African Journal Of Mathematical Physics **1** *(2004)61-68*

On Manifolds of G_2 Holonomy in M- theory compactifications

Adil Belhaj[1,2*]

1. ITF and Physics Department, Universita Autonoma de Madrid, Spain,

2. Groupement National de Physique des hautes Energies, GNPHE, Siege focal, FSR, Rabat, Morocco.

Abstract

In this work, we present quotients $X_7 = \frac{X_8}{U(1)}$ of G_2 holonomy form toric" hyper-Kahler eight real dimensional manifolds X_8 using $N = 4$ sigma model, with $U(1)^r$ gauge symmetry, with ADE Cartan vector charges. In particular, for the A_r Cartan matrix, the quotient space is a cone on a S^2 bundle over r intersecting $\mathbf{WCP}^2_{(1,2,1)}$ weighted projective spaces according to the A_r Dynkin diagram.

I. INTRODUCTION

The idea underlying the string theory is that the fundamental objects (like electrons) of physics are no longer considered to be point particles of zero dimension but rather extended objects of dimension one, string[1]. During its classical motion, the string, being closed or open, generates, in space time, a two dimensional surface which is called worldsheet*. The classical model of the string theory can be seen as a two dimensional field theory. At quantum level, the constraintes of conformal invariance require that the space time dimension be 26 instead of our space time that we observe. This type of model, which is called bosonic string is not consistent due to the presence of tachyon and the absence of the fermions in the spectrum. Both problems can be solved in the context of the ten dimesional superstring theory obtained by adding fermions in the world-sheet. In terms of weak coupling perturbation theory there appear to be only five different consistent superstring models known as Type IIA, Type IIB, $SO(32)$ Heterotic, $E_8 \times E_8$ Heterotic and Type I $SO(32)$. The spectrum of massless states of these models contains, in general, the dilaton (ϕ), graviton ($g_{\mu\nu}$) and anti-symmetric gauge tensors generalizing the notion of vector potential A_μ to $A_{\mu_1 \ldots, \mu_{p+1}}$ $(p+1)$ forms, depending on the theory on question. The type II superstings do not contain non abelian gauge symmetries. They do however contain Dbranes objects which are coupled to $(p+1)$ gauge forms. These non perturbative objects are characterized by the fact that open strings can end on them. The letters have Chan-Paton gauge degrees of freedom located at the end points and so they contains the usual gauge theories. To make contact with our 4-dimensional world we need to compactify the 10-dimensional superstring theory on a 6-dimensional compact manifold. This way of doing assumes that the extra dimensions are compact are not observable on our scale. In this way, we have learn that superstring propagating not on ten dimensional Minkowski space but a four dimensional space-time times a six dimensional compact manifold: $M_4 \times X_6$. One way could simply be to consider the extra 6 dimensions as 6-dimensional Torus $X_6 = T^6 = (S^1)^6$. As it turns out this would preserve too much supersymmetry. To preserve the minimal amount of supersymmetry, either $N = 1$ from heterotic superstrings , or $N = 2$ from type II superstrings, in 4 dimensions, we need to compactify on a special kind of 6-manifold called a Calabi-Yau manifold with $SU(3)$ holonomy. The compactification is not a way to overcome the problem of the spacetime dimension but also it offers

* GNPHE Publication 2004, *e-mail: ajmp@fsr.ac.ma*

the possibility of connecting the various models in lower dimensions via duality symmetry. Before talking on these symmetries in lower dimensions, it should be interesting to recall the so called S-duality in ten dimensions. The latter relates the strong coupling limit of one theory to the weak coupling limit of another theory. For example the $SO(32)$ heterotic superstring and the Type I superstring models are S-dual in 10 dimensions which means that the strong coupling limit of the $SO(32)$ heterotic superstring is the weakly coupled type I superstring and visa versa. The other S-duality in 10 dimensions called $Sl(2,Z)$ is the self duality of the IIB superstring. In this way, the strong coupling limit of the IIB string is another weakly coupled IIB string theory.This non perturbative duality has been confirmed using solitonic D-branes physics.

After compactification to nine dimensions, the first kind of duality that one has is called T-duality relating type IIA compactified on a circle with radius R_{IIA}, to type IIB theory compactified on a circle with radius $R_{IIA} = \frac{1}{R_{IIB}}$. Note that the two heterotic superstrings are equivalent under T-duality. It turns out that T-duality, up compactification in nine dimensions, leaves in principal three independent consistent superstring models. However these models with different number of space-time supersymmetries can be related after compactifications on different manifolds leading in lower dimensions to the same number of supersymmetries. The non trivial example of this feature arises in six dimensions connecting type IIA superstrings on K3 and heterotic superstrings on T^4. The latters have both the same physical moduli spaces and $N=2$ supersymmetry. This duality is also confirmed using wrapped D2-branes on vanishing 2-cycles of K3. The latters lead to massless state particles giving rise to enhanced non abelian gauge symmetries in the IIA supersting side. It was observed that this duality leads to the first indication to non perturbative understanding of the superstring theory. Is was shown that, this duality can be pushed up two dimensions by using the adiabatic argument. This equivalence leads to the first exact solution for the Coulomb branch $N=2$ quantum field theory. These results has been developed using the geometric engineering method which based on the local mirror symmetry and the toric geometry realization of three dimensional Calabi-Yau's. One of the consequence of superstring dualities is that all superstring models are equivalent in the sense that they correspond to different limits in moduli space of the same theory, called M-theory. M-theory is considered nowadays as the best candidate for the unification of the weak and strong coupling sectors of superstring models[2-4]. It is described, at low energies, by an eleven dimensional supergravity theory. The compactification of this theory on a seven real dimensional manifold X_7 of G_2 holonomy gives rise to an $N=1$ models with four supercharges in four dimensions, i.e $N=1$ $D=4$ [4-11]. In this regard, the four dimensional resulting physics models depend on geometric properties of X_7. For instance, if X_7 is smooth, the low energy theory contains, in addition to $N=1$ supergravity, only abelian gauge group and neutral chiral multiplets. However, non abelian gauge symmetries with chiral fermions can be obtained using singular manifold compactifications [10,11]. Following[12], a beautiful analysis for building singular spaces of G_2 holonomy is to consider the quotient of a conical hyper-Kahler eight manifolds X_8 by a $U(1)$ symmetry. This approach, which is called the unfolding of the singularity, guarantees the G_2 holonomy group of the quotient space $\frac{X_8}{U(1)}$. A remarkable feature of this method, which may be related to two dimensional ($2D$) $N=4$ sigma model Calabi-Yau fourfolds construction $CY^4 = X_8$, is that the $\frac{X_8}{U(1)}$ space solutions differ by what kind of $U(1)$ symmetry is chosen and moreover the matter fields, in four dimensions, are obtained using similar techniques of the geometric engineering of quantum field theory [12-15].

The aim of this work is to contribute in this direction by considering models with $N=4$ $2D$ sigma model ADE Cartan matrix gauge charges for building X_8 manifolds. Actually, these examples go beyond of the first example studied in [11], where the matrix charge of the hypermultiplets ϕ_i, under $U(1)^r$ gauge symmetry, was

$$q_i^a = -\delta_{i-1}^a + \delta_i^a, \quad a = 1,\ldots,r, \qquad (1.1)$$

and may give a link between the corresponding geometries and the structure of ADE Lie algebras as in the case of K3 surfaces.

It was suggested in [11] that the analysis may be adapted to others examples of X_8 manifolds, in particular toric-hyper-Kahler manifolds. In this paper, we would like to present a new class of these X_8 manifolds, which will be called toric-hyper-Kahler eight manifods X_8, with the Calabi-Yau condition in sigma model construction

$$\sum_i q_i^a = 0, \qquad (1.2)$$

and their quotients by a $U(1)$ group symmetry using toric geometry circle actions. Our way involves two steps:

First we introduce the ADE Cartan matrices as $2D$ $N=4$ $U(1)^r$ linear sigma model matrix gauge charges. Second mimicking the analysis of[12] and using toric geometry circle actions, we discuss the construction of a new class of the quotients $\frac{X_8}{U(1)}$ of G_2 holonomy group. In particular, for the A_r Cartan matrix, the quotient space is a cone on a S^2 bundle over r intersecting $\mathbf{WCP}^2_{(1,2,1)}$ weighted projective spaces according to the A_r Dynkin geometry.

The organization of this paper is as follows: In section 2, we give an overview on aspects of $2D$ $N=4$ linear sigma model. Then we give examples illustrating the field theoretical construction of hyper-Kahler manifolds. In section 3, we introduce the ADE Cartan matrices as matrix gauge charges in the $2D$ $N=4$ field theory construction of X_8 manifolds. For the A_r Lie algebra, the moduli space of the classical theory is given by the cotangent bundle over r intersecting $\mathbf{WPC}^2_{1,2,1}$ weighted projective spaces according to the A_r Dynkin graph, extending the A_r singularity of K3 surfaces described by $N=2$ type IIA superstring sigma model. In section 4, we identify the $U(1)$ symmetry group with the toric geometry circle actions of X_8 to present quotients $X_7 = \frac{X_8}{U(1)}$ of G_2 holonomy. For the A_r Cartan matrix gauge charge, the geometry is a cone on a S^2 bundle over r intersecting $\mathbf{WCP}^2_{(1,2,1)}$ weighted projective spaces according to the A_r Dynkin diagram. Our conclusion will be given in section 5.

II. $N = 4$ SIGMA MODEL APPROACH

In this section we review the main lines of the $N=4$ sigma model approach for building the hyper-Kahler manifolds involved in the study of superstring, M-theory and F-theory, compactifications, Yang Mills small instantan singularities and more general in supersymmetric field theories with eight supercharges [16,17]. For this purpose, consider $2D$ $N=4$ supersymmetric $U(1)^r$ gauge theory with n hypermultiplets ϕ_i $(i=1,\dots,n)$ of a matrix charge q_i^a $(a=1,\dots,r)$, under $U(1)^r$ gauge symmetry, and r 3-vectors FI coupling $\vec{\xi}_a$. The equations defining the hyper-Kahler moduli space of this classical gauge theory are given by the following D-terms

$$\sum_i q_a^i[\phi_i^\alpha\bar{\phi}_{i\beta} + \phi_{i\beta}\bar{\phi}_i^\alpha] = \vec{\xi}_a\vec{\sigma}^\alpha_\beta. \qquad (2.1)$$

The double index (i,α) of ϕ_i^α's refer to component field doublets of the n hypermultiplets, and $\vec{\sigma}^\alpha_\beta$ are the traceless 2×2 Pauli matrices. For later use it is interesting to note the following points:

(1) Equations (2.1) have a formal analogy with the D-flatness equations of $2D$ $N=2$ $U(1)^r$ toric sigma model involved in the study of type II superstring compactifications on ALE spaces with ADE singularities[13]. The latters are given by:

$$\sum_{i=1}^n q_i^a|x_i|^2 = R_a, \quad a=1,\dots,r \qquad (2.2)$$

where r is the rank of the ADE Lie algebras and where q_i^a, up some details, the minus of the corresponding Cartan matrices satisfying the Calabi-Yau condition $\sum_{i=1}^n q_i^a = 0$.

(2) For each $U(1)$ factor, there are three real constraint equations transforming as an iso-triplet of $SU(2)$ R-symmetry $(SU(2)_R)$ acting on the hyper-Kahler structures.

(3) Using the $SU(2)_R$ transformations

$$\begin{aligned}\phi^\alpha &= \varepsilon^{\alpha\beta}\phi_\beta, \quad \varepsilon_{12} = \varepsilon^{21} = 1 \qquad (2.3)\\ \overline{(\phi^\alpha)} &= \overline{\phi}_\alpha,\end{aligned}$$

and replacing the Pauli matrices by their expressions, the identities (2.1) can be split as follows:

$$\sum_{i=1}^n q_i^a(|\phi_i^1|^2 - |\phi_i^2|^2) = \xi_a^3 \qquad (2.4)$$

$$\sum_{i=1}^n q_i^a\phi_i^1\overline{\phi}_i^2 = \xi_a^1 + i\xi^2{}_a \qquad (2.5)$$

$$\sum_{i=1}^n q_i^a\phi_i^2\overline{\phi}_i^1 = \xi_a^1 - i\xi^2{}_a. \qquad (2.6)$$

Note that these equations have similar features of the description of [16] leaving only half the supersymmetry of the gauge model.

(4) After dividing the moduli space of zero energy states of the classical gauge theory (2.1) by the action of the $U(1)^r$ gauge symmetry, we find precisely a toric-hyper-Kahler variety $X_{4(k-r)}$ of $4(k-r)$ real dimensions. This construction is called the hyper-Kahler quotient extending the Kahler one involved in $2D$ $N=2$ toric sigma model.

(5) The solutions of eqs (2.1) depend on the values of the FI couplings. For the case where $\xi^1 = \xi^2 = 0$ and $\xi^3 > 0$, it is not difficult to see that eqs (2.1) describe the cotangent bundle over a toric variety defined by

$$\sum_{i=1}^n q_i^a|\phi_i^1|^2 = \xi_a^3. \qquad (2.7)$$

Indeed, if we set all $\phi_i^2 = 0$, the ϕ_i^1's, modulo the complexified $U(1)^r$ gauge group, determine a toric variety $\frac{C^n}{C^{*r}}$ of $2(n-r)$ real dimensions, see equations (2.2). The equations (2.6-7) mean that the

ϕ_i^2's define the cotangent fiber directions over the toric variety given by (2.8). To see this feature, we assume that $\xi_a^1 = \xi_a^2 = 0$ and we set $q_i^a = 1$, so we have

$$\sum_{i=1}^{n}(|\phi_i^1|^2 - |\phi_i^2|^2) = \xi^3 \quad (2.8)$$

and

$$\sum_{i=1}^{n}\phi_i^1\overline{\phi}_i^2 = 0. \quad (2.9)$$

Equation (2.10) means that ϕ_i^2 parameterizes the cotangent directions over a $\mathbf{CP^{n-1}}$ projective space defined by $\sum_{i=1}^{n}|\phi_i^1|^2 = \xi^3$. In what follows, we give two extra examples illustrating this analysis and reconsidering the example given in [11]. In the first example, we consider a $2D$ $N = 4$ $U(1)$ linear sigma model with two hypermultiplets of a vector charge $(1, -1)$. The D-flatness conditions of this model read as

$$(|\phi_1^1|^2 - |\phi_2^1|^2) - (|\phi_1^2|^2 - |\phi_2^2|^2) = \xi^3 \quad (2.10)$$
$$\phi_1^1\overline{\phi}_1^2 - \phi_2^1\overline{\phi}_2^2 = 0 \quad (2.11)$$
$$\phi_1^2\overline{\phi}_1^1 - \phi_2^2\overline{\phi}_2^1 = 0. \quad (2.12)$$

Permuting the role of ϕ_2^1 and $\bar{\phi}_2^2$, and making the following field change $\varphi_1 = \phi_1^1$ $\varphi_2 = -\bar{\phi}_2^2$ $\psi_1 = \phi_1^2$ and $\psi_2 = \bar{\phi}_2^1$, the constraint equations (2.9-11) become

$$\xi^3 = |\varphi_1|^2 + |\varphi_2|^2) - (|\psi_1|^2 + |\psi_2|^2) \quad (2.13)$$
$$\varphi_1 0 = \overline{\psi}_1 + \varphi_2\overline{\psi}_2 \quad (2.14)$$
$$0 = \overline{\varphi_1}\psi_1 + \overline{\varphi_2}\psi_2 \quad (2.15)$$

and describe a cotangent bundle over a $\mathbf{CP^1}$ projective. In this way, the $\mathbf{CP^1}$ is defined by the following equation:

$$\xi^3 = |\varphi_1|^2 + |\varphi_2|^2. \quad (2.16)$$

Recall, in passing, that the cotangent bundle over $\mathbf{CP^1}$, which is known by the resolved A_1 singularity of K3 surfaces, is isomorphic to $\frac{C^2}{Z_2}$ and plays a crucial role in the study of the non perturbative limit of type II superstring dynamics in six and four dimensions [13,14,15]. The second example we want to consider deals with the generalization of the first one. This concerns a $2D$ $N = 4$ $U(1)^r$ linear sigma model with $(r + 1)$ hypermultiplets of a matrix charge satisfying (1.1). Using the same procedure, the D-flatness conditions (2.1) become:

$$\xi_a^3 = (|\varphi_{a-1}|^2 + |\varphi_a|^2) - (|\psi_{a-1}|^2 + |\psi_a|^2) \quad (2.17)$$
$$0 = \psi_{a-1}\overline{\varphi}_{a-1} + \varphi_a\overline{\psi}_a \quad (2.18)$$
$$0 = \varphi_{a-1}\overline{\psi}_{a-1} + \psi_a\overline{\varphi}_a. \quad (2.19)$$

The solution of these equations describes the cotangent bundle over r intersecting complex curves $\mathbf{CP^1}$. In the limit when all ξ_a^3 go to zero, the CP^1's shrink and one ends with the A_r singularity of local K3 surfaces. Note that this example has been used in [11] to construct seven real dimensional manifolds X_7 of G_2 holonomy group from the quotient of X_8 hyper-Kahler eight real dimensional manifolds by an $U(1)$ group symmetry. These eight dimensional spaces are obtained using $2D$ $N = 4$ $U(1)^{(r-1)}$ linear sigma model with $(r + 1)$ hypermultiplets, where the missing $U(1)$ invariance is explored to get the quotient $\frac{X_8}{U(1)}$ of G_2 holonomy group [11]. In what follows we want to give a new class of X_8 manifolds, which will be called toric-hyper-Kahler Calabi-Yau fourfolds $(CY^4 = X_8)$ by introducing the ADE Cartan matrices instead of the gauge matrix charge given in equation (1.1).

III. TORIC-HYPER-KAHLER EIGHT MANIFOLDS WITH CALABI-YAU CONDITION

We start this section by recalling that complex Calabi-Yau manifolds are the best ingredients for obtaining semi-realistic models of superstrings/M/F-theory [18,19,20], with minimal supercharges in lower dimensions. In particular, for later use, Calabi-Yau fourfolds, compact, non compact, singular or non-singular, are considered as ways for getting $N = 1$ supersymmetric models in four dimensions from the F-theory compactifications [20,21]. From M-theory side, compatifications on manifolds of G_2 holonomy can be effectively described by four dimensional $N = 1$ supersymmetry. Furthermore, from supersymmetry breaking viewpoint, the above geometries, which preserve both the same supercharges in particular $\frac{1}{8}$ of initial ones of the uncompactified theory, have a similar role in superstrings and M-theory compactifications. From this physical argument and the string duality results, connecting type IIA and type IIB strings, we think that there are, at least, two natural questions. The latters are as follows: (1) Exist there a four dimensional duality connecting M-theory on manifolds of G_2 holonomy and F-theory on Calabi-Yau fourfolds?. (2) Or exist there a link between the corresponding geometries,(manifolds of G_2 holonomy and Calabi-Yau fourfolds)?. These questions, which are quite similar to the link between M-theory on manifolds of G_2 holonomy and heterotic strings on Calabi-Yau threefolds, need deeper study. However, here we try to give a modest comment on the the second

one; while the first one will be dealt with in future work. This comment is based on the following known points:
(i) Manifolds with G_2 holonomy can be constructed as $U(1)$ quotients of eight manifolds.
(ii) The maximal group of automorphisms in eight dimensions is $SO(8)$. Using Dynkin geometries, this group, including the SU(4) group, can give the G_2 group.
(iii) Eight manifolds can have hyper-Kahler constructions in terms of $N=4$ sigma model.
Combining these points with the Calabi-Yau condtuion, $\sum_i q_i^a = 0$, in sigma model approach, one may say that seven real dimensional manifolds of G_2 holonomy group may be constructed from Calabi-Yau fourfolds geometries, in particular from their quotients by one finite circle, preserving the supercharges. In this present study, using similar ideas of [11], we would like to discuss the construction of seven dimensional manifolds with G_2 holonomy group from Calabi-Yau fourfolds geometry physics data, but with a different realization of the $U(1)$ group symmetry for obtaining the quotient. This study involves two steps. First we will introduce, in the field theoretical construction of Calabi-Yau fourfolds, the ADE Cartan matrices as $2D$ $N=4$ linear sigma model matrix gauge charges. Second, mimicking the method of [11] and using toric geometry circle actions, we will discuss quotients $\frac{X_8}{U(1)}$ of G_2 Holonomy group which will be given in the next section. Roughly speaking the toric-hyper- Calabi-Yau fourfolds $CY^4 = X_8$ may be viewed as the moduli space of $2D$ $N=4$ supersymmetric $U(1)^r$ gauge theory with $(r+2)$ ϕ_i^α hyper-multiplets $(4(r+2-r)=8)$ with a matrix charge q_a^i with the Calabi-Yau condition (2.3). In what follows, we will consider a matrix charge going beyond the equation (1.1). Our choice will be given by ADE Cartan matrices. For simplicity, we first consider the A_r Lie algebra where the Cartan matrix is given by

$$q_i^a = -2\delta_i^a + \delta_{i-1}^a + \delta_{i+1}^a, \quad a=1,\ldots,r, \tag{3.1}$$

satisfying the Calabi-Yau condition $\sum_i q_a^i = 0$. Putting these equations into the D-flatness equations(2.1), one gets the following system of $3r$ equations:

$$(|\phi_{a-1}^1|^2 + |\phi_{a+1}^1|^2 - 2|\phi_a^1|^2) - (|\phi_{a-1}^2|^2 + |\phi_{a+1}^2|^2 - 2|\phi_a^2|^2) = \xi_a \tag{3.2}$$

$$\phi_{a-1}^1\overline{\phi^2}_{a-1} + \phi_{a+1}^1\overline{\phi^2}_{a+1} - 2\phi_a^1\overline{\phi^2}_a = 0 \tag{3.3}$$

$$\phi_{a-1}^2\overline{\phi^1}_{a-1} + \phi_{a+1}^2\overline{\phi^1}_{a+1} - 2\phi_a^2\overline{\phi^1}_a = 0. \tag{3.4}$$

We first solve these equations for the simple example of $U(1)$ gauge theory. Then we will give the result for the $U(1)^r$ gauge model. For $r=1$, the above equations reduce to

$$(|\phi_0^1|^2 + |\phi_2^1|^2 - 2|\phi_1^1|^2) - (|\phi_0^2|^2 + |\phi_2^2|^2 - 2|\phi_1^2|^2) = \xi \tag{3.5}$$

$$\phi_0^1\overline{\phi^2}_0 + \phi_2^1\overline{\phi^2}_2 - 2\phi_1^1\overline{\phi^2}_1 = 0 \tag{3.6}$$

$$\phi_0^2\overline{\phi^1}_0 + \phi_2^2\overline{\phi^1}_2 - 2\phi_1^2\overline{\phi^1}_1 = 0. \tag{3.7}$$

To handle these D-terms equations, it should be interesting to note that they are quite similar to equations (2-9-10), and also (2.11-13). After permuting the role of ϕ_2^1 and $\overline{\phi^2}_2$, equations may be rewritten as

$$(|\phi_0^1|^2 + |\phi_2^1|^2 + 2|\overline{-\phi_1^2}|^2) - (|\phi_0^2|^2 + |\phi_1^2|^2 + 2|\phi_1^1|^2) = \xi \tag{3.8}$$

$$\phi_0^1\overline{\phi^2}_0 + \phi_2^1\overline{\phi^2}_2 + 2\phi_1^1\overline{(-\phi^2{}_1)} = 0 \tag{3.9}$$

$$\phi_0^2\overline{\phi^1}_0 + \phi_2^2\overline{\phi^1}_2 + 2(-\phi_1^2)\overline{\phi^1}_1 = 0. \tag{3.10}$$

Making the following field changes

$$\phi_0^1 = \varphi_1, \quad \phi_0^2 = \psi_1$$
$$\phi_1^1 = \varphi_2, \quad \phi_1^2 = \psi_2$$
$$-\overline{\phi}_1^2 = \varphi_3, \quad \overline{\phi}_1^1 = \psi_1,$$

the above equations become

$$(|\varphi_1|^2 + |\varphi_3|^2 + 2|\varphi_2|^2) - (|\psi_1|^2 + |\psi_3|^2 + 2|\psi_2|^2) = \xi^3 \tag{3.11}$$

$$\varphi_1\overline{\psi_1} + \varphi_3\overline{\psi_3} + 2\varphi_2\overline{\psi_2} = 0 \tag{3.12}$$

$$\overline{\varphi_1}\psi_1 + \overline{\varphi_3}\psi_3 + 2\overline{\varphi_2}\psi_2 = 0. \tag{3.13}$$

Using similar analysis of the previous section, one sees that the above equations describe a cotangent bunble over $\mathbf{WCP^2_{1,2,1}}$ weighted projective space. A way to to see this feature is to use the link between $N=2$ sigma model and toric geometry technics. Indeed, taking $\psi_1 = \psi_2 = \psi_3 = 0$, equations (3.11-13) reduce to

$$|\varphi_1|^2 + |\varphi_3|^2 + 2|\varphi_2|^2 = \xi^3 \tag{3.14}$$

and can be encoded in a toric diagram. In this diagram, one has three vectors v_1, v_2 and v_3 in Z^2 lattice such that

$$v_1 + v_3 + 2v_2 = 0 \tag{3.15}$$

where the coefficients of v_i are exactly the ones of φ_i in (3.14). Using toric geometry language, equation (3.15) defines naturally a $\mathbf{WCP^2_{1,2,1}}$ weighted projective space, where ξ^3 is a Kahler real parameter controlling its seize. The equation (3.11-13), for generic value of ψ_i, can be interpreted to mean that ψ_i parameterizes the fiber cotangent directions over $\mathbf{WCP^2_{1,2,1}}$. Since the subset of (3.11) with $\psi_i = 0$ is a $\mathbf{WCP^2_{1,2,1}}$ weighted projective space and $\varphi_1\overline{\psi_1} + \varphi_3\overline{\psi_3} + 2\varphi_2\overline{\psi_2} = 0$, thus the the space of solutions of (3.11-13) is isomorphic to the cotangent space over $\mathbf{WCP^2_{1,2,1}}$, $T^*(\mathbf{WCP^2_{1,2,1}})$. In the general case corresponding to the $U(1)^r$ gauge theory, if we take the all ξ_a's are no zero, it not too difficult to see that equations (3.2-4) describe the cotangent bundle over r intersecting $\mathbf{WCP^2_{1,2,1}}$ weighted projective spaces. This means that the base geometry, of the cotangent bundle, consists of r intersecting $\mathbf{WCP^2_{1,2,1}}$ according to the A_r Dynkin diagram, instead of one projective space in the case of $U(1)$ gauge theory. In the limit that some ξ_a's are zero, we obtain a singular geometry. Actually, this geometry may be used to extend the intersecting $\mathbf{CP^1}$ projective spaces of ALE spaces involved in the geometric engineering method of the quantum field theory [13,14,15]. We will conclude this section by noting that this analysis of the A_r Lie algebra may be extended to the others DE Lie algebras. However, these algebras contain trivalent vertex Dynkin geometries which complicates the computation. Recall that the trivalent Dynkin geometry involves a central node intersecting three other nodes once; moreover this geometry has been used in the geometric engineering of quantum field theories, in particular in the introduction of fundamental matters in a chain of SU product gauge group with $N = 2$ bifundamental matters. In toric sigma model approach, the corresponding vector charge, up the Calabi-Yau condition, is given by

$$q_i = (0, \dots, -2, 1, 1, 1, 0, \dots, 0, -1),$$

instead of the bivalent geometry (3.1). A priori there are different ways one may follow to overcome this problem. A naive way is to delete these trivalent vertices. In this case, the D-flatness constraint equations have similar solutions of the A_n Lie algebra. However a tricky method is to leave and use the trivalent geometry results involved in the elliptic fibrations singularities over the complex plane. In this way, the base geometry of X_8 may be given by three chains of intersecting $\mathbf{WCP^2_{1,2,1}}$ according to the trivalent geometry.

In what follows we would like to discuss the corresponding seven real dimensional manifolds of G_2 holonomy group. Similarly to the ideas of [11], we should look for a $U(1)$ group symmetry acting on X_8. As mentioned before, there are many ways one may follow to choosing the $U(1)$ group action of X_8. In this regard, the solutions differ by what kind of $U(1)$ symmetry is chosen. Two kinds of solutions are given in [11]. But here we will consider another way. The latter is inspired from the toric geometry circle actions.

IV. ON THE QUOTIENT SPACE $X_7 = \frac{X_8}{U(1)}$ OF G_2 HOLONOMY

Having constructed toric-hyper-Kahler Calabi-Yau fourfolds X_8 associated to ADE Cartan matrices sigma model gauge charges. We are now in the position to carry out quotient spaces $X_7 = \frac{X_8}{U(1)}$ of G_2 holonomy group using circle actions involved in toric varieties. Before doing this, let us tell some things about toric geometry. The letter is a powerful tool for studying n-dimensional complex manifolds exhibiting toric circle actions $U(1)^n$ which allow to encode the geometric properties of the complex spaces in terms of simple combinatorial data of polytopes Δ of the R^n real space [22,23,24,25]. The simple example of toric varieties is the complex plane $\mathbf{C}$. The latter admits a $U(1)$ toric action

$$z \to ze^{i\theta}, \tag{4.1}$$

which has a fixed point at $z = 0$. Thus the toric geometry of $\mathbf{C}$ can be viewed as a circle fibered on a half line parameterized by $|z|$. The circle, which determined by the action of θ, shrinks at $z = 0$. This realization can be generalized easily to $\mathbf{C^n}$ space where we have a T^n fibration, parameterized by the angular coordinates θ_i, over a n-dimensional real base parameterized by $|z_i|$. The second example we want to give is the $\mathbf{CP^1}$ projective space. This space has also a $U(1)$ toric action having two fixed points describing respectively north and south poles of the two sphere $S^2 \sim \mathbf{CP^1}$. Thus the toric geometry of $\mathbf{CP^1}$ is given by an interval fibered by S^1 with zero size at the endpoints of the interval. Using these ideas, the cotangent bundle over $\mathbf{CP^1}$ can be also viewed as a toric space. In this way, we have two circle actions on this space. The first one is the one corresponding to the action on the $\mathbf{CP^1}$ base space and the other circle acts on the fiber cotangent direction. Our next example will be the two complex dimensional projective space $\mathbf{CP^2}$. The latter has a $U(1)^2$ toric action exhibiting three fixed points defining a triangle in the $\mathbf{R^2}$ real space. The toric geometry of this manifold is described by a triangle of $\mathbf{R^2}$ fibered by a two real dimensional torus T^2 which degenerates to a S^1 circle on the three edges and shrinks to a point on the endpoints. The cotangent bundle over $\mathbf{CP^2}$ is a 4-dimensional (eight real) local toric geometry, where we have two extra circle actions coming from the fiber cotangent directions. Note that this analysis is similar for the $\mathbf{WCP^2}$, in particular $\mathbf{WCP^2_{(1,2,1)}}$, and can be extended easily to higher dimensional (weighted) projective spaces. In what follows we will consider the above toric geometry circle actions to identify the $U(1)$ group symmetry of quotient spaces $X_7 = \frac{X_8}{U(1)}$.

Let us consider the simple example of the $U(1)$ gauge theory with three hypermultiplets. In this case the geometry X_8 can be viewed as $\mathbf{C^2}$ bundle over a $\mathbf{WCP^2_{(1,2,1)}}$. This manifold has four toric geometry circle actions $U(1)^2_{base} \times U(1)^2_{fiber}$; two of them correspond to the $\mathbf{WCP^2_{(1,2,1)}}$'s base space denoted by $U(1)^2_{base}$ while the remaining ones $U(1)^2_{fiber}$ act on the fiber cotangent directions. Next we want to divide by one finite circle toric geometry action for obtaining seven real manifolds. Mimicking the analysis of [11] and identifying the $U(1)$ group symmetry of the quotient with one finite fiber circle action

$$X_7 = \frac{X_8}{U(1)_{fiber}}, \tag{4.2}$$

we can obtain a 7-dimensional geometry. Since $\frac{\mathbf{C^2}}{U(1)} = \mathbf{R}^+ \times \mathbf{C}$, the quotient space is now a $\mathbf{R}^+ \times \mathbf{C}$ bundle over a $\mathbf{WCP^2_{(1,2,1)}}$. By compactifying the $\mathbf{C}$ complex plane, which can be done by adding a point at infinity, this space will be a $\mathbf{R}^+ \times S^2$ bundle over $\mathbf{WCP^2_{(1,2,1)}}$. Similarly to [11], this geometry is a cone on a S^2 bundle over $\mathbf{WCP^2_{(1,2,1)}}$ of G_2 holonomy. More generally, if we consider the $U(1)^r$ gauge theory with the A_r Cartan matrix gauge charges and $(r+2)$ hypermultiplets, then the quotient space is a cone on a S^2 bundle over r intersecting $\mathbf{WCP^2_{(1,2,1)}}$ weighted projective spaces according to the A_r Dynkin diagram.

The singularities
A naive way to study the singularities of these X_7 manifolds is to consider the identification structure of the weighted projective spaces. The latters are not generally smooth because non trivial fixed points under the variable identifications lead to singularities. To see this feature, consider the identification structure of $\mathbf{WCP^2_{1,2,1}}$ defined by introducing three homogeneous complex coordinates z_1, z_2, z_3 not all of them simultaneously zero with a projective relation:

$$(z_1, z_2, z_3) \equiv (\lambda z_1, \lambda^2 z_2, \lambda z_3). \tag{4.3}$$

Note, in passing, that these (z_1, z_2, z_3) homogeneous complex coordinates can be related respectively to ψ_1, ψ_2 and ψ_3 fields of the sigma model construction. Finally, it is not hard to show that this space is singular. Indeed, if we take $\lambda = -1$, equation (4.3) reduces to

$$(z_1, z_2, z_3) \equiv (-z_1, z_2, -z_3), \tag{4.4}$$

and so we have a Z_2 orbifold singularity at $(z_1, z_2, z_3) = (0, z_2, 0)$.

V. CONCLUSION

In this paper, we have contributed in the M-theory compactifications to four dimensions. This involves the compactification on seven manifolds of G_2 holonomy group, leading to $N = 1$ four dimensional supersymmetric models. In particular, we have constructed a new class of toric-hyper-Kahler eight manifolds giving G_2 holonomy spaces after dividing by one finite toric geometry circle action. This building has been proceeded in two steps. We have first introduced the ADE Cartan matrices as matrix gauge charges in the $N = 4$ $2D$ field theoretical construction of toric-hyper-Kahler eight manifolds X_8. In particular, the solution for the A_r Lie algebra is described by the cotangent bundle over r intersecting $\mathbf{WCP^2_{1,2,1}}$ weighted projective spaces according to the A_r Dynkin diagram. Actually these spaces may extend the geometry of A_r ALE space, described by $2D$ $N = 2$ type IIA superstring sigma model used in the geometric engineering method. Second we have used the toric geometry circle actions of X_8 to build quotients $X_7 = \frac{X_8}{U(1)}$ of G_2 holonomy group.

Acknowledgments
Adil Belhaj would like to thank E.H. Saidi for valuable discussions. This work is supported by SARS, programme de soutien à la recherche scientifique; Université Mohammed V-Agdal, Rabat.

---

[1] C. Vafa, Lectures on strings and dualities, hep-th/9702201.
[2] E. Witten, String theory dynamics in various dimensions; Nucl.Phys. **B 443** (1995)85, hep-th/9503124.
[3] P.K Townsend, The eleven-dimensional supermembrane revisited; Phys. Lett **B 350** (1995)184, hep-th/9501068.
[4] C. Gomez, Lectures presented at the Workshop on Noncommutative Geometry, Superstrings and Particle Physics. Rabat -Morocco, (May 11-12 2001).
[5] P.K Townsend and G.Papadopoulos, Compactification of D=11 supergravity on spaces of exceptional holonomy; Phys. Lett **B 357** (1995)472, hep-th/9506150.
[6] B. Acharya, On realizing $N = 1$ super Yang-Mills in M-theory, hep-th/0011089.
[7] M.F.Atiyah and E.Witten, M-theory dynamics on a manifold of G_2 Holonomy; hep-th/0107177.
[8] M. Cvetic, G. Shiu, Angel M. Uranga, Chiral Type II Orientifold Constructions as M Theory on G_2

holonomy spaces, hep-th/0111179.
[9] M. Cvetic, G.W. Gibbons, H. Lu, C.N. Pope, A G_2 Unification of the Deformed and Resolved Conifolds, hep-th/0112138.
[10] M. Cvetic, G.W. Gibbons, H. Lu, C.N. Pope, M-theory Conifolds, hep-th/0112098.
[11] E. Witten, Anomaly Cancellation On Manifolds Of G_2 Holonomy, hep-th/0108165.
[12] B. Acharya, E. Witten, Chiral Fermions from Manifolds of G_2 Holonomy, hep-th/0109152
[13] S. Katz, C. Vafa; Nucl.Phys. B497 (1997) 146-154, hep-th/9606086.
[14] S. Katz, P. Mayr and C. Vafa, Mirror symmetry and exact solution of 4d N=2 gauge theories I; Adv, Theor. Math. Phys **1**(1998)53.
[15] P. Mayr, Geometric Construction of N=2 gauge theories; Spring Workshop on Superstrings and Related Matters, (ICTP, Trieste, March 1999).
[16] A.Belhaj, A. EL Fallah and E.H. Saidi, On non simply laced mirror geometries in type II strings; Class. Quantum. Grav.**17** (1999)515-532.
[17] S. Gukov, C. Vafa, E. Witten, CFT's From Calabi-Yau Four-folds; Nucl.Phys. B584 (2000) 69-108; Erratum-ibid. B608 (2001) 477-478,hep-th/9906070.
[18] A. Belhaj and E. H. Saidi, Hyper-Kahler Singularities in Superstrings Compactification and 2d N=4 Conformal Field Theory; Class.Quant.Grav. 18 (2001) 57-82, hep-th/0002205.
A. Belhaj and E.H.Saidi, On HyperKahler Singularities; Mod. Phys. Lett. A, Vol. 15, No. **29** (2000) pp. 1767-1779, hep-th/0007143
[19] B. R. Grenne, String Theory on Calabi-Yau Manifolds; **TASI** lectures 1996, hep-th/9702155.
[20] S.Theisen, Introduction to Calabi-Yau manifolds; Lectures presented at Spring school on superstring theories and related matters, ICTP, Trieste, Italy, (2001).
[21] D. R. Morrison, C. Vafa, Compactifications of F-Theory on Calabi–Yau Threefolds – II; Nucl.Phys. B476 (1996) 437-469,hep-th/9603161.
[22] C.Vafa, On N =1 Yang-Mills in Four Dimensions; Adv theor Math. Phys **2** (1998).
[23] W. Fulton, Introduction to Toric varieties; Annals of Math. Studies, No .131, Princeton University Press, 1993.
[24] N.C. Leung and C. Vafa; Adv .Theo. Math. Phys 2(1998) 91, hep-th/9711013.
[25] A. Belhaj and E.H. Saidi, Toric Geometry, Enhanced non Simply Laced Gauge Symmetries in Superstrings and F-theory Compactifications; hep-th/0012131.
[26] A. Belhaj, Mirror symmetry and Landau- Ginzburg Calabi-Yau superpotentials in F-theory; J.Phys. A35 (2002) 965-984, hep-th/0112005.