has the Hamiltonian of the form [20]

$$H=\frac{1}{2m}e^{-\frac{\gamma t}{m}}p^2+\frac{1}{2}m\omega^2e^{\frac{\gamma t}{m}}q^2 \tag{9}$$

and its classical equation of within solution takes the form [21]

$$\ddot{q}(t)+\frac{\dot{m}(t)}{m(t)}\dot{q}(t)+\omega^2(t)q(t)=0 \tag{10}$$

comparing Eq. (10) with Eq. (3) show that $\frac{\dot{m}(t)}{m(t)}=\gamma\cos\gamma(t,t_0)$ for our model and γ for Ref. [20]. The classical dynamics gene-rated by Eq. (10) with mass m(t)
defined as

$$m(t)=m\exp(\sin\gamma(t-t_0) \tag{11}$$

is given as

$$\begin{pmatrix} q(t)\\ p(t)\end{pmatrix}=\begin{pmatrix}\sigma_{qq} & \sigma_{qp}\\ \sigma_{pq} & \sigma_{pp}\end{pmatrix}\begin{pmatrix} q_0\\ p_0\end{pmatrix} \tag{12}$$

where the 2 × 2 matrix

$$T(t_1,t_0)=\begin{pmatrix}\sigma_{qq} & \sigma_{qp}\\ \sigma_{pq} & \sigma_{pp}\end{pmatrix}, \tag{13}$$

and reads for underdamped $\gamma<\omega$ as

$$\sigma_{qq}(t,t_0)=e^{-\gamma(t-t_0)\cos\gamma(t-t_0)}\left(\cos\Omega t+\frac{\gamma}{\Omega}\sin\Omega t\right),$$

$$\sigma_{qp}(t,t_0)=\frac{e^{-\gamma(t-t_0)\cos\gamma(t-t_0)}}{m\Omega}\sin\Omega(t-t_0)$$

$$\sigma_{qp}(t,t_0)=-\frac{\sin\Omega(t-t_0)e^{-\gamma(t-t_0)\cos\gamma(t-t_0)}}{\Omega}$$

$$\sigma_{pp}(t,t_0)=e^{\gamma(t-t_0)\cos\gamma(t-t_0)}\left(\cos\Omega(t-t_0)-\frac{\gamma}{\Omega}\sin\Omega(t-t_0)\right) \tag{14}$$

For the overdamped case $\gamma>\omega$, we have

$$\sigma_{qq}(t)=e^{-\gamma(t-t_0)\cos\beta\gamma(t-t_0)}\left(\cosh\Omega(t-t_0)+\frac{\gamma}{\Omega}\sinh\Omega(t-t_0)\right)$$

$$\sigma_{qp}(t) = \frac{e^{-\gamma(t-t_0)\cos\gamma(t-t_0)}}{m\Omega}\sin\Omega(t-t_0)$$

$$\sigma_{pq}(t) = \left(-\frac{m\omega^2}{\Omega}\sinh\Omega(t-t_0)\right)e^{\gamma(t-t_0)\cos\gamma(t-t_0)}$$

$$\sigma_{pp}(t) = e^{\gamma(t-t_0)\cos\gamma(t-t_0)}\left(\cosh\Omega(t-t_0) - \frac{\gamma}{\Omega}\sinh\Omega(t-t_0)\right) \qquad (15)$$

with the help of Eq. (13), we define the uncertain relation for the underdamped case as

$$\langle\Delta p(t)\Delta q(t-t_0)\rangle = \frac{\hbar}{2}\left[\sigma_{qq}^2\sigma_{pp}^2 - \frac{1}{4}m^2\gamma^4\sigma_{qp}^2\sigma_{pq}^2\right]^{\frac{1}{2}}$$

$$= \frac{\hbar\omega}{2\Omega}\left[1 + \frac{\gamma^2}{2\Omega^2}\sin^2\Omega(t-t_0) + \left(\frac{\gamma^2}{2\Omega^2} - 1\right)\sin^2 2\Omega(t-t_0)\right.$$

$$\left. + \left(\frac{\gamma^2}{2\Omega^2} + \frac{\gamma^4}{4\Omega^4}\right)\sin^4\Omega(t-t_0)\right]^{\frac{1}{2}} \qquad (16)$$

Similarly we obtain the uncertainty relation for the overdamped $\gamma > \omega$ as

$$\langle\Delta p(t-t_0)\Delta q(t-t_0)\rangle = \frac{\hbar}{2}\left[1 + \left\{\left(1 + \frac{\gamma^2}{2\Omega^2}\right)\sinh^2\Omega(t-t_0)\right.\right.$$

$$\left.\left. - \frac{\gamma^2}{2\Omega^2}\sin^2 2\Omega(t-t_0) + \left(1 + \frac{\gamma^2}{2\Omega^2} + \frac{\gamma^4}{4\Omega^4}\right)\sinh^4\Omega(t-t_0)\right\}\right]^{\frac{1}{2}} \qquad (17)$$

Here, Eq. (15) and Eq. (16) give the minimum uncertainty relation. The formulas derived reduces to those of the simple Harmonic oscillator (SHO) when $\gamma = 0$. At t = t0 and (t − t0)→∞ the minimum uncertainty relation is preserved as

$$\langle\Delta p(0)\Delta q(0)\rangle = \langle\Delta p(\infty)\Delta q(\infty)\rangle = \frac{\hbar}{2} \qquad (18)$$

Even though our classical dynamics breaks the local composition law [16],

$$\sigma(t_2, t_{0)} = \sigma(t_2, t_1)o\sigma(t, t_0) \qquad (19)$$

for t0 ≤ t1 ≤ t2 , but the classical dynamics still preserved the basic minimum uncertainty as in Eq. (17).

III. PROPAGTOR AND THE WAVE-FUNCTION

The Feynman path integral takes the form [21]

$$K(q,t;q_0,t_0)=\int_{q_0=q_0(t_0)}^{q=q(t)} Dq(t)e^{\frac{i}{\hbar}S[q(t)]}$$

$$=\int Dq(t)e^{\frac{i}{\hbar}\int L[q,\dot{q}]dt}, \qquad (20)$$

and the propagator in related to the time-dependent Schrödinger wave function as [15]

$$K(q,t;q_0,t_0)=\left[\frac{i}{2\pi\hbar}\frac{\partial^2}{\partial q_0\partial q}s_{cl}\left[q,t;q_0,t_0\right]\right]^{\frac{1}{2}} e^{\frac{i}{\hbar}}s_{cl}\left[q,t;q_0,t_0\right] \qquad (22)$$

where

$$F(q,t;q_0,t_0)=\left[\frac{i}{2\pi\hbar}\frac{\partial^2}{\partial q_0\partial q}s_{cl}\left[q,t;q_0,t_0\right]\right]^{\frac{1}{2}}, \qquad (23)$$

is the prefactor referred to as the Van Vleek-Pauli Morete determinant [22-23].
Using Eqs. (2, 11, 20 − 23), the damped Harmonic oscillator takes an explicit form as [24]

$$K(q,t;q_0,t_0)=\left(\frac{m\Omega e^{\sin\gamma(t_1-t_0)}}{2\pi i\hbar\sin\Omega(t-t_0)}\right)^{\frac{1}{2}}$$

$$\times\exp\left[\frac{im}{2\hbar}\left(aq^2+2bq_0^2+2qq_0c\right)\right], \qquad (24)$$

where we obtain the coefficients a, b, c as [24]

$$a=\left(-\frac{\gamma}{2}+\Omega\cot\Omega(t-t_0)\right)e^{\sin\gamma(t-t_0)} \qquad (25)$$

$$b=\left(\frac{\gamma}{2}+\Omega\cot\Omega(t-t_0)\right) \qquad (26)$$

$$c=\left(-\frac{\Omega}{\sin\Omega(t-t_0)}e^{\sin\gamma(t-t_0)}\right) \qquad (27)$$

substituting Eqs. (25 − 27) in Eq. (24) yield [16, 25]

$$K(q,t;q_0,t_0)=\left(\frac{m\Omega e^{\sin\gamma(t_1-t_0)}}{2\pi i\hbar\sin\Omega(t-t_0)}\right)^{\frac{1}{2}}$$

$$\times\exp\left[\frac{im}{4\hbar}\left(\gamma\left(q_0^2e^{\sin\gamma t_0}-e^{\sin\gamma t}q^2\right)+\frac{2\Omega}{\sin\gamma(t-t_0)}\right[\right.$$

$$\left[\left(q_0^2 e^{\sin\gamma t_0} + e^{\sin\gamma t} q^2\right)\right]\cos\Omega(t-t_0) - 2qq_0 e^{\sin\gamma(t+t_0)}\Big)\Big] \qquad (28)$$

The time-independent wave function is obtained from Eq. (28) as [15, 24 – 26]

$$\psi_n(q,t,t_0) = \int_{-\infty}^{\infty} dq_0 k(q,t,q_0,t_0)\psi\left(q,0\right), \qquad (29)$$

$$= \frac{N}{(2^n n!)^{\frac{1}{2}}}\exp\left\{-i\left[\left(n+\frac{1}{2}\right)\cot^{-1}\left(\frac{\gamma}{2\Omega}+\cot\Omega(t-t_0)\right)\right]\right\}$$

$$\times H_n\left[Dq\right]\exp\left[-A(q-q_0)^2\right] \qquad (30)$$

where D, A and N take the values

$$D(t) = \frac{\alpha e^{\sin\gamma(t+t_0)}}{\eta(t)\sin\Omega(t-t_0)}, \qquad (31)$$

with

$$\eta^2(t) = \frac{\gamma^2}{4\Omega^2} + \frac{\gamma}{\Omega}\cos\Omega(t-t_0) + \cos ec^2\Omega(t-t_0) \qquad (32)$$

$$A(t) = \frac{m\Omega}{2\hbar}e^{\sin\gamma(t_-t_0)}\left[\frac{1}{\eta^2(t)\sin^2\Omega(t-t_0)}\right.$$

$$\left. +i\left(\frac{\gamma}{2\Omega} - \cot\Omega(t-t_0) + \frac{\frac{\gamma}{2\Omega}+\cot\Omega(t-t_0)}{\eta^2(t)\sin^2\Omega(t-t_0)}\right)\right] \qquad (33)$$

and

$$N(t) = \left(\frac{m\Omega}{\pi\hbar}\right)^{\frac{1}{4}}\frac{e^{\sin\frac{\gamma}{2}(t+t_0)}}{\eta(t)\sqrt{\sin\Omega(t-t_0)}} \qquad (34)$$

In order to determine how a Gaussian wavepacket evolves for the propagator, we consider the Gaussian wavepacket of the form [16, 27]

$$\psi_0(q,t_0) = N(t)e^{-\frac{(q-q_0)^2}{\gamma(t-t_0)}} \qquad (35)$$

and we find the probability $|\psi_0(q,t_0)|^2$ as

$$|\psi_0(q,t_0)|^2 = \frac{1}{\sqrt{2\pi\sigma_t^2}}\exp\left[-\frac{\left[q-q_0e^{-\sin\gamma t}\left(\cos\Omega t+\frac{\gamma}{2\Omega}\sin\Omega t\right)\right]^2}{2\sigma_t^2}\right], \qquad (36)$$

where

$$\sigma_t^2 = \sigma_0^2 e^{-\sin\gamma t}\left[\left(\cos\Omega t + \frac{\gamma}{2\Omega}\sin\Omega t\right)^2 + \left(\frac{\hbar\cos\gamma(t-t_0)}{m_0\sigma_0\gamma}\right)^2\right] \qquad (37)$$

The expectation value of q is obtained from Eq. (36) as

$$\langle q\rangle = \int_{-\infty}^{\infty} q\,|\psi|^2\,dq$$

$$= q_0 e^{-\sin\gamma t}\left(\cos\Omega(t-t_0) + \frac{\gamma}{2\Omega}\sin\Omega(t-t_0)\right) \qquad (38)$$

The variance $\sigma_t^2(t-t_0)$ and the expectation value $\langle q(t-t_0)\rangle$ reduces as follows for t→t0;

$$\sigma_t^2(t-t_0) = \sigma_0^2\left[1 + \left(\frac{\hbar}{m_0\sigma_0\gamma}\right)^2\right], \qquad (39)$$

and

$$\langle q(t-t_0)\rangle = q_0 \qquad (40)$$

IV. CONSLUSION

We have constructed the propagator for a damped quantum mechanical systems with an arbitrary time responsible for effective memory. The classical dynamics of this system is used to evaluate the uncertainty relation. Even though this system violates the basic composition law, the system still preserved the minimum uncertainty relations.
As a check, using Eq. (14), we have

$$\sigma\left(t,t_0\right) \neq \sigma\left(t,t_1\right) o\sigma\left(t_1,t_0\right) \qquad (41)$$

for *t0* ≤ *t1* ≤ *t*. But the composition law holds for standard Caldirola-Kanai oscillator as

$$\sigma^{ck}\left(t,t_0\right) = \sigma^{ck}\left(t,t_1\right) o\sigma^{ck}\left(t_1,t_0\right) \qquad (42)$$

As we observed before our model holds for γt→0. The wavepacket evolution Eq. (36) depends explicitly on the standard deviation and (t-t0). We conclude that the minimum value of the variance and the certain expectation is obtained when (t-t0)→0.

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