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The Central Disintegration Measure Of The Regular Representation On Non Unimodular Locally Compact Group

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abstract

The article generalizes to non-unimodular locally compact groups some basic facts that were previously only known in the unimodular case. The main results concern Gelfand measures, they determine the support of the central disintegration measure of the natural regular representation. The proof of this result uses the notion of μ-non-degeneracy of representations.
Key words: Locally compact group- Non-unimodular group- Gelfand measure- μ-non-degenerate representation- Central disintegration measure.

I. INTRODUCTION

Let G a locally compact group, not necessarily unimodular group, dx the left Haar measure on G. Δ the modular function, and $\hat{G}$ is the set of all unitary irreducible and continuous representations of G.

$$L_1^{\mu}(G) = \{f \in L_1(G)/\ f = \ \mu * f = f * \mu\} = \mu * L_1(G) * \mu$$

$$M_1^{\mu}(G) = \{\nu \in M_1(G)/\ \nu = \mu * \nu = \nu * \mu\} = \mu * M_1(G) * \mu$$

are respectively closed subalgebras of $L_1(G)$ and $M_1(G)$ associated to a bounded symmetric idempotent measure μ.
μ is called Gelfand measure on G if and only if the algebra $L_1^{\mu}(G)$ is commutative. Suppose that G is non-unimodular and μ a Gelfand measure on G.
In paragraph 1, we give some definitions and fix notations.
In pragraph 2, we denote

$$\mathbf{H} = \mathbf{L_2^{\langle\mu\rangle}}(\mathbf{G}) = \mathbf{L_2}(\mathbf{G}) * \mathbf{\Delta^{1/2}}\boldsymbol{\mu}$$

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and put $(\rho(x)(f))(y) = f(x^{-1}y)$ the restriction of the left regular representation of G to $\mathbf{H}$. Then we prove that the representation $(\rho, \mathbf{H})$ is μ- non degenerate.
In the paragraph 3, we will show that the central disintegration measure σ of ρ is concentrated in

$$\hat{G}_\mu := \left\{\pi \in \widehat{G}\colon\ \pi(\mu) \neq 0\right\}.$$

II. NOTATIONS AND DEFINITIONS

Let G a locally compact goup, not necessarily unimodular group, dx the left Haar measure on G. Δ the modular function, and $\hat{G}$ is the set of all unitary irreducible and continuous representations of G. $K(G)$ is the set of continuous functions on G with compact support. $C(G)$ is the Banach space of continuous functions on G with values in $\mathbf{C}$. $C_b(G)$ the Banach space of bounded continuous functions, with the norm $\|f\|_\infty = sup|f(x)|$ for every x in G, and f in $C_b(G)$.
Functional spaces related to the Haar measure are denoted $L_p(G)$,$1 \leq p \leq \infty$. We know that $K(G)$ is dense in each of the spaces $L_1(G)$ and $L_2(G)$.
$M_1(G)$ the Banach algebra of bounded complex measures on G. We have:

$$L_1^\mu(G) = \{f \in L_1(G)/\ f =\ \mu * f = f * \mu\} = \mu * L_1(G) * \mu$$

$$M_1^\mu(G) = \{\nu \in M_1(G)/\ \nu = \mu * \nu = \nu * \mu\} = \mu * M_1(G) * \mu$$

which are respectively closed subalgebras of $L_1(G)$ and $M_1(G)$.
For every complex function f of G, we have the following notations, definitions and remarks:
(1) If f and g are complex functions in $L_1(G)$ then:

$$f * g(x) := \int f(y)g(y^{-1}x)dy = \int f(xy)g(y^{-1})dy.$$

(2) $\varphi \in C_b(G)$,$f \in L_1(G)$, and ν, μ elements of $M_1(G)$, then:

$$\langle \varphi, \mu \rangle = \int \varphi(x)d\mu(x)$$

$$\mu * f(x) = \int f(y^{-1}x)d\mu(y)$$

$$f * \nu(x) = \int f(xy^{-1})\Delta^{-1}(y)d\nu(y)$$

$$\langle \varphi, \mu * \nu \rangle = \int\int \varphi(xy)d\mu(x)d\nu(y)$$

$$\mu * f * \nu(x) = \int\int f(t^{-1}xs)\Delta^{-1}(s)d\mu(t)d\nu(s)$$

(3) $f \in L_1(G)$ and $a \in G$, then:

$$\int f(x)dx = \int f(x^{-1})\Delta^{-1}(x)dx$$

$$\int f(xa)dx = \Delta^{-1}(a)\int f(x)dx$$

$$\int f(ax)dx = \int f(x)dx$$

(4) If f isa complex function on G and a is an element of G, then:

$$\overline{f}(x) = \overline{f(x)}$$

$$\overset{\vee}{f}(x) = f(x^{-1})$$

$$\widetilde{f}(x) = \overline{f(x^{-1})}$$

$$f^{\star}(x) = \Delta^{-1}(x)\widetilde{f}(x)(f \in L_1(G))$$

$$_af(x) = (L_a(f))(x) = f(a^{-1}x)$$

$$f_a(x) = (R_a(f))(x) = f(xa)$$

(5) If μ is an element of $M_1(G)$ then:

$$\|\mu\| := |\mu|(G) = |\mu|(x)$$

The measures $\overset{\vee}{\mu}$, $\overline{\mu}$, $\mu^{\star}$, μ_a, and $_a\mu$ are defined as follows:

$$\left\langle \varphi, \overset{\vee}{\mu} \right\rangle := \left\langle \overset{\vee}{\varphi},\ \mu \right\rangle$$

$$\langle \varphi, \overline{\mu} \rangle := \overline{\langle \overline{\varphi}, \mu \rangle}$$

$$\langle \varphi, \mu^{\star} \rangle := \overline{\langle \widetilde{\varphi},\ \mu \rangle} := \left\langle \overset{\vee}{\varphi},\ \overline{\mu} \right\rangle$$

$$\langle \varphi,\ \mu_a \rangle := \langle \varphi_{a^{-1}}, \mu \rangle$$

$$\langle \varphi,_a \mu \rangle := \langle _{a^{-1}}\varphi, \mu \rangle$$

If $G^{'}$ is another topological locally compact group, and if θ is a continuous application on G with values in $G^{'}$. We denoteμ^{θ} the image measure of μ by θ: it is an element of $M_1(G^{'})$defined for every function ψ element of $C_b(G^{'})$ by:

$$\left\langle \psi,\ \mu^{\theta} \right\rangle := \langle \psi \circ \theta,\ \mu \rangle$$

(6) $L_2(G)$ is the space of square integrable functions with complex values defined on G, where we have the product :

$$\langle f, g \rangle = \int f(x)\overline{g(x)}dx$$

The support of μ an element of $M_1(G)$ will be denoted by $Supp(\mu)$, and H_{μ} the closed subgroup generated by the support of μ.We have also the following result:
If G is a topological locally compact group , μ and ν elements of $M_1(G)$, then $\mu = \nu$ if and only if $\pi(\mu) = \pi(\nu)$, for every representation π of G [God].
(7) H a Hilbert space over **C** which is separable with finite or infinite dimension, with the product $\langle ., . \rangle^{'}$. $L_{\infty}(H)$ is the Banach algebra of linear bounded operators in H, and $L^{\perp}$ is the orthogonal of L in H,

where L is a Hilbert subspace of H.
$L_\infty(H)$ and $L_2(H)$ are respectively the space of bounded operators$'$.
ρ the regular representation of a locally compact group G is by definition the homomorphism from G to $GL(L_2(G))$ (the group of invertible linear transformations of $L_2(G)$), which is defined for every x in G and f in $L_2(G)$ as follows:

$$(\rho(x)(f))(y) = f(x^{-1}y).$$

ρ satisfies:

$$\rho(st) = \rho(s)\rho(t)$$

$$\rho(s^{-1}) = \rho(s)^{-1} = \rho(s)^\star$$

$\rho(e) = I$, where s and t are element of G, and I the identity map of $L_2(G)$.
We have also the following:

Theorem and definition:
Let μ an element of $M_1(G)$ such that $\mu = \mu^\star = \mu * \mu$, then the following properties are equivalent:
(i) the algebra $L_1^\mu(G)$ is commutative
(ii) the algebra $M_1^\mu(G)$ is commutative
(iii) for every representation π element of $\widehat{G}$, the operator $\pi(\mu)$ is of rank $\preceq 1$
A measure μ element of $M_1(G)$ such that $\mu = \mu^\star = \mu * \mu$,with property (i), (ii), or (iii) is called Gelfand measure on G [Akk].
As a particular case, the pair (G, K) is a Gelfand pair if and only if the normalised Haar measure of K is a Gelfand measure on G.

III. μ-NON DEGENERACY OF THE REGULAR REPRESENTATION

Let μ an element of $M_1(G)$ such that $\mu = \mu^\star = \mu * \mu$
Theorem 3.1.
For every function f in $L_2(G)$ we have:
1. $f * \Delta^{1/2}\mu$ is an element of $L_2(G)$.
2. $U : L_2(G) \to L_2(G), f \to f * \Delta^{1/2}\mu$ is linear and continuous.
Proof.
1. Since we have the isometry $f \mapsto \Delta^{-1/2}\widetilde{f}$, and μ an element of $M_1(G)$, then we have:
$\mu * f \in L_2(G)$ for every f in $L_2(G)$, then:

$$\Delta^{-1/2}\,\widetilde{(\mu * f)} = \Delta^{-1/2}\widetilde{f} * \Delta^{1/2}\mu^\star = \Delta^{-1/2}\widetilde{f} * \Delta^{1/2}\mu$$

(we take into account that $\mu = \mu^\star$)
Thus, for every f in $L_2(G)$, $f * \Delta^{1/2}\mu$ is an element of $L_2(G)$.
2. Since we have $d\mu = hd\,|\mu|$, where $|h| = 1\ \mu - p.p$ (or $\mu = h\,|\mu|$where $|h| = 1\ \mu - p.p$) [Die], then using Cauchy-Schwartz inequality,
and by Fubini-Tonnelli theorem, we obtain:

$$\left\| f * \Delta^{1/2}\mu \right\|_2 \leq \|\mu\| \, \|f\|_2$$

which shows that U is continuous.
Put $\mathbf{H} = \mathbf{L_2^{\langle\mu\rangle}}(\mathbf{G}) = L_2(G) * \Delta^{1/2}\mu$, then we have
Proposition 3.1.
The space $\mathbf{H}$ is closed in $L_2(G)$

Proof.
Let (g_n) a sequence of elements of **H**, such that

$$(g_n) \to h$$

$$g_n = f_n * \Delta^{1/2}\mu$$

where (f_n) is a sequence of element of $L_2(G)$,
then we have:

$$U(f_n * \Delta^{1/2}\mu) \to U(h)$$

then:

$$(f_n * \Delta^{1/2}\mu) * \Delta^{1/2}\mu \to U(h)$$

i.e

$$(f_n * \Delta^{1/2}\mu) * \Delta^{1/2}\mu \to h * \Delta^{1/2}\mu$$

If we take into account that $\mu = \mu * \mu$ we obtain:

$$(f_n * \Delta^{1/2}\mu) * \Delta^{1/2}\mu = f_n * \Delta^{1/2}\mu$$

Thus:

$$h = h * \Delta^{1/2}\mu$$

Which means that h is an element of **H**, and that $\overline{\mathbf{H}} = L_2(G)$
then:
the space **H** is closed in $L_2(G)$.

Proposition 3.2.
For every x and y in G, and for every f element of **H**,the restriction of the left regular representation of G to **H**, $(\rho(x)(f))(y) = f(x^{-1}y)$ is an unitary representation.

Proof.
Let f an element of **H**,

$$\rho(x)(f) * \Delta^{1/2}\mu(y) = \int f(x^{-1}yz^{-1})\Delta^{-1/2}(z)d\mu(z) = (f * \Delta^{1/2}\mu)(x^{-1}y)$$

Then, $\rho(x)(f) * \Delta^{1/2}\mu(y) = f(x^{-1}y)$
i.e: $\rho(x)(f) * \Delta^{1/2}\mu = \rho(x)(f)$
Thus, $f \in \mathbf{H} \Rightarrow \boldsymbol{\rho}(\mathbf{x})(\mathbf{f}) \in \mathbf{H}$
On the other hand, since $\|\rho(x)(f)\|_2 = \|f(x)\|$, we get that the restriction of the left regular representation of G to **H**, $(\rho(x)(f))(y) = f(x^{-1}y)$ is an unitary representation.

Lemma 3.1.
f an element of $K(G)$, and let consider the operator A_f such that:
$A_f\ (g) = f * g$, for every g in $L_2(G)$.
If we denote $A_f^\star$ the adjoint operator of A_f related to the product in $L_2(G)$,
then we have:

$$(A_f^\star)(h) = f^\star * h$$

for every h in $L_2(G)$.

Proof.
Let g and h elements of $L_2(G)$, then we have:

$$\langle f * g, h\rangle = \int (f * g)(x)\overline{h}(x)dx = \int\int f(xy)g(y^{-1})\overline{h}(x)dydx = \int(\int f(xy)g(y^{-1})dy)\overline{h}(x)dx$$

(Fubini theorem)
Then:
$\langle f*g,h\rangle = \int(\int f(xy^{-1})g(y)\Delta^{-1}(y)\overline{h}(x)dy)dx = \int(\int f(xy^{-1})\Delta^{-1}(y)\overline{h}(x)dx)g(y)dy$ (Fubini theorem)
Thus:

$$\langle f*g,h\rangle = \int \overline{(\int \overline{f}(y^{-1}x)\Delta^{-1}(y)h(x)dx)}g(y)dy = \int \overline{(\int \overline{\overset{\vee}{f}(yx)}\Delta^{-1}(xy)h(x^{-1})dx)}g(y)dy$$

i.e

$$\langle f*g,h\rangle = \int \overline{(\int \overset{\vee}{\bar{f}}(yx^{-1})\Delta^{-1}(yx^{-1})\Delta^{-1}(x)h(x)dx)}g(y)dy$$

Then:

$$\langle f*g,h\rangle = \int \overline{(\int f^{\star}(yx^{-1})\Delta(x^{-1})h(x)dx)}g(y)dy = \int \overline{(\int f^{\star}(yx)h(x^{-1})dx)}g(y)dy$$

i.e

$$\langle f*g,h\rangle = \int g(y)\overline{f^{\star}*h}(y)dy = \langle g,\ f^{\star}*\ h\rangle$$

Thus:
$(A_f^{\star})(h) = f^{\star}*\, h$, for every h in $L_2(G)$.

Lemma 3.2.
For every g and h in $L_2(G)$, we have:
$\langle g,\ \mu*h\rangle = \langle \mu*g,h\rangle$.

Proof.

$$\langle g,\ \mu*h\rangle = \int g(x)\overline{(\mu*h)(x)}dx = \int g(x)\overline{(\int h(y^{-1}x)d\mu(y))}dx$$

i.e $\langle g,\ \mu*h\rangle = \int g(x)\overline{(\int \overset{\vee}{h}(x^{-1}y)d\mu(y))}dx$
Since we have $\langle \varphi,\ \overline{\mu}\rangle := \overline{\langle\overline{\varphi},\mu\rangle}$,
then:

$$\langle g,\ \mu*h\rangle = \int g(x)(\int \widetilde{h}(x^{-1}y)d\overline{\mu}(y))dx = \int\int g(x)\overset{\vee}{\overline{h}}(x^{-1}y)d\overline{\mu}(y))dx$$

and also since we have

$$\langle \varphi,\mu^{\star}\rangle := \left\langle \overset{\vee}{\varphi},\ \overline{\mu}\right\rangle,$$

we obtain:

$$\langle g,\ \mu*h\rangle = \int\int g(yx)\overline{h}(x^{-1})d\mu(y))dx = \int\int g(yx)\overset{\vee}{\overline{h}}(x)d\mu(y))dx$$

Then:

$$\langle g,\ \mu*h\rangle = \int\int g(yx)\overline{h(x)}d\overline{\mu}(y) = \int(\int g(y^{-1}x)d\mu^{\star}(y))\overline{h(x)}dx$$

Thus:
$\langle g,\ \mu*h\rangle = \langle \mu*g,h\rangle$,for every g and h in $L_2(G)$
For every g and h in $L_2(G)$ that $\langle g,\ \mu*h\rangle = \langle \mu*g,h\rangle$.

Theorem 3.2.
The representation $(\boldsymbol{\rho}, \mathbf{H})$ is μ-non degenerate.

Proof.
Let H_μ the closed vectorial subspace of $\mathbf{H}$ invariant by ρ, and generated by $Im(\rho(\mu))$

$$H_\mu = \overline{\left\{ f * \ \mu \ * g \ * \ \Delta^{1/2}\mu, \ f \ , \ g \text{ elements of } K(G) \right\}}$$

Let k an element of H_μ such that :

$$\left\langle f * \ \mu \ * g \ * \ \Delta^{1/2}\mu, \ k \right\rangle = 0$$

Then, for every f and g elements of $K(G)$, we have:

$$\left\langle \mu \ * g * \ \Delta^{1/2}\mu \ , f^{\star} * \ k \right\rangle = 0$$

i.e :

$$\left\langle g, \mu \ * (f * \ k) * \ \Delta^{1/2}\mu \right\rangle = 0$$

Thus for every element of $K(G)$, we have:

$$\mu * (f * k) * \Delta^{1/2}\mu = 0$$

Since k is an element of H_μ, which is a closed subspace invariant by ρ and generated by Im$\rho(\mu)$, then by [Kab], there exists a unit k element of H_μ such that: $h = \Delta^{-1/2}\widetilde{h}$ (we can take as an example, $h = \Delta^{-1/2}\widetilde{\varphi} * \varphi$, where φ is a unit)
Then, it is easy to see that $h = \Delta^{-1/2}\widetilde{h}$.
Thus,

$$k(x) = \Delta^{-1/2}(x)\overline{k(x^{-1})} = \Delta^{-1/2}(x)\overline{(k * \Delta^{-1/2}\mu)(x^{-1})}$$

Then,

$$k(x) = \Delta^{-1/2}(x)\overline{(\int k(x^{-1}y^{-1})\Delta^{1/2}(y)\Delta^{-1}(y)d\mu(y))} = \overline{\int \overset{\vee}{k}(yx)\Delta^{-1/2}(yx)d\mu(y)}$$

Thus,

$$k(x) = \int \widetilde{k}(yx)\Delta^{-1/2}(yx)d\overline{\mu}(y) = \int k(yx)d\overline{\mu}(y) = \int k(y^{-1}x)d\mu(y)$$

Then, $k(x) = \mu * \, k(x)$
Then we have $k = 0$,
i.e $H_\mu^{\perp} = 0$,
This means that the representation $(\rho, \mathbf{H})$ is μ- non degenerate.

IV. THE CENTRAL DISINTEGRATION MEASURE OF THE REGULAR REPRESENTATION

Theorem 4.1.
If μ is a Gelfand measure, then the central disintegration measure σ of ρ is concentrated in $\hat{G}_\mu := \left\{ \pi \in \widehat{G}\colon \ \pi(\mu) \neq 0 \right\}$

Proof.
Since we have in [Akk] the following:

Proposition:
If $\mu \in M_1(G)$ is a Gelfand measure on the group G, and $(\pi, \mathbf{H})$ is an unitary continuouis representation which is μ- non degenerate, then the central desintegration measure of π is concentrated in $\hat{G}_\mu := \left\{\pi \in \widehat{G} : \pi(\mu) \neq 0\right\}$.

Theorem 4.2.
If μ is a Gelfand measure, then the representation ρ is type I.

Proof.
Since we have in [Akk] the following:

Proposition:
If $\mu \in M_1(G)$ is a Gelfand measure on the group G, then every μ- non degenerate unitary continuous representation of G is type I.

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