



Some properties of Newton polygons of polynomial ordinary differential equations

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abstract

We show in this paper some properties of the Newton polygon of a polynomial ordinary differential equation. We give the relation between the Newton polygons of a differential polynomial and its partial derivatives. Newton polygons of sums and evaluations of differential polynomials are also described.

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I. INTRODUCTION

Newton polygons construct a useful tool for solving polynomial ordinary differential equations. Earlier, Briot and Bouquet¹ used Newton polygon methods for solving first order and first degree ordinary differential equations. Recently, Grigoriev and Singer⁸, Cano^{2,3} and Della Dora et. al.⁶ described Newton polygon algorithms for computing Puiseux series and series with real exponents as solutions of polynomial differential equations. Newton polygons are also useful for computing Newton-Puiseux expansions for the roots of polynomials and for factoring polynomials over fields of formal power series^{4,5}.

In this paper, we describe some useful properties of Newton polygons of differential polynomials. Section II gives the definition of the Newton polygon of a polynomial ordinary differential equation. Section III describes the relation between the Newton polygons of a differential polynomial and its partial derivatives. Section IV gives the Newton polygons of the sum of two differential polynomials. Section V establishes Newton polygon of different evaluations of differential polynomials.

Let K be a field and \overline{K} be an algebraic closure of K . Let L and \mathcal{L} be the two following fields:

$$L = \cup_{\nu \in \mathbb{N}^*} K((x^{\frac{1}{\nu}})), \quad \mathcal{L} = \cup_{\nu \in \mathbb{N}^*} \overline{K}((x^{\frac{1}{\nu}}))$$

which are the fields of fraction-power series of x over K (resp. \overline{K}), i.e., the fields of Puiseux series of x with coefficients in K (resp. \overline{K}). Each element $\psi \in L$ (resp. $\psi \in \mathcal{L}$) can be represented in the form $\psi = \sum_{i \in \mathbb{Q}} c_i x^i$, $c_i \in K$ (resp. $c_i \in \overline{K}$). The order of ψ is defined by $ord(\psi) := \min\{i \in \mathbb{Q}, c_i \neq 0\}$. The fields L and \mathcal{L} are differential fields with the differentiation operator

$$\frac{d}{dx}(\psi) = \sum_{i \in \mathbb{Q}} i c_i x^{i-1}.$$

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Let y_0, \dots, y_n be new variables algebraically independent over K and $F(y_0, \dots, y_n)$ be a polynomial in the variables y_0, \dots, y_n with coefficients in L . This polynomial defines an ordinary differential equation $F(y, \frac{dy}{dx}, \dots, \frac{d^n y}{dx^n}) = 0$ which will be denoted by $F(y) = 0$. We can write F in the form:

$$F = \sum_{i \in \mathbb{Q}, \alpha \in A} f_{i,\alpha} x^i y_0^{\alpha_0} \dots y_n^{\alpha_n}, \quad f_{i,\alpha} \in K$$

where $\alpha = (\alpha_0, \dots, \alpha_n)$ belongs to a finite subset A of \mathbb{N}^{n+1} . The order of F is defined by $ord(F) := \min \{i \in \mathbb{Q}; f_{i,\alpha} \neq 0 \text{ for a certain } \alpha\}$. We define the degree of F w.r.t. x by $deg_x(F) = \max \{i \in \mathbb{Q}; f_{i,\alpha} \neq 0 \text{ for a certain } \alpha\}$ (it can be equal to $+\infty$).

II. NEWTON POLYGONS OF POLYNOMIAL DIFFERENTIAL EQUATIONS

We define the Newton polygon of F as follow. For every couple $(i, \alpha) \in \mathbb{Q} \times A$ such that $f_{i,\alpha} \neq 0$ (i.e., every existing term in F), we mark the point

$$P_{i,\alpha} := (i - \alpha_1 - 2\alpha_2 - \dots - n\alpha_n, \alpha_0 + \alpha_1 + \dots + \alpha_n) \in \mathbb{Q} \times \mathbb{N}.$$

We denote by $P(F)$ the set of all the points $P_{i,\alpha}$. The convex hull of these points and the point $(+\infty, 0)$ in the plane \mathbb{R}^2 is denoted by $\mathcal{N}(F)$ and is called the Newton polygon of the differential equation $F(y) = 0$ in the neighborhood of $x = 0$. If $deg_{y_0, \dots, y_n}(F) = m$, then $\mathcal{N}(F)$ is located between the two horizontal lines $y = 0$ and $y = m$. For each $(a, b) \in \mathbb{Q}^2 \setminus \{(0, 0)\}$, we define the set

$$N(F, a, b) := \{(u, v) \in P(F), \forall (u', v') \in P(F), \quad au' + bv' \geq au + bv\}.$$

A point $P_{i,\alpha} \in P(F)$ is a vertex of the Newton polygon $\mathcal{N}(F)$ if there exist $(a, b) \in \mathbb{Q}^2 \setminus \{(0, 0)\}$ such that $N(F, a, b) = \{P_{i,\alpha}\}$. We remark that $\mathcal{N}(F)$ has a finite number of vertices. A pair of different vertices $e = (P_{i,\alpha}, P_{i',\alpha'})$ forms an edge of $\mathcal{N}(F)$ if there exist $(a, b) \in \mathbb{Q}^2 \setminus \{(0, 0)\}$ such that $e \subset N(F, a, b)$. We denote by $E(F)$ (resp. $V(F)$) the set of all the edges e (resp. all the vertices p) of $\mathcal{N}(F)$ for which $a > 0$ and $b \geq 0$ in the previous definitions. It is easy to prove that if $e \in E(F)$, then there exists a unique pair $(a(e), b(e)) \in \mathbb{Z}^2$ such that $GCD(a(e), b(e)) = 1$, $a(e) > 0$, $b(e) \geq 0$ and $e \subset N(F, a(e), b(e))$ where GCD is an abbreviation of "Greatest Common Divisor". By the inclination of a line we mean the negative inverse of its geometric slope. If $e \in E(F)$, we can prove that the fraction $\mu_e = \frac{b(e)}{a(e)} \in \mathbb{Q}$ is the inclination of the straight line passing through the edge e . If $p \in V(F)$ and $N(F, a, b) = \{p\}$ for a certain $(a, b) \in \mathbb{Q}^2 \setminus \{(0, 0)\}$, then the fraction $\mu = \frac{b}{a} \in \mathbb{Q}$ is the inclination of a straight line which intersects $\mathcal{N}(F)$ exactly in the vertex p .

For each edge $e \in E(F)$, we define the univariate polynomial (in a new variable Z)

$$H_{(F,e)}(Z) = \sum_{P_{i,\alpha} \in N(F,a(e),b(e))} f_{i,\alpha} Z^{\alpha_0 + \alpha_1 + \dots + \alpha_n} (\mu_e)_1^{\alpha_1} \dots (\mu_e)_n^{\alpha_n} \in K[Z],$$

where $(\mu_e)_k := \mu_e(\mu_e - 1) \dots (\mu_e - k + 1)$ for any positive integer k . We call $H_{(F,e)}(Z)$ the *characteristic* polynomial of F associated to the edge $e \in E(F)$. Its degree is at most $m = deg_{y_0, \dots, y_n}(F)$.

If $\psi \in \mathcal{L}$ is a solution of the differential equation $F(y) = 0$ such that $ord(\psi) = \mu$, i.e., ψ has the form $\psi = \sum_{i \in \mathbb{Q}, i \geq \mu} c_i x^i$, $c_i \in \overline{K}$, then there exists an edge $e \in E(F)$ such that $\mu_e = \mu$ and $H_{(F,e)}(c_\mu) = 0$, i.e. c_μ is a root of the polynomial $H_{(F,e)}$ in \overline{K} . This condition is called a necessary initial condition to have a solution of $F(y) = 0$ in the form of ψ (see Lemma 1 of³). Namely, $H_{(F,e)}(c_\mu)$ is equal to the coefficient of the lowest term in the expansion of $F(\psi(x))$ with indeterminates μ and c_μ .

For each vertex $p = (u, v) \in V(F)$, let $\mu_1 < \mu_2$ be the inclinations of the adjacent edges at p in $\mathcal{N}(F)$. It is easy to prove that for all rational number $\mu = \frac{b}{a}$, $a \in \mathbb{N}^*$, $b \in \mathbb{N}$ such that $N(F, a, b) = \{p\}$, we have $\mu_1 < \mu < \mu_2$. We associate to p the polynomial

$$h_{(F,p)}(\mu) = \sum_{P_{i,\alpha} = p} f_{i,\alpha} (\mu)_1^{\alpha_1} \dots (\mu)_n^{\alpha_n} \in K[\mu],$$

which is called the *indicial* polynomial of F associated to the vertex p (here μ is considered as an indeterminate). Let $H_{(F,p)}(Z) = Z^v h_{(F,p)}(\mu)$ defined as above for edges $e \in E(F)$.

Remark 1 et $p = (u, v) \in V(F)$ and e be the edge of $\mathcal{N}(F)$ descending from p , then $h_{(F,p)}(\mu_e)$ is the coefficient of the monomial Z^v in the expansion of the characteristic polynomial of F associated to e .

III. NEWTON POLYGONS OF PARTIAL DERIVATIVES OF DIFFERENTIAL POLYNOMIALS

Write F in the form

$$F = F_0 + \dots + F_m$$

where $m = \deg_{y_0, \dots, y_n}(F)$ and $F_s = \sum_{i \in \mathbb{Q}, |\alpha|=s} f_{i,\alpha} x^i y_0^{\alpha_0} \dots y_n^{\alpha_n}$ is the homogeneous part of F of degree s with respect to the indeterminates y_0, \dots, y_n , $\alpha = (\alpha_0, \dots, \alpha_n) \in A$ and $|\alpha| = \alpha_0 + \dots + \alpha_n$ is the norm of α . Then the ordinate of any point of $P(F_s)$ is equal to s and

$$P(F) = \cup_{0 \leq s \leq m} P(F_s).$$

Let $0 \leq j \leq n$. If there exists an integer $k \geq 1$ such that for all $1 \leq s \leq m$ (such that $F_s \neq 0$), $D_{s,j} := \deg_{y_j}(F_s) \geq k$ then we can easily prove that $P(\frac{\partial^k F}{\partial y_j^k})$ is the translation of $P(F)$ defined by the point $(kj, -k)$, i.e.,

$$P\left(\frac{\partial^k F}{\partial y_j^k}\right) = P(F) + \{(kj, -k)\} \text{ and then } \mathcal{N}\left(\frac{\partial^k F}{\partial y_j^k}\right) = \mathcal{N}(F) + \{(kj, -k)\}.$$

For any $(a, b) \in \mathbb{Q}^2 \setminus \{(0, 0)\}$, we have

$$N\left(\frac{\partial^k F}{\partial y_j^k}, a, b\right) = N(F, a, b) + \{(kj, -k)\}.$$

Thus the edges of $\mathcal{N}\left(\frac{\partial^k F}{\partial y_j^k}\right)$ are exactly the translation of those of $\mathcal{N}(F)$. For each $e \in E\left(\frac{\partial^k F}{\partial y_j^k}\right)$, its characteristic polynomial is

$$H_{\left(\frac{\partial^k F}{\partial y_j^k}, e\right)}(Z) = \sum_{P_{i,\alpha} \in N(F,a(e),b(e))} f_{i,\alpha} Z^{\alpha_0 + \alpha_1 + \dots + \alpha_n - k} (\alpha_j)_k (\mu_e)_1^{\alpha_1} \dots (\mu_e)_j^{\alpha_j - k} \dots (\mu_e)_n^{\alpha_n} \in K[Z].$$

For each $p \in V\left(\frac{\partial^k F}{\partial y_j^k}\right)$, its indicial polynomial is

$$h_{\left(\frac{\partial^k F}{\partial y_j^k}, p\right)}(\mu) = \sum_{P_{i,\alpha} = p} f_{i,\alpha} (\alpha_j)_k (\mu_e)_1^{\alpha_1} \dots (\mu_e)_j^{\alpha_j - k} \dots (\mu_e)_n^{\alpha_n} \in K[\mu].$$

Let $0 \leq j_1 \neq j_2 \leq n$. If there exist integers $k_1, k_2 \geq 1$ such that for all $1 \leq s \leq m$, $D_{s,j_2} \geq k_2$ and $\deg_{y_{j_1}}\left(\frac{\partial^{k_2} F_s}{\partial y_{j_2}^{k_2}}\right) \geq k_1$ then

$$P\left(\frac{\partial^{k_1+k_2} F}{\partial y_{j_1}^{k_1} \partial y_{j_2}^{k_2}}\right) = P(F) + \{(k_1 j_1 + k_2 j_2, -k_1 - k_2)\}$$

and then

$$\mathcal{N}\left(\frac{\partial^{k_1+k_2} F}{\partial y_{j_1}^{k_1} \partial y_{j_2}^{k_2}}\right) = \mathcal{N}(F) + \{(k_1 j_1 + k_2 j_2, -k_1 - k_2)\}.$$

For any $(a, b) \in \mathbb{Q}^2 \setminus \{(0, 0)\}$, we have

$$N\left(\frac{\partial^{k_1+k_2} F}{\partial y_{j_1}^{k_1} \partial y_{j_2}^{k_2}}, a, b\right) = N(F, a, b) + \{(k_1 j_1 + k_2 j_2, -k_1 - k_2)\}.$$

For each $e \in E\left(\frac{\partial^{k_1+k_2} F}{\partial y_{j_1}^{k_1} \partial y_{j_2}^{k_2}}\right)$, its characteristic polynomial is $H\left(\frac{\partial^{k_1+k_2} F}{\partial y_{j_1}^{k_1} \partial y_{j_2}^{k_2}}, e\right)(Z) =$

$$\sum_{P_{i,\alpha} \in N(F,a(e),b(e))} f_{i,\alpha} Z^{\alpha_0+\alpha_1+\dots+\alpha_n-k_1-k_2} (\alpha_{j_1})_{k_1} (\alpha_{j_2})_{k_2} (\mu_e)_1^{\alpha_1} \dots (\mu_e)_{j_1}^{\alpha_{j_1}-k_1} \dots (\mu_e)_{j_2}^{\alpha_{j_2}-k_2} \dots (\mu_e)_n^{\alpha_n}.$$

Theorem 1 Let $k \geq 1$ be an integer such that for all $1 \leq s \leq m$ (such that $F_s \neq 0$) and for all $0 \leq j \leq n$ we have $D_{s,j} \geq k$. For any $e \in E(F)$, the k -th derivative of $H_{(F,e)} \in K[Z]$ is given by the formula

$$H_{(F,e)}^{(k)}(Z) = \sum_{0 \leq k_0, \dots, k_n \leq n} (\mu_e)_1^{k_1} \dots (\mu_e)_n^{k_n} H\left(\frac{\partial^k F}{\partial y_0^{k_0} \dots \partial y_n^{k_n}}, e\right)(Z).$$

where the sum ranges over all the partitions (k_0, \dots, k_n) of k , i.e., $k_0 + \dots + k_n = k$.

Proof. By induction on k taking into account the above discussion. \square

IV. NEWTON POLYGONS OF SUMS OF DIFFERENTIAL POLYNOMIALS

Let F and G be two differential polynomials of degree less or equal than m . Write $F = F_0 + \dots + F_m$ and $G = G_0 + \dots + G_m$ where F_s (resp. G_s) is the homogeneous part of F (resp. G) of degree s with respect to the indeterminates y_0, \dots, y_n . For each $0 \leq s \leq m$ such that $F_s + G_s \neq 0$, let $P_s = (s_1, s)$ be the point of the plane defined by $s_1 := \min\{u; (u, s) \in P(F_s + G_s)\}$. Then it is easy to prove that the convex hull of all the points P_s for all $0 \leq s \leq m$ and $(+\infty, 0)$ in the plane \mathbb{R}^2 is the Newton polygon of the differential polynomial $F + G$.

V. NEWTON POLYGONS OF EVALUATIONS OF DIFFERENTIAL POLYNOMIALS

Let F be a differential polynomial as in Section I. Let $0 \neq c \in \overline{K}$, $\mu \in \mathbb{Q}$ and $G(y) = F(cx^\mu + y)$ be the differential polynomial obtained from F by replacing y_k by $c(\mu)_k x^{\mu-k} + y_k$ for all $0 \leq k \leq n$ (where $(\mu)_0 := 1$). In this section, we will construct the Newton polygon of the differential equation $G(y) = 0$ for different values of c and μ .

For each differential monomial $m(y) = f_{i,\alpha} x^i y_0^{\alpha_0} \dots y_n^{\alpha_n}$ of F with corresponding point $p \in P(F)$, compute

$$m(cx^\mu + y) = f_{i,\alpha} x^i (cx^\mu + y_0)^{\alpha_0} (c\mu x^{\mu-1} + y_1)^{\alpha_1} \dots (c(\mu)_n x^{\mu-n} + y_n)^{\alpha_n}.$$

Remark that the corresponding points of the differential monomials of $m(cx^\mu + y)$ have ordinate less or equal than $s = \alpha_0 + \alpha_1 + \dots + \alpha_n$ and lie in the line passing through p with inclination μ . There are two possibilities for μ :

Theorem 2 If $\mu = \mu_e$ is the inclination of an edge $e \in E(F)$. For any $0 \leq s \leq m = \deg_{y_0, \dots, y_n}(F)$, the vertex of $\mathcal{N}(G)$ of ordinate s corresponds to the differential monomial of G with coefficient equals to

$$q_s(c, \mu_e) := \sum_{0 \leq k_0 \leq \dots \leq k_n \leq n} \frac{1}{k_0! \dots k_n!} H\left(\frac{\partial^s F}{\partial y_0^{k_0} \dots \partial y_n^{k_n}}, e\right)(c).$$

where the sum ranges over all the partitions (k_0, \dots, k_n) of s i.e., $k_0 + \dots + k_n = s$. Its x -coordinate is the minimum of the quantities $i + \mu(\alpha_0 + \dots + \alpha_n - s) - \alpha_1 - 2\alpha_2 - \dots - n\alpha_n$ for $i \in \mathbb{Q}$ and $\alpha \in A$.

If $\mu_1 < \mu < \mu_2$ where μ_1 and μ_2 are the inclinations of the two adjacent edges of a vertex $p = (u, v) \in V(F)$, then for any $0 \leq s \leq m$, the vertex of $\mathcal{N}(G)$ of ordinate s corresponds to the differential monomial of G with coefficient equals to

$$c^{v-s} \sum_{0 \leq k_0 \leq \dots \leq k_n \leq n} \frac{1}{k_0! \dots k_n!} h\left(\frac{\partial^s F}{\partial y_0^{k_0} \dots \partial y_n^{k_n}}, p\right)(\mu).$$

where the sum ranges over all the partitions (k_0, \dots, k_n) of s .

Proof. Let $\mu = \mu_e$ for $e \in E(F)$ and compute

$$G(y) = \sum_{i \in \mathbb{Q}, \alpha \in A} f_{i,\alpha} x^i (cx^\mu + y_0)^{\alpha_0} (c\mu x^{\mu-1} + y_1)^{\alpha_1} \cdots (c(\mu)_n x^{\mu-n} + y_n)^{\alpha_n}.$$

For each $0 \leq s \leq m$, compute G_s the homogeneous part of G of degree s in y_0, \dots, y_n . We remark that for fixed i and α , all the differential monomials of G_s have the same corresponding point which is

$$(i + \mu(\alpha_0 + \cdots + \alpha_n - s) - \alpha_1 - 2\alpha_2 - \cdots - n\alpha_n, s).$$

The x-coordinate of the vertex of $\mathcal{N}(G)$ of ordinate s is the minimum of the x-coordinates $i + \mu(\alpha_0 + \cdots + \alpha_n - s) - \alpha_1 - 2\alpha_2 - \cdots - n\alpha_n$ for $i \in \mathbb{Q}$ and $\alpha \in A$. This minimum is realized by the points $P_{i,\alpha} \in \mathcal{N}(F, a(e), b(e))$. This proves the lemma taking into account the formula for the characteristic polynomial of the derivatives of F in Section III (see Theorem 1). \square

The following Corollary is a generalization of Lemma 2.2 of⁷ which deals with the Newton polygon of the Riccati equation associated to a linear ordinary differential equation.

Corollary 3 Let $\mu = \mu_e$ be the inclination of an edge $e \in E(F)$. The edges of $\mathcal{N}(G)$, situated above the edge e are the same as in $\mathcal{N}(F)$. Let an integer $s_1 \geq 0$ be such that $q_s(c, \mu_e) = 0$ for all $0 \leq s < s_1$ and $q_{s_1}(c, \mu_e) \neq 0$ then $\mathcal{N}(G)$ has a vertex of ordinate s_1 and it has at least s_1 edges with inclination greater than μ_e .

Proof. Let p_{s_1} be the vertex of $\mathcal{N}(G)$ of ordinate s_1 and x-coordinate $x_{p_{s_1}}$ the minimum of the values $i + \mu(\alpha_0 + \cdots + \alpha_n - s_1) - \alpha_1 - 2\alpha_2 - \cdots - n\alpha_n$ for $i \in \mathbb{Q}$ and $\alpha \in A$ (by Theorem 2). We have $q_{s_1-1}(c, \mu_e) = 0$, then the x-coordinate of the vertex p_{s_1-1} of ordinate $s_1 - 1$ is strictly less than the minimum of the values $i + \mu(\alpha_0 + \cdots + \alpha_n - s_1 + 1) - \alpha_1 - 2\alpha_2 - \cdots - n\alpha_n$ for $i \in \mathbb{Q}$ and $\alpha \in A$. Thus the inclination of the edge joining p_{s_1} and p_{s_1-1} is greater than μ . \square

Corollary 4 Let $\mu = \mu_e$ be the inclination of an edge $e \in E(F)$. If $H_{(F,e)}(c) = 0$ then the intersection point of the straight line passing through e with the x-axis is not a vertex of $\mathcal{N}(G)$ and $\mathcal{N}(G)$ has an edge with inclination greater than μ_e .

Proof. We have $q_0(c, \mu_e) = H_{(F,e)}(c) = 0$, then $s_1 \geq 1$. This proves the corollary by applying Corollary 3. \square

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