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Algebraic points on some Fermat curves and some quotients of Fermat curves: Progress

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abstract

In this work we speak about progress of research on the algebraic points on some curves. The main results completes previous works obtained on some Fermat curves and their quotients.

I. INTRODUCTION

Let C be a smooth projective plane curve defined over $\mathbb{Q}$. For all extension K of $\mathbb{Q}$, we denote by $C(K)$ the set of K-rational points on C, and by $C^{(d)}(\mathbb{Q})$ the set

$$C^{(d)}(\mathbb{Q}) = \bigcup_{[K:\mathbb{Q}]\leq d} C(K)$$

The degree of an algebraic point is the degree of its field of definition over $\mathbb{Q}$ i.e

$$\deg(R) = [\mathbb{Q}(R) : \mathbb{Q}]$$

It is well know by Faltings in [Fa], that if C has genus $g \geq 2$ than $C(\mathbb{Q})$ is finite.
We recall some know results about algebraic points on some Fermat curves and some quotients of Fermat curves; see [Ab-Ha], [De-Fah], [Fr].
Theorem [De-Kl]
Let C be a smooth projective plane curve defined by an equation of degree d with rational coefficients.
1- If $d \geq 7$, the curve C hase only finitely many points whose field of definition has degree $\leq d-2$ over $\mathbb{Q}$; i.e $C^{(d-2)}(\mathbb{Q})$ is finite.
2- If $d \geq 8$, all but finitely many points of C whose field of definition has degree $\leq d-1$ over $\mathbb{Q}$ arise as the intersection of C with a rational line through a rational point of C.
We first describe the set of algebraic points of degree less than 10 on the Fermat septic; this result completes the work of P. Tzermias [Tz1].
We then determine explicitly algebraic points of a given degree on some quotients of Fermat curves of degree 5, 7 or 11; this result completes previous work of Gross and Rohrlich [Gr-Ro] who gave a description of points of degree at most 2.
It should be noted that by a result of Gross an Rohrlich [Gr-Ro], the Mordell-Weil groups $J_p(\mathbb{Q})$ are infinite for $p \geq 11$. My work use an argument that the Mordell-Weil groups are finite.
By the results in [Fad] and in [Gr-Ro] the Mordell-Weil groups $J_p(\mathbb{Q})$ are finite in the cases:
i) $F_p = \left\{(X, Y, Z) \in \mathbb{P}^2\left(\overline{\mathbb{Q}}\right) : X^p + Y^p + Z^p = 0\right\}$ for $p = 5$ or 7.
ii) $C_{r,\,s}(p) : y^p = x^r(x-1)^s$, $1 \leq r, s, r+s \leq p-1$; for $p = 5$ or 7 and for $p = 11$ and $r = s$.

II. WE STUDY ALGEBRAIC POINTS ON THE FERMAT SEPTIC, I.E ON THE SMOOTH PLANE CURVE

$$F_7 = \left\{ (X,\, Y,\, Z) \in \mathbb{P}^2\left(\overline{\mathbb{Q}}\right) : X^7 + Y^7 + Z^7 = 0 \right\}$$

In this work, which is based on [Tz1], we describe the set of algebraic points of degree less than 10 on the Fermat septic. This result completes the work of Tzermias [Tz1].
It has been conjectured (see for example [Kl-Tz]) that all points on the latter set lie on the line $X + Y + Z = 0$.
By the work of Gross and Rohrlich [Gr-Ro], there are exactly five algebraic points of degree at most 3 on F_7, namely
$a = (0,\; -1,\, 1)$; $b = (-1,\, 0,\, 1)$, $\infty = (-1,\, 1,\, 0)$; $p = (\eta,\, \overline{\eta},\; -1)$, $\overline{p} = (\overline{\eta},\, \eta,\; -1)$
where η is a primitive 6−th root of unity in $\overline{\mathbb{Q}}$, and $\overline{\eta}$ is the complex conjugate of η. Note that the above five points are the only points of intersection of F_7 with the line $X + Y + Z = 0$.
We recall some know results:
$F_7^{(3)}(\mathbb{Q}) = \{a,\, b,\, \infty,\, p,\, \overline{p}\}$ [Gr-Ro]
$F_7^{(5)}(\mathbb{Q}) = \{a,\, b,\, \infty,\, p,\, \overline{p}\}$ [Tz1]
In [Sa1] we describe the set of algebraic points of degree less than 6 (resp. 4) on the Fermat curve of degree 7 (resp. 5).
Our main result is the following theorem:
Theorem: The algebraic points of degree 6 (resp. 4) *over* $\mathbb{Q}$ *on the Fermat curve of degree* 7 (resp. 5) *arise as the intersection of* F_7 (rep. F_5) *with a rational line through a, b or* ∞*.*
In [Sa3] we describe the set of algebraic points of degree at most 10 on the Fermat curve of degree 7. This work develops and generalizes our previous note [Sa1]. Our main result is the following theorem:
Theorem: The algebraic points of degree 7 *over* $\mathbb{Q}$ *on* F_7 *arise as the intersection of* F_7 *with a rational line.*
There is no algebraic points of degree 8 *or* 9 *over* $\mathbb{Q}$ *on* F_7*.*
The algebraic points of degree 10 *over* $\mathbb{Q}$ *on* F_7 *arise as the intersection of* F_7 *with a rational conic C through* $\{a,\, b\}$ *or* $\{a,\, \infty\}$ *or* $\{b,\, \infty\}$*; i.e*

$$F_7.C = R_1 + \cdots + R_{10} + 2Q_1 + 2Q_2$$

with $Q_1 \neq Q_2$ *and* $Q_1,\, Q_2 \in \{a,\, b,\, \infty\}$*;* $R_1,\, \cdots,\, R_{10}$ *the Galois conjugates of a point on* F_7 *of degree* 10 *over* $\mathbb{Q}$*.*

III. WE DETERMINE EXPLICITLY ALGEBRAIC POINTS OF A GIVEN DEGREE ON SOME QUOTIENTS OF FERMAT CURVES:

$$C_{r,\, s}(p) : y^p = x^r\,(x-1)^s$$

p is an odd prime and $r,\, s$ integers with $1 \leq r,\, s,\, r+s \leq p-1$.
It is well know [see [Gr-Ro] , [Tz2]] that the latter curves $C_{r,\, s}(p) : y^p = x^r\,(x-1)^s$ are the quotients of Fermat curves F_p. We recall [see [Fad] and [Gr-Ro]] that the Mordell-Weil groups $J_p(\mathbb{Q})$ are finite for $p = 5$ or 7 and for $p = 11$ and $r = s$.
By the work of Gross and Rohrlich [Gr-Ro] $C_{r,\, s}(5)$ and $C_{1,\, 1}(5)$ are birationally equivalent over $\mathbb{Q}$; $C_{r,\, s}(7)$ and $C_{1,\, 1}(7)$ are birationally equivalent over $\mathbb{Q}$; or $C_{r,\, s}(7)$ and $C_{1,\, 2}(7)$ are birationally equivalent over $\mathbb{Q}$.
We will use the following notation:
$P_0 = (0,\, 0,\, 1)$; $P_1 = (1,\, 0,\, 1)$, $P_\infty = (1,\, 0,\, 0)$; $P_\eta = (\eta,\, \overline{\eta},\, 1)$, $\overline{P}_\eta = (\overline{\eta},\, \eta,\, 1)$
where η is a primitive 6−th root of unity in $\overline{\mathbb{Q}}$, and $\overline{\eta}$ is the complex conjugate of η.
1) Algebraic points on the curve

$$C_{1,\, 2}(7) : y^7 = x\,(x-1)^2$$

We determine the set of algebraic points of degree at most 3 on the Klein quartic curve. This result extends a previous result given by Hurwitz [Hu] who described the set of rational points.
The curve $C_{1,\,2}(7) : y^7 = x(x-1)^2$ is birationally isomorphic to the Klein quartic curve given by the projective equation

$$\mathcal{K} : X^3Y + Y^3Z + Z^3X = 0$$

and the affine equation

$$\mathcal{K} : u^3v + v^3 + u = 0$$

This isomorphism is defined explicitly in projective coordinates by

$$\begin{aligned} \Psi : \mathcal{K} &\longrightarrow C_{1,\,2}(7) \\ (X,\, Y,\, Z) &\longmapsto \left(\frac{-XZ^2}{Y^3},\, \frac{-X}{Y},\, 1\right) \end{aligned}$$

and in affine coordinates by

$$\begin{aligned} \Psi : \mathcal{K} &\longrightarrow C_{1,\,2}(7) \\ (u,\, v) &\longmapsto \left(\frac{-u}{v^3},\, \frac{-u}{v}\right) \end{aligned}$$

and we have

$$\begin{aligned} \Psi^{-1} : C_{1,\,2}(7) &\longrightarrow \mathcal{K} \\ (x,\, y) &\longmapsto \left(\frac{y^7 + x^2 - x}{(xy)^2},\, \frac{1-x}{y^3}\right) \end{aligned}$$

Theorem [Hurwitz [Hu]]*: The set of rational points on* $C_{1,\,2}(7)$ *is*

$$C_{1,\,2}(7)(\mathbb{Q}) = \{P_\infty,\, P_0,\, P_1\}.$$

Theorem [Tzermias [Tz3]]*: The set of quadratic points on* $C_{1,\,2}(7)$ *is*

$$\mathcal{O} = \{P_\eta,\, \overline{P}_\eta\}.$$

Our main result is the following theorem
Theorem: The set of cubic points on $C_{1,\,2}(7)$ *is the union of the following sets:*
$P_1 = \left\{(x,\, \lambda) \mid \lambda \in \mathbb{Q}^*, \text{ and } x \text{ a root of } x(x-1)^2 = \lambda^7\right\}$;
$P_2 = \left\{\left(1 + \lambda y^2,\, y\right) \mid \lambda \in \mathbb{Q}^*, \text{ and } y \text{ a root of } y^3 = \lambda^2\left(1 + \lambda y^2\right)\right\}$;
$P_3 = \left\{\left(1 + \lambda y^3,\, y\right) \mid \lambda \in \mathbb{Q}^*, \text{ and } y \text{ a root of } y = \lambda^2\left(1 + \lambda y^3\right)\right\}$;
$P_4 = \left\{\left(1 + y^4,\, y\right) \mid\ y \text{ a root of } y^3 + y^2 - 1 = 0\right\}$;
$P_5 = \left\{(1 - y,\, y) \mid\ y \text{ a root of } y^3 + y^2 - 1 = 0\right\}$;
$P_6 = \left\{\left(y^3 - y^2 + 1,\, y\right) \mid\ y \text{ a root of } y^3 - 2y^2 - y + 1 = 0\right\}$;
$P_7 = \left\{\left(-y^2,\, y\right) \mid\ y \text{ a root of } y^3 + 2y^2 + y + 1 = 0\right\}$.
On the Klein quartic curve $\mathcal{K}$, we have:
Theorem: The set of quadratic points on K is

$$\mathcal{O}_{\mathcal{K}} = \left\{\Psi^{-1}(P_\eta),\, \Psi^{-1}\left(\overline{P}_\eta\right)\right\}$$

$\Psi^{-1}(P_\eta) = (-\eta,\, -\overline{\eta},\, 1)$, $\Psi^{-1}\left(\overline{P}_\eta\right) = (-\overline{\eta},\, -\eta,\, 1)$.
Theorem: The set of cubic points on $\mathcal{K}$ *is the union of the following sets:*
$A_1 = \left\{(u,\, v,\, 1) \mid\ v \in \mathbb{Q}^*,\, u^3v + u + v^3 = 0\right\}$;
$A_2 = \left\{(u,\, v,\, 1) \mid\ u \in \mathbb{Q}^*,\, u^3v + u + v^3 = 0\right\}$;
$A_3 = \left\{(s,\, 1,\, t) \mid\ s \in \mathbb{Q}^*,\, t^3s + t + s^3 = 0\right\}$;

$A_4 = \left\{ \left(\dfrac{\alpha^5 + \alpha^4 + 1}{(1 + \alpha^4)^2}, \; -\alpha \right) \mid \alpha \text{ a root of } \alpha^3 + \alpha^2 - 1 = 0 \right\};$
$A_5 = \left\{ \left(\dfrac{\alpha^6 + \alpha - 1}{\alpha (1 - \alpha)^2}, \; \dfrac{1}{\alpha^2} \right) \mid \alpha \text{ a root of } \alpha^3 + \alpha^2 - 1 = 0 \right\};$
$A_6 = \left\{ \left(\dfrac{\alpha^5 + \alpha^4 - 2\alpha^3 + \alpha^2 + \alpha - 1}{\alpha (1 - \alpha)^2}, \; \dfrac{1 - \alpha}{\alpha} \right) \mid \alpha \text{ a root of } \alpha^3 - 2\alpha^2 - \alpha + 1 = 0 \right\};$
$A_7 = \left\{ \left(\dfrac{\alpha^5 + \alpha^2 + 1}{\alpha^4}, \; \dfrac{1 + \alpha^2}{\alpha^3} \right) \mid \alpha \text{ a root of } \alpha^3 + 2\alpha^2 + \alpha + 1 = 0 \right\}.$
We have:
$A_4 = \Psi^{-1}(P_4)$; $A_5 = \Psi^{-1}(P_5)$; $A_6 = \Psi^{-1}(P_6)$; $A_7 = \Psi^{-1}(P_7)$.
2) Algebraic points on the curves

$$C_{1,\,1}(5);\; C_{1,\,1}(7) \text{ and } C_{1,\,1}(11).$$

We determine explicitly algebraic points of given degree on these curves for completes previous work of Gross and Rohrlich [Gr-Ro] who gave a description of points of degree at most two.
Theorem [Goss and Rohrlich [Gr-Ro]]*:*

$$C^2_{1,\,1}(p)(\mathbb{Q}) = \left\{ \left(\frac{1}{2} \pm \sqrt{y^p + \frac{1}{4}}, \; y \right) \mid y \in \mathbb{Q} \right\} \cup \{P_\infty\}$$

for $p = 5, 7$ *or* 11*.*
Our main result is the following theorem
Theorem: For $p = 5, 7$ *or* 11 *and* $l \geq 1$*, we have:*

$$C^l_{1,\,1}(p)(\mathbb{Q}) = \left(\bigcup_{0 \leq m \leq p-1} N_m \right) \bigcup \left(\bigcup_{0 \leq \delta \leq \frac{l}{2}} M_\delta \right)$$

with

$$N_m = \left\{ \begin{array}{c} \left(-\dfrac{y^{p-m} g(y)}{h(y)}, \; y \right) \mid h(y) \neq 0, \; 0 \leq \deg(h) \leq \dfrac{l-m}{2}, \\ 0 \leq \deg(g) \leq \dfrac{l-p+m}{2}, \; y \text{ a root of} \\ y^m [h(y)]^2 - g(y) [y^{p-m} g(y) + h(y)] = 0 \end{array} \right\}$$

$M_\delta = \{(x, y) \mid [\mathbb{Q}[y] : \mathbb{Q}] = \delta$ *and* x *a root of* $x(x-1) = y^p\}$.
We finish by describing the subjacent principle of the method used for the demonstration of the principal theorems.
One supposes given a point $\infty \in C(\mathbb{Q})$ and the jacobian embedding

$$\begin{array}{rl} j : C & \longrightarrow J \\ P & \longmapsto [P - \infty] \end{array}.$$

The method supposes that one knows or determines the structure of the group $J(\mathbb{Q})$ and that this one be finite:

$$J(\mathbb{Q}) \simeq (\mathbb{Z}/N_1\mathbb{Z}) \times \cdots \times (\mathbb{Z}/N_s\mathbb{Z}).$$

One then chooses $D_1, \cdots, D_s$ the dividers on C defined on $\mathbb{Q}$ such as $j(D_i)$ be the order N_i and $j(D_1)$, $\cdots$, $j(D_s)$ generate $J(\mathbb{Q})$. If R is an algebraic point of degree k over $\mathbb{Q}$ and $R_1, \cdots, R_k$ its Galois conjugates, then $j(R_1 + \cdots + R_k)$ belongs to $J(\mathbb{Q})$ and consequently it exists $0 \leq m_i \leq N_i - 1$ such as

$$j(R_1 + \cdots + R_k) = m_1 j(D_1) + \cdots + m_s\, j(D_s).$$

The Abel-Jacobi theorem involves the existence of a rational function f defined on $\mathbb{Q}$ such as

$$R_1 + \cdots + R_k - m_1 D_1 - \cdots - m_s D_s + \left(\sum_{1 \le i \le s} m_i \deg D_i - k \right) \infty = div\,(f).$$

The function f thus has prescribed poles, and if one knows how to analyze spaces

$$\mathcal{L}\,(D) = \left\{ f \in \overline{\mathbb{Q}}\,(C) \mid div\,(f) + D \ge 0 \right\}$$

(note: if D is defined on $\mathbb{Q}$ then $\mathcal{L}_{\overline{\mathbb{Q}}}\,(D) = \mathcal{L}_{\mathbb{Q}}\,(D) \otimes \overline{\mathbb{Q}}$)

i.e the linear system $|D| = \mathbb{P}\mathcal{L}\,(D)$, one can deduced the restrictions on the R_i, and even in the good cases an explicit description.

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