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Magnetic Laplacians on differential forms of the hyperbolic disc and Landau Levels

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Abstract

The aim of this paper is to introduce the "Magnetic Laplacian" $\square^p_{\nu,\rho}$; $\nu > 0$ acting on differential p-forms, $p = 0, 1, 2$, of the hyperbolic disc D_ρ of radius $\rho > 0$ and to give the asymptotic behaviour of its point spectrum when $\rho \longmapsto +\infty$. For $p = 0$ we show that it leads to the Landau Levels of the free Hamiltonian of a spinless electron moving on the plane under the action of a uniform magnetic field. For $p = 1$, we recover the point spectrum of the so-called Pauli-Schrödinger operators for a particle $\frac{1}{2}$-spin moving in the plane under a constant magnetic field of magnitude 4ν.

Key Words: Magnetic Laplacian; Differential forms; Point spectrum; Landau Levels.

1 Introduction and overview

On the upper half plane $\mathcal{P}$ with rectangular coordinates (x, y), $x \in \mathbb{R}$, $y > 0$, A. Comtet (see [C]) had considered, up to a constant, the following geometrical and differential items parameterized by $\rho > 0$ and summarized as follows[1]:

$\mathcal{P} = \{(x,y) \in \mathbb{R}^2 \quad / \quad y > 0\}$	
Metric	$ds^2(\rho) = \frac{\rho^2}{4y^2}\left(dx^2 + dy^2\right)$
Scalar Curvature	$S(\rho) = -\frac{4}{\rho^2}$
Volume form	$d\mu(\rho)(x,y) = \frac{\rho^2}{4y^2}dxdy$
Potential vector	$A_\nu(\rho) = \nu\rho^2\frac{dx}{y}; \quad \nu > 0$
Magnetic field	$B_\nu(\rho) = 4\nu d\mu(\rho); \quad \nu > 0$
Hilbert space	$L^2(\mathcal{P}, \frac{\rho^2}{4y^2}dxdy)$
Hamiltonian	$H_\nu(\rho) = 4\left\{\frac{y^2}{\rho^2}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) - 2i\nu y\frac{\partial}{\partial x} - \nu^2\rho^2\right\}, \quad \nu > 0$

Table 1

[1] GNPHE Publication 2004, *e-mail: ajmp@fsr.ac.ma*

and he had shown that the point spectrum of $H_\nu(\rho)$ in $L^2(\mathcal{P}, \frac{\rho^2}{4y^2}dxdy)$ tends to the Landau levels

$$-4\nu(2l+1); \qquad l = 0, 1, 2, \cdots$$

that constitute the point spectrum in $L^2(\mathbb{R}^2, dxdy)$ of the following Schrödinger operator $H_\nu(\infty)$,

$$H_\nu(\infty) = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) + 4\sqrt{-1}\nu\left(y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}\right) - 4\nu^2(x^2+y^2)$$

where the plane $\mathbb{R}^2$ has been viewed as the Wick rotated form of two dimensional space-time over which a constant external magnetic field was acting.
The Hamiltonian $H_\nu(\rho)$,

$$H_\nu(\rho) = 4\left\{\frac{y^2}{\rho^2}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) - 2i\nu y\frac{\partial}{\partial x} - \nu^2\rho^2\right\}; \quad \nu > 0 \tag{1.1}$$

describes the movement of a charged particle on $\mathcal{P}$ when a uniform magnetic field $B_\nu(\rho)$ is perpendicularly applied to the x-y plane (with $B_\nu(\rho)$ a fictitious third dimension $(0,0,4\nu)$). Here the magnetic field should be thought of as a 2-form $B_\nu(\rho) = dA_\nu(\rho)$. Hence, when $A_\nu(\rho)$ is linear, $B_\nu(\rho)$ is the magnitude times the volume form.

According to Comtet's result, one can ask that what the connection (at least formal) there exists between the Hamiltonians $H_\nu(\rho)$ on $\mathcal{P}$ and $H_\nu(\infty)$ on $\mathbb{R}^2$. In this context, we note that a part of the scalar curvature $S(\rho)$ (see Table 1), there is no apparent formal convergence when $\rho \longmapsto +\infty$ of all other items on $\mathcal{P}$ to those on $\mathbb{R}^2$. But using Cayley transform $\mathcal{C}_\rho : w \longmapsto \rho\frac{w-i}{w+i}$, one can get an immediate formal convergence by transforming all structures on $\mathcal{P}$ to the hyperbolic disc D_ρ of radius $\rho > 0$ (see Table 2 in section 2).

The purpose of this paper is to introduce the Magnetic Laplacians $\square^p_{\nu,\rho}$ (resp. $\square^p_{\nu,\infty}$), $\nu > 0$ acting on differential p-forms of the complex disc D_ρ (resp. complex plane $\mathbb{C} = \mathbb{R}^2$) and to give explicitly their expressions (Proposition 1 in Section 2). In addition, we will establish the analogous of the Comtet's result in the case of the differential 1-forms by giving the point spectrum of $\square^1_{\nu,\rho}$ and studying its asymptotic behaviour when $\rho \longmapsto +\infty$ (see Section 3). Finally, we conclude this introduction by mentioning some remarks:

Remark 1.1 *For $\rho = 1$, one can note that the metric $ds^2(1)$, up to the factor $1/4$, is exactly the usual Poincaré metric on $\mathcal{P}$. Also the Schrödinger operator $H_\nu(1)$ has been studied in the Literature of both Mathematics and Physics by many authors (see, e.g., [A-H-S 1; A-P; C; C-H; Gr; I-M]). Further, it can be rewritten in the form*

$$-4y^2\left(\sqrt{-1}\frac{\partial}{\partial x} + \frac{\nu}{y}\right)^2 + 4y^2\frac{\partial^2}{\partial y^2}$$

and it is the same operator as the Maass Laplacian D_ν given by

$$D_\nu = y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) - 2\sqrt{-1}\nu y\frac{\partial}{\partial x}$$

which plays an important role in several fields of Mathematics. Needless to say there has been much work on it (see, e.g., [E; F; M; P]).

Remark 1.2 *The operator $H_\nu(\infty)$ is in the form*

$$H_\nu(\infty) = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) + 4\nu L_3 - 4\nu^2(x^2+y^2)$$

where L_3 is the 3-angular momentum. Further, it can be rewritten in the complex coordinates $z = x + iy \in \mathbb{C}$ as follows:

$$4\left\{\frac{\partial^2}{\partial z \partial \bar{z}} + \nu\left(z\frac{\partial}{\partial z} - \bar{z}\frac{\partial}{\partial \bar{z}}\right)\right\} - 4\nu^2|z|^2 \tag{1.2}$$

and has Landau degeneracy which, following [A-H-B 2], is a characteristic of any system of a non-zero total charge in a constant magnetic field.

Remark 1.3 *Concerning quantum mechanical magnetic bottles in the sense of [A-H-B 1], one can note that the gauge $A_\nu(\rho) = \nu\rho^2 \frac{dx}{y}$ (resp. $A_\nu(\infty) = 2\nu(ydx - xdy)$), canonically associated to the metric $ds^2(\rho)$ (resp. $ds^2(\infty) = dxdy$), is a magnetic bottle of second kind.*

2 Magnetic Laplacians on differential p-forms, $p = 0, 1, 2$

In this section, we will introduce a family of differential operators of magnetic Schrödinger type that we will calling here Magnetic Laplacians. To do this, the following table regroups some basic notations, that we need, on geometrical elements of the complex plane $\mathbb{C}$ and the complex disc $D_\rho = \{z = x + iy \in \mathbb{C} \,/\, |z|^2 = x^2 + y^2 < \rho^2\}$ of radius $\rho > 0$.

	on D_ρ; $\rho > 0$	on $\mathbb{C}$
Hermitian metric	$ds^2_\rho = (1 - \lvert\frac{z}{\rho}\rvert^2)^{-2} dz \otimes d\bar{z}$	$ds^2_\infty = dz \otimes d\bar{z}$
Scalar curvature	$S_\rho = -\frac{4}{\rho^2}$	$S_\infty = 0$
Volume form	$d\mu_\rho(z) = \frac{\sqrt{-1}}{2}(1 - \lvert\frac{z}{\rho}\rvert^2)^{-2} dz \wedge d\bar{z}$	$d\mu_\infty(z) = \frac{\sqrt{-1}}{2} dz \wedge d\bar{z}$
Potential vector	$A_{\nu,\rho}(z) = \nu\sqrt{-1}(1 - \lvert\frac{z}{\rho}\rvert^2)^{-1}(\bar{z}dz - zd\bar{z})$	$A_{\nu,\infty}(z) = \nu\sqrt{-1}(\bar{z}dz - zd\bar{z})$
Magnetic field	$B_{\nu,\rho}(z) = -4\nu d\mu_\rho(z)$	$B_{\nu,\infty}(z) = -4\nu d\mu_\rho(z)$

Table 2.

At once, one can note that $dA_{\nu,\rho}(z) = B_{\nu,\rho}(z)$ and $dA_{\nu,\infty}(z) = B_{\nu,\infty}(z)$. Also the magnetic fields on D_ρ and $\mathbb{C}$ have the same magnitude and same direction (perpendicular to the plane). In addition, we will consider the following spaces

	on D_ρ, $\rho > 0$	on $\mathbb{C}$
Space of p-forms	$\mathcal{C}^\infty(D_\rho, \Lambda^p)$	$\mathcal{C}^\infty(\mathbb{C}, \Lambda^p)$
Hilbert space	$L^2(D_\rho, \Lambda^p)$	$L^2(\mathbb{C}, \Lambda^p)$

Table 3.

Here we recall that a differential form ω on D_ρ (resp. on $\mathbb{C}$) can be written in the following general form

$$\omega(z) = \overbrace{f_{0,0}(z)}^{0-form} \quad + \quad \overbrace{f_{1,0}(z)dz + f_{0,1}(z)d\bar{z}}^{1-form} \quad + \quad \overbrace{f_{1,1}(z)dz \wedge d\bar{z}}^{2-form}$$

with $f_{i,j}$ are in $\mathcal{C}^\infty(D_\rho)$ (resp. $\mathcal{C}^\infty(\mathbb{C})$).

However, the Hilbert space $L^2(D_\rho, \Lambda^p)$ (resp. $L^2(\mathbb{C}, \Lambda^p)$), $p = 0, 1, 2$ is obtained as the completion of $\mathcal{C}_c^\infty(D_\rho, \Lambda^p)$ (resp. $\mathcal{C}_c^\infty(\mathbb{C}, \Lambda^p)$) the $\mathcal{C}^\infty$-differential p-forms that are compactly supported with respect to the natural norm $\|\alpha\|$ induced from the metric ds_ρ^2 (resp. ds_∞^2) given by:

$$\langle\langle \alpha, \beta \rangle\rangle = \int_{D_\rho} \alpha \wedge \star\beta; \quad \alpha, \beta \in \mathcal{C}_c^\infty(D_\rho, \Lambda^p) \tag{2.1}$$

$$\langle\langle \alpha, \beta \rangle\rangle = \int_{\mathbb{C}} \alpha \wedge \star\beta; \quad \alpha, \beta \in \mathcal{C}_c^\infty(\mathbb{C}, \Lambda^p) \tag{2.2}$$

where $\star$ denotes the Hodge star operator[2] associated to ds_ρ^2 (resp. ds_∞^2) and acts on 1, dz, $d\bar{z}$, $dz \wedge d\bar{z}$ by:

	on D_ρ, $\rho > 0$	on $\mathbb{C}$
$\star 1 =$	$\frac{\sqrt{-1}}{2}\left(1 - \lvert\frac{z}{\rho}\rvert^2\right)^{-2} dz \wedge d\bar{z}$	$\frac{\sqrt{-1}}{2} dz \wedge d\bar{z}$
$\star dz =$	$\sqrt{-1} d\bar{z}$	$\sqrt{-1} d\bar{z}$
$\star d\bar{z} =$	$-\sqrt{-1} dz$	$-\sqrt{-1} dz$
$\star(dz \wedge d\bar{z}) =$	$2\sqrt{-1}\left(1 - \lvert\frac{z}{\rho}\rvert^2\right)^2$	$2\sqrt{-1}$

Table 4.

Hence we have

	on D_ρ, $\rho > 0$	on $\mathbb{C}$
Hermtian scalar product		
0-forms $\langle\langle f, g \rangle\rangle =$	$\int_{D_\rho} f(z)\bar{g}(z)(1 - \lvert\frac{z}{\rho}\rvert^2)^{-2} d\lambda(z)$	$\int_{\mathbb{C}} f(z)\bar{g}(z) d\lambda(z)$
1-forms $\langle\langle \alpha, \beta \rangle\rangle =$	$2\int_{D_\rho} f_\alpha(z)\bar{f}_\beta(z) d\lambda(z) +$	$2\int_{\mathbb{C}} f_\alpha(z)\bar{f}_\beta(z) d\lambda(z) +$
$\alpha = f_\alpha dz + g_\alpha d\bar{z}$	$+2\int_{D_\rho} g_\alpha(z)\bar{g}_\beta(z) d\lambda(z)$	$+2\int_{\mathbb{C}} g_\alpha(z)\bar{g}_\beta(z) d\lambda(z)$
$\beta = f_\beta dz + g_\beta d\bar{z}$		
2-forms		
$\langle\langle f dz \wedge d\bar{z}, g dz \wedge d\bar{z} \rangle\rangle =$	$\int_{D_\rho} f(z)\bar{g}(z)(1 - \lvert\frac{z}{\rho}\rvert^2)^2 d\lambda(z)$	$\int_{\mathbb{C}} f(z)\bar{g}(z) d\lambda(z)$
Hilbert Spaces		
0-forms	$L^2(D_\rho, (1 - \lvert\frac{z}{\rho}\rvert^2)^{-2} d\lambda)$	$L^2(\mathbb{C}, d\lambda)$
1-forms	$L^2(D_\rho, d\lambda)dz \oplus L^2(D_\rho, d\lambda)d\bar{z}$	$L^2(\mathbb{C}, d\lambda)dz \oplus L^2(\mathbb{C}, d\lambda)d\bar{z}$
2-forms	$L^2(D_\rho, (1 - \lvert\frac{z}{\rho}\rvert^2)^2 d\lambda) dz \wedge d\bar{z}$	$L^2(\mathbb{C}, d\lambda) dz \wedge d\bar{z}$

Table 5.

where $d\lambda$ denotes here the usual Lebesgue measure.

Now one can construct the following differential items:

[2]For the definition and some properties of this operator see for example [W]

	on D_ρ, $\rho > 0$	on $\mathbb{C}$
Exterior multiplication Coderivation Magnetic Laplacians	$(extA_{\nu,\rho})\omega = A_{\nu,\rho} \wedge \omega$ $\nabla^\nu_\rho = d + \sqrt{-1}A_{\nu,\rho}$ $\Box_{\nu,\rho} = (\nabla^\nu_\rho)^*\nabla^\nu_\rho + \nabla^\nu_\rho(\nabla^\nu_\rho)^*$	$(extA_{\nu,\infty})\omega = A_{\nu,\infty} \wedge \omega$ $\nabla^\nu_\infty = d + \sqrt{-1}A_{\nu,\infty}$ $\Box_{\nu,\infty} = (\nabla^\nu_\infty)^*\nabla^\nu_\infty + \nabla^\nu_\infty(\nabla^\nu_\infty)^*$

Table 6.

Here d is the usual exterior derivation and the adjoint operation * is taken with respect to the Hermitian scalar products given in Table 5.

Remark 2.1 *The above operators $\Box^p_{\nu,\rho}$ and $\Box^p_{\nu,\infty}$ are self-adjoint and elliptic second order differential operators. Furthermore, they are invariant by some action of the appropriate group of motions on $\mathcal{C}^\infty$-differential forms (see [G-I 1]).*

Remark 2.2 *The above Laplacians corresponding to $\nu = 0$ ("no magnetic Laplacian") reduce to the the usual Laplace-Beltrami operator $d^*d + dd^*$ acting on differential forms.*

Remark 2.3 *For $p = 0$, the Laplacian $\Box^0_{\nu,\rho}$ is unitary equivalent to $H_\nu(\rho)$ (that given on $\mathcal{P}$ by (1.1)). More precisely, we have*

$$\left[\Box^0_{\nu,\rho} f\right](z) = -4\left(\frac{\bar{w}-i}{w+i}\right)^{-\nu\rho^2} H_\nu(\rho)\left[\left(\frac{\bar{w}-i}{w+i}\right)^{\nu\rho^2} f\left(\rho.\frac{w-i}{w+i}\right)\right]; \quad f \in \mathcal{C}^\infty(D_\rho), w \in \mathcal{P}$$

Moreover, $H_\nu(\rho)$ can be rewritten in the form $-(\nabla_{A_\nu(\rho)})^*\nabla_{A_\nu(\rho)}$ where $\nabla_{A_\nu(\rho)} = d + \sqrt{-1}ext(A_\nu(\rho))$ ($A_\nu(\rho)$ is as given in Table 1) with respect to the inner product

$$(f, g) = \int_{\mathcal{P}} f(z)\bar{g}(z)\frac{\rho^2}{4y^2}dxdy; \quad f, g \in \mathcal{C}^\infty_c(\mathcal{P})$$

Therefore, the analogous of Comtet's result, mentioned in the introduction, holds for $\Box^0_{\nu,\rho}$.

Remark 2.4 *In high dimension, the intrinsic definition given to the Magnetic Laplacian (in one dimensional see Table 6) works well. In this case the L^2-Spectral theory and its asymptotic behaviour of such Laplacian acting only on functions ($p = 0$) has been completely studied by the authors in [G-I 1] and [G-I 2]. Also for $\rho = 1$, a Selberg trace formulae for heat and wave kernels of such Laplacians "shifted" on compact forms had been established in [A-I].*

Remark 2.5 *By using some elementary properties of the Hodge star operator, one can establish that $\star\Box^p_{\nu,\bullet} = \Box^{2-p}_{-\nu,\bullet}\star$. Thus the study of $\Box^p_{\nu,\bullet}$ reduces further to $p = 0, 1$ and $\nu > 0$. More exactly, one can get by a straightforward computation that*

$$\Box^2_{\nu,\rho}(fdz \wedge d\bar{z}) = \left[(1 - |\frac{z}{\rho}|^2)^{-2}\Box^0_{\nu,\rho}\left((1 - |\frac{z}{\rho}|^2)^2 f\right)\right]dz \wedge d\bar{z}$$

and

$$\Box^2_{\nu,\infty}(fdz \wedge d\bar{z}) = \left[\Box^0_{\nu,\rho} f\right]dz \wedge d\bar{z}$$

Hence and in view of Remark 2.3, we will concentrate to the study of the case $p = 1$.

Now the following proposition gives the explicit expressions in the z-complex coordinates of the $\Box^1_{\nu,\rho}$ and $\Box^1_{\nu,\infty}$ ($p = 1$) viewed as matricial differential operators.

Proposition 1 *Let $\nu > 0$. Then the action of the Magnetic Laplacians $\Box^1_{\nu,\rho}$ and $\Box^1_{\nu,\infty}$ on differential 1-forms $\omega = fdz + gd\bar{z}$ of D_ρ and $\mathbb{C}$ respectively, identified here to the vector column $\begin{pmatrix} f \\ g \end{pmatrix}$, are given explicitly by:*

i)

$$\Box^1_{\nu,\rho}\begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} \Delta^{(\nu,\nu+\frac{2}{\rho^2})} & 0 \\ 0 & \Delta^{(\nu-\frac{2}{\rho^2},\nu)} \end{pmatrix}\begin{pmatrix} f \\ g \end{pmatrix} - 4\nu\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} f \\ g \end{pmatrix}$$

where $\Delta^{(a,b)} = -4\Big(1-|\frac{z}{\rho}|^2\Big)\Big\{\Big(1-|\frac{z}{\rho}|^2\Big)\frac{\partial^2}{\partial z\partial\bar{z}} + az\frac{\partial}{\partial z} - b\bar{z}\frac{\partial}{\partial\bar{z}}\Big\} + 4ab|z|^2,$ *for* $a, b \in \mathbb{R}$.

ii)

$$\Box^1_{\nu,\infty}\begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} \Delta^{\nu} & 0 \\ 0 & \Delta^{\nu} \end{pmatrix}\begin{pmatrix} f \\ g \end{pmatrix} - 4\nu\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} f \\ g \end{pmatrix}$$

with $\Delta^{\nu} = -4\Big\{\frac{\partial^2}{\partial z\partial\bar{z}} + \nu z\frac{\partial}{\partial z} - \nu\bar{z}\frac{\partial}{\partial\bar{z}}\Big\} + 4\nu^2|z|^2$.

Sketch of the Proof. First note that $\Box^1_{\nu,\rho}$ and $\Box^1_{\nu,\infty}$ can be rewritten in the following forms

$$\begin{aligned} \Box^1_{\nu,\rho} &= [d,d^*]_+ \sqrt{-1}\Big([d^*, extA_{\nu,\rho}]_- [(extA_{\nu,\rho})^*, d]_+\Big) + [extA_{\nu,\rho}, (extA_{\nu,\rho})^*]_+ \\ \Box^1_{\nu,\infty} &= [d,d^*]_+ \sqrt{-1}\Big([d^*, extA_{\nu,\infty}]_- [(extA_{\nu,\infty})^*, d]_+\Big) + [extA_{\nu,\infty}, (extA_{\nu,\infty})^*]_+ \end{aligned}$$

where $[A,B]_+ = AB + BA$ is the anti-commutator operator between A and B. Second, by a straightforward computation, using the results of Table 4 and the facts that $d^* = -\star d\star$ and $(extA)^* = \star(extA)\star$, we get the desired result of the proposition. (For more details see [G]).

Remark 2.6 *According to Proposition 1, the matrix* $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} =: \sigma_3$ *is the third spin-Pauli matrix. In addition, the operators $\Box^1_{\nu,\infty}$ give rise to the so-called Pauli-Schrödinger operators for a spin particle in the plane $\mathbb{C} = \mathbb{R}^2$ moving under a constant magnetic field.*

Remark 2.7 *Adding to the fact that the Hilbert space $L^2(D_\rho, \Lambda^1)$ (resp. $L^2(\mathbb{C}, \Lambda^1)$) splits as $L^2(D_\rho, \Lambda^1) = L^2(D_\rho, d\lambda)dz \oplus L^2(D_\rho, d\lambda)d\bar{z}$ (resp. $L^2(\mathbb{C}, \Lambda^1) = L^2(\mathbb{C}, d\lambda)dz \oplus L^2(\mathbb{C}, d\lambda)d\bar{z}$), see Table 5, the above proposition shows that the Magnetic Laplacian $\Box^1_{\nu,\rho}$ (resp. $\Box^1_{\nu,\infty}$) can be also splited in terms of usual differential operators acting on functions. More exactly, we have*

$$\Box^1_{\nu,\rho}(fdz + gd\bar{z}) = \Big[\Box^{(1,0)}_{\nu,\rho} f\Big]dz + \Big[\Box^{(0,1)}_{\nu,\rho} g\Big]d\bar{z}$$

and

$$\Box^1_{\nu,\infty}(fdz + gd\bar{z}) = \Big[\Box^{(1,0)}_{\nu,\infty} f\Big]dz + \Big[\Box^{(0,1)}_{\nu,\infty} g\Big]d\bar{z}$$

with

$$\Box^{(1,0)}_{\nu,\rho} = \Delta^{(\nu,\nu+\frac{2}{\rho^2})} - 4\nu \quad \textit{and} \quad \Box^{(0,1)}_{\nu,\rho} = \Delta^{(\nu-\frac{2}{\rho^2},\nu)} + 4\nu$$

and

$$\Box^{(1,0)}_{\nu,\infty} = \Delta^{\nu} - 4\nu \quad \textit{and} \quad \Box^{(0,1)}_{\nu,\infty} = \Delta^{\nu} + 4\nu$$

Now, with the help of the above explicit expression of $\Box^1_{\nu,\rho}$ (resp. $\Box^1_{\nu,\infty}$), one can do a concrete spectral theory of such operators acting on 1-forms (for further results see [G]). We limit here to give only its point spectrum and then to show that the analogous of Comtet's result ($p = 0$) holds also for $p = 1$. Namely, our main result is the following:

Proposition 2 *Let $\square^1_{\nu,\rho}$ (resp. $\square^1_{\nu,\infty}$) be the Magnetic Laplacian on D_ρ (resp. on $\mathbb{C}$) as given in Proposition 1. Then, we have*

i)- The point spectrum of $\square^1_{\nu,\rho}$ is given by $\sigma_d(\Delta^1_{\nu,\rho}) = \sigma_d(\square^{(1,0)}_{\nu,\rho}) \cup \sigma_d(\square^{(0,1)}_{\nu,\rho})$ *with*

$$\begin{aligned}\sigma_d(\square^{(1,0)}_{\nu,\rho}) &= \left\{8\nu l - 4\frac{l(l-1)}{\rho^2};\ l = 0,1,2,\cdots;\ 0 \le l < \nu\rho^2 + \frac{1}{2}\right\} \\ \sigma_d(\square^{(0,1)}_{\nu,\rho}) &= \left\{8\nu(l+1) - 4\frac{(l+1)(l+2)}{\rho^2};\ l = 0,1,2,\cdots;\ 0 \le l < \nu\rho^2 + \frac{3}{2}\right\}\end{aligned}$$

ii)- The point spectrum of $\square^1_{\nu,\infty}$ is given by

$$\sigma_d(\square^1_{\nu,\infty}) = \{8\nu l; \quad l = 0,1,2,...\}$$

iii)- The point spectrum of $\square^1_{\nu,\rho}$ converges to that of $\square^1_{\nu,\infty}$ when ρ goes to infinity.

For the proof of this proposition, we proceed as in [G-I 1] for the case of the Magnetic Laplacians acting on functions. Essentially, it is based on the resolution of some partial differential equations and on the computation of some integrals of special functions as Gauss hypergeometric function and the confluent hypergeometric function (as main reference we cite [N-O]).

Remark 2.8 *The point spectrum of $\square^1_{\nu,\infty}$ is in the form $\sigma_d(\square^1_{\nu,\infty}) = \sigma_d(\square^{(1,0)}_{\nu,\infty}) \cup \sigma_d(\square^{(0,1)}_{\nu,\infty})$ where*

$$\begin{aligned}\sigma_d(\square^{(1,0)}_{\nu,\infty}) &= \{8\nu l; \quad l = 0,1,2,...\} \\ \sigma_d(\square^{(0,1)}_{\nu,\rho}) &= \{8\nu(l+1); \quad l = 0,1,2,\cdots\}\end{aligned}$$

Remark 2.9 *For $\rho = 1$ the study of L^2-Spectral theory of $\square^1_{\nu,1}$ can be done directly by decomposing $\square^1_{\nu,1}$ in terms of Maass Laplacians $\Delta_{\alpha,\beta}$ that studied concretely in [B-I].*

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