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The conformal invariance of the Poisson-Lie group $SU(2)$ by dressing transformation

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abstract

We show the conformal invariance of the Poisson-Lie group $SU(2)$ by dressing transformations. This construction gives in particular a Poisson cohomology class of the group $SU(2)$.

I. INTRODUCTION

Denote by $SU(2)$, the special unitary group defined by :

$$SU(2)=\left\{\begin{pmatrix}\alpha & \beta\\ -\overline{\beta} & \overline{\alpha}\end{pmatrix}\ /\ \alpha,\beta\in\mathbb{C},\ \alpha\overline{\alpha}+\beta\overline{\beta}=1\right\}.$$

Let $\alpha = x + iy$ and $\beta = z + it$, $SU(2)$ can be identified with the unit sphere $\mathbb{S}^3$ in $\mathbb{R}^4$.
The Lie algebra $su(2)$ of group $SU(2)$ is defined by:

$$su(2)=\{S\in\mathbb{C}^{2\times 2}/{}^{t}\overline{S}+S=0\ and\, Tr\,(S)=0\}.$$

Let

$$X=\begin{pmatrix}i & 0\\ 0 & -i\end{pmatrix}\ ;\ Y=\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\ ;\ Z=\begin{pmatrix}0 & i\\ i & 0\end{pmatrix},$$

a basis of $su(2)$. The Lie bracket on $su(2)$ is defined by:

$$[Z,X]=2Y\ \ ;\ [Z,Y]=-2X;\ [X,Y]=2Z.$$

The fields left invariant associated to this basis had this local expression :

$$\begin{aligned}\widetilde{X}&=-y\partial_x+x\partial_y+t\partial_z-z\partial_t\\ \widetilde{Y}&=-z\partial_x-t\partial_y+x\partial_z+y\partial_t\\ \widetilde{Z}&=-t\partial_x+z\partial_y-y\partial_z+x\partial_t,\end{aligned}$$

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and the left invariant 1-forms are written:

$$\begin{aligned}
\widetilde{\alpha} &= -ydx + xdy + tdz - zdt \\
\widetilde{\beta} &= -zdx - tdy + xdz + ydt \\
\widetilde{\delta} &= -tdx + zdy - ydz + xdt.
\end{aligned}$$

Now, we recall the Lie-Poisson structure given on $SU(2)$ $(^{BGS}, ^{O})$. Since $su(2)$ is simple, any Poisson structure on $SU(2)$ is given by a solution of Yang-Baxter equation, these solutions are of the form

$$Q = k.X \wedge Y, \quad k \in \mathbb{R}_+^*.$$

So any bialgebra structure of $su(2)$ is written

$$\begin{aligned}
p(X) &= -2k.Z \wedge X, \\
p(Y) &= 2k.Y \wedge Z, \\
p(Z) &= 0.
\end{aligned}$$

Then, the Poisson Lie structure on $SU(2)$ are given by

$$P(s) = R_{s_*}Q - L_{s_*}Q, \quad s \in SU(2),$$

where R_{s_*} (resp: L_{s_*}) is the differential of the right translation (resp: right), for s in neutral element e the group $SU(2)$.
Then using $\alpha = x + iy$, $\beta = z + it$ and $x^2 + y^2 + z^2 + t^2 = 1$ one gets :

$$P(x, y, z, t) = P_1 \widetilde{Y} \wedge \widetilde{Z} + P_2 \widetilde{Z} \wedge \widetilde{X} + P_3 \widetilde{X} \wedge \widetilde{Y},$$

then

$$\begin{aligned}
P_1 &= 2k(xz - yt) \\
P_2 &= -2k(xy + zt) \\
P_3 &= 2k(y^2 + z^2).
\end{aligned}$$

Dressing transformation on SU(2)

In this section we recall the dressing transformation on the group $SU(2)$ and we explicit its fundamental fields.
As p is a Poisson tensor, the dual map of p gives a Lie algebra structure on $su^*(2)$. Recall that the action of $su^*(2)$ in $SU(2)$, namely dressing transformation, in the left invariant version, is a Lie algebra anti-homomorphism :

$$\begin{aligned}
\lambda : su^*(2) &\longrightarrow \chi\,(SU(2)) \\
\xi &\longmapsto i_{\widetilde{\xi}}P,
\end{aligned}$$

where $\widetilde{\xi}$ is the left-invariant 1-forme on $SU(2)$ whose value at the neutral element e is ξ and $i_{\widetilde{\xi}}P$ denote the interior product of P by $\widetilde{\xi}$.
The Poisson action within the meaning of Lu-Weinstein is expressed by the following commutative diagram $(*)^{LW}$.

$$\begin{tikzcd}
su^*(2) \arrow[r, "\lambda"] \arrow[d, "\delta"'] \arrow[dr, phantom, "\circlearrowleft"] & \Gamma\,(T \quad SU\,(2)) \arrow[d, "{[P, .]_s}"] \\
\Lambda^2 su^*(2) \arrow[r, "\lambda \wedge \lambda"] & \Gamma\left(\Lambda^2 T \quad SU\,(2)\right)
\end{tikzcd}$$

where, δ is the transpose map of the Lie bracket on $su(2)$ and $[\, ,\,]$ denote the Shcouten bracket.
The fundamental vector field of the dressing action, generate the characteristic distribution of P are given by :

$$\begin{aligned} U_1 &= i_{\widetilde{\alpha}}P = P_3\widetilde{Y} - P_2\widetilde{Z},\\ U_2 &= i_{\widetilde{\beta}}P = -P_3\widetilde{X} + P_1\widetilde{Z},\\ U_3 &= i_{\widetilde{\gamma}}P = P_2\widetilde{X} - P_1\widetilde{Y}. \end{aligned}$$

II. THE MAIN RESULTS

An algebra 1-cocycle

Our main results are given in the following theorem :

Theorem 1. *There exists a linear mapping* $\mu : su^*(2)) \longrightarrow C^\infty(SU(2))$ *such hat*

$$[P, \lambda(\xi)]_s = \mu(\xi).P, \quad \xi \in su^*(2).$$

Let (α, β, δ) *a base algebra of* $su^*(2)$, μ *is given by :*

$$\mu(a\alpha + b\beta + c\delta) = -2aP_1 - 2bP_2 - 2cP_3,$$

with a, b and c are reals.

Proof of Theorem 1
Let $\xi = a\alpha + b\beta + c\gamma$, an element of the algebra $su^*(2)$. We have

$$\begin{aligned} [P, \lambda(\xi)] &= L_{\lambda(\xi)}P\\ &= aL_{\lambda(\alpha)}P + bL_{\lambda(\beta)}P + cL_{\lambda(\gamma)}P,\\ &= aL_{U_1} + bL_{U_2}P + cL_{U_3}P. \end{aligned}$$

The proof of the theorem follows from:

Lemma 1. *With the above notation, we have:*

$$L_{U_i}P = -2P_i \cdot P \quad i = 1, 2, 3$$

Proof. Using the commutative diagram $(*)$ we get

$$L_{U_1}P = [P, \lambda(\alpha)] = \lambda \wedge \lambda(\delta(\alpha)),$$

As mentioned before, δ is the transpose of the Lie bracket of $su(2)$, then we have

$$\delta(\alpha) = -2\gamma \wedge \beta,$$

thus

$$\begin{aligned} \lambda \wedge \lambda(\delta(\alpha)) &= \lambda \wedge \lambda(-2\gamma \wedge \beta)\\ &= -2\lambda(\gamma) \wedge \lambda(\beta)\\ &= -2U_3 \wedge U_2\\ &= -2(P_2\widetilde{X} - P_1\widetilde{Y}) \wedge (-P_3\widetilde{X} + P_1\widetilde{Z}). \end{aligned}$$

A simple calculation gives us $\lambda \wedge \lambda(\delta(\alpha)) = -2P_1 \cdot P$, thus $L_{U_1}P = -2P_1 \cdot P$. Similarly we find $L_{U_2}P = -2P_2 \cdot P$ and $L_{U_3}P = -2P_3 \cdot P$. □

Now, seeing that the dressing transformation λ give a representation of $su^*(2)$ on $C^\infty(SU(2))$ defined by

$$\begin{aligned} su^*(2) \times C^\infty(SU(2)) &\longrightarrow C^\infty(SU(2))\\ (\xi, f) &\longmapsto \lambda(\xi)(f) := \xi \cdot f, \end{aligned}$$

which defines a $su^*(2)-$module structure on $C^\infty(SU(2))$
Let $[\,,\,]_*$ denote the Lie bracket on $su^*(2)$ induced by d_eP^V. We have

Proposition 1. *The map μ is a 1-cocycle of $su^*(2)$ with values in $su^*(2)-$module, $C^\infty(SU(2))$. i.e.*

$$\mu([\xi,\eta]_*) = \xi.\mu(\eta) - \eta.\mu(\xi), \quad \xi,\eta \in su^*(2).$$

Proof. For each $\xi, \eta \in su^*(2)$, we have

$$L_{\lambda([\xi,\eta]_*)}P = L_{\lambda(\xi)}PL_{\lambda(\eta)}P - L_{\lambda(\eta)}PL_{\lambda(\xi)}P,$$

this implies that

$$\begin{aligned}\lambda([\xi,\eta]_*)P &= L_{\lambda(\xi)}(\mu(\lambda)P) - L_{\lambda(\eta)}(\mu(\xi)P)\\ &= \lambda(\xi)\mu(\eta)P + \mu(\eta)L_{\lambda(\xi)}P - \lambda(\eta)\mu(\xi)P - \mu(\xi)L_{\lambda(\eta)}P\\ &= \lambda(\xi)\mu(\eta)P + \mu(\eta)\mu(\xi)P - \lambda(\eta)\mu(\xi)P - \mu(\xi)\mu(\eta)P.\end{aligned}$$

□

A class of Poisson cohomology

The first Poisson cohomology group $H^1_P(SU(2))$ is the quotient of the space of Poisson vector fields (i.e., the vector fields X such that $L_XP = 0$) by the space of Hamiltonian vector fields(i.e., vector field of the type $[P, f] = X_{-f}$).
Let $\mu^* : SU(2) \longrightarrow su(2)$ the dual map of μ defined by

$$\forall \xi \in su^*(2), \qquad <\mu^*(x), \xi >= \mu(\xi)(x).$$

This allows us to define the vector field X_p associates to P by:

$$X_P(s) = R_{s_*}\mu^*(s), \quad s \in SU(2).$$

We use the expression of μ given in the Theorem 1, we find

$$X_p = -2P_1\widetilde{X} - 2P_2\widetilde{Y} - 2P_3\widetilde{Z}.$$

We will be check that X_P does not be a Hamiltonian vector field, consequently, it defines a not trivial element of $H^1_P(SU(2))$.

Proposition 2. *The vector field X_p defines a non trivial class $[X_p] \in H^1_P(SU(2))$.*

Proof. In fact, for $X_p = -2P_1\widetilde{X} - 2P_2\widetilde{Y} - 2P_3\widetilde{Z}$, one gets

$$L_{X_p}P = -2L_{P_1\widetilde{X}}P - 2L_{P_2\widetilde{Y}}P - 2L_{P_3\widetilde{Z}}P.$$

We denote by $\mathcal{X}_{P_i}$ the Hamiltonian field associated to P_i, $i = 1, 2, 3$, we have

$$\begin{aligned}L_{P_1\widetilde{X}}P &= P_1L_{\widetilde{X}}P + \mathcal{X}_{P_1} \wedge \widetilde{X}\\ L_{P_2\widetilde{Y}}P &= P_2L_{\widetilde{Y}}P + \mathcal{X}_{P_2} \wedge \widetilde{Y}\\ L_{P_3\widetilde{Z}}P &= P_3L_{\widetilde{Z}}P + \mathcal{X}_{P_3} \wedge \widetilde{Z}.\end{aligned}$$

Recall that for every left invariant field $\widetilde{U}$, $L_{\widetilde{U}}P$ is a left invariant tensor, in particular we have

$$\begin{aligned}L_{\widetilde{X}}P &= C^1_{23}\widetilde{Y} \wedge \widetilde{Z} + C^1_{31}\widetilde{Z} \wedge \widetilde{X} + C^1_{12}\widetilde{X} \wedge \widetilde{Y}\\ L_{\widetilde{Y}}P &= C^2_{23}\widetilde{Y} \wedge \widetilde{Z} + C^2_{31}\widetilde{Z} \wedge \widetilde{X} + C^2_{12}\widetilde{X} \wedge \widetilde{Y}\\ L_{\widetilde{Z}}P &= C^3_{23}\widetilde{Y} \wedge \widetilde{Z} + C^3_{31}\widetilde{Z} \wedge \widetilde{X} + C^3_{12}\widetilde{X} \wedge \widetilde{Y},\end{aligned}$$

with, $(C_{ij}^k)_{1\leq i,j,k\leq 3}$ are the structure's constants of $su^*(2)$.
Secondly, the $su^*(2)-$module structure allows us to write

$$\mathcal{X}_{P_i} = (\alpha.P_i)\widetilde{X} + (\beta.P_i)\widetilde{Y} + (\delta.P_i)\widetilde{Z}, \qquad i = 1,2,3,$$

with (α, β, δ) is a base of $su^*(2)$.
A simple calculation allows us to conclude that $L_{X_P}P = 0$.
Now, we show that $[X_p]$ is not zero. Recall that for a smooth function $f \in C^\infty(SU(2))$, the Hamiltonian field associated to f is expressed by :

$$\mathcal{X}_f = U_1(f)\widetilde{X} + U_2(f)\widetilde{Y} + U_3(f)\widetilde{Z}.$$

We consider the local coordinates $(u, v, w) = (y, z, \arctan\frac{t}{x})$ introduced in^{BGS}, around the identity $e = (0, 0, 1)$, the equation $\mathcal{X}_f = X_P$ in this coordinates is expressed by

$$\begin{cases} u\partial_u f + v\partial_v f - \frac{P_2(u,v,w)}{P_1(u,v,w)}\frac{1}{1-u^2-v^2}\partial_w f &= -2 \\ u\partial_u f + v\partial_v f + \frac{P_1(u,v,w)}{P_2(u,v,w)}\frac{1}{1-u^2-v^2}\partial_w f &= -2 \\ u\partial_u f + v\partial_v f &= \frac{2P_3(u,v,w)}{1-u^2-v^2}. \end{cases}$$

It is easy to see that this system didn't solutions and then $[X_p] \neq 0$. □

A group 1-cocycle

Proposition 3. *$\mu^* : SU(2) \to su(2)$ is a 1-cocycle of $SU(2)$ with values in $su(2)$ i.e.,*

$$\mu^*(sr) = Ad_s\mu^*(r) + \mu^*(s), \qquad s \text{ and } r \in SU(2).$$

Proof. By Theorem 1, we have

$$\mu(a\alpha + b\beta + c\delta) = -2aP_1 - 2bP_2 - 2cP_3.$$

By definition of μ^*, we have

$$\mu^*(x) = -2P_1(x)X - 2P_2(x)Y - 2P_3(x)Z\,, \qquad x \in SU(2).$$

Recall that, the structure of algebra $su(2)$ is given by

$$[Z, X] = 2Y\ ;\ [Z, Y] = -2X;\ [X, Y] = 2Z,$$

thus

$$\mu^*(x) = P_1(x)[Y, Z] + P_2(x)[Z, X] + P_3(x)[X, Y].$$

We get $x = exp(t\xi)$, with $\xi \in su(2)$. Then, by derivative, we obtain

$$d_e\mu^*(\xi) = d_eP_1(\xi)[Y, Z] + d_eP_2)\xi)[Z, X] + d_eP_3(\xi)[X, Y].$$

If we denote by L, the bracket on $su(2)$, we conclude that $d_e\mu^* = -L \circ p$.
On the other hand, it is easy to see that, for any $\xi_1, \xi_2, \xi_3 \in su(2)$, we have

$$L(ad_{\xi_1}(\xi_2 \wedge \xi_3) = ad_{\xi_1}(L(\xi_2 \wedge \xi_3)).$$

It's follows that $d_e\mu^*$ is a 1-cocycle of $su(2)$ with values in $su(2)$ i.e.,

$$d_e\mu^*([X, Y]) = ad_X d_e\mu^*(Y) - ad_Y d_e\mu^*(X).$$

since $SU(2)$ is connected and simply connected, the proof is complete. □

As $su(2)$ is semi-simple, by Whithead Lemma, we have

Corollary 1. *There is a 1-cocycle $S \in su(2)$ such that*

$$\mu^*(s) = S - Ad_sS, \quad s \in SU(2).$$

Indeed, we find

$$S = -2k.Z, \quad k \in \mathbb{R}^*.$$

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