



Type IIB String Backgrounds on Parallelizable PP-Waves and Conformal Liouville Theory

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Abstract

The scope of this work concerns the adaptation of the parallelizability pp-wave (Ppp-wave) process to $D = 10$ type IIB string backgrounds in the presence of the non-trivial anti-self dual R-R 5-form \mathcal{F} . This is important in the sense that it gives rise to some unsuspected properties. In fact, exact solutions of type IIB string backgrounds on Ppp-waves are discussed. For the u -dependence of the dilaton field Φ , we establish explicitly a correspondence between type IIB supergravity equations of motion and $2d$ -conformal Liouville field theory. We show also that the corresponding conserved conformal current $T(\Phi)$ coincides exactly with the trace of the symmetric matrix μ_{ij} appearing in the quadratic front factor $F = \mu_{ij}x^i x^j$ of the Ppp-wave. Furthermore, we consider the transverse space dependence of the dilaton Φ and show that the supergravity equations are easily solved for the linear realization of the dilaton field. Other remarkable properties related to this case are also discussed.

I. INTRODUCTION

Recently, there has been an increasing interest in string theory on pp-waves*. The motivation comes from the fact that pp-waves spacetime provides an example of exact string theory background with all α' corrections vanishing^{1,2}. The existence of a covariantly constant null killing vector greatly simplifies the quantization of a string light cone gauge^{2,7}. In this context, a particular interest has been devoted to the study of $D = 10$ type II string theory, especially type IIB, on pp-waves, both in NS-NS and R-R sectors³⁻⁶.

Based on the fact that all ten dimensional parallelizable solutions (related with the existence of torsion which makes the manifold flat) of type II or

heterotic supergravities are also α' exact⁷, the parallelizable pp-wave backgrounds have been recently presented in⁸. In the same article, the most general parallelizable pp-wave backgrounds which are non-dilatonic solutions in the NS-NS sector of type IIA and IIB string theories are considered. After that, Figueroa-O'Farrill has classified all the parallelizable NS-NS backgrounds with the dilaton field turned off⁹, a work which was completed later in type II supergravity with non trivial dilaton field in¹⁰. The classification of the simply connected supersymmetric parallelizable backgrounds of heterotic supergravity was recently done in¹¹, where the authors construct all parallelizable backgrounds of ten dimensional type I supergravity coupled to supersymmetric Yang Mills.

The scope of this work concerns the adaptation of the parallelizability pp-wave process to type IIB string backgrounds in $D = 10$ in the presence of the anti self dual R-R 5-form \mathcal{F} . This is a promis-

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ing topics attracting a lot of attention recently and showing a big flexibility to open up on new research ideas. Thus, techniques developed in $2d$ -conformal field theories can be applied to complete several studies about pp-waves and supergravity solutions. We find among other that in the light cone dependence of the dilaton field Φ , the ten dimensional supergravity equations of motion are related to $2d$ -conformal Liouville theory.

The outline of this paper is as follows: In section 2, we give an overview of pp-waves and parallelizability. The main steps concerning the proof of the theorem relating the parallelizable pp-waves backgrounds and homogeneous plane waves are also presented. In section 3, we consider $D = 10$ type IIB supergravity equations of motion, with non trivial R-R 5-form \mathcal{F} , and observe that they are simplified with the parallelizable pp-waves conditions. Assuming that the R-R 5-form reads as $\mathcal{F} = du \wedge \varphi(x^i)$, with $\varphi(x^i)$ is an anti self dual closed 4-form in the transverse eight dimensional space, we focus to study the u -dependence of the dilaton Φ . This gives rise to a relationship between type IIB supergravity and conformal Liouville field theory, whose general solution is explicitly derived. We show also that the corresponding conserved conformal current $T(\Phi)$, coincides exactly with the trace of the symmetric matrix μ_{ij} appearing in the quadratic front factor of the parallelizable pp-wave $F = \mu_{ij}x^ix^j$. The other cases corresponding to the x^i -dependence of the dilaton Φ , and the transverse space case are also discussed. Section 4 is devoted to our conclusion.

II. PP-WAVES AND PARALLELIZABILITY : A REVIEW.

Being given the importance of supergravity theories which are α' -exact solutions, we focus in what follows to review two essential notions namely the pp-waves and the parallelizability and discuss the parallelizable pp-waves as well as the homogeneous plane waves. For other important aspects of these topics, the reader is referred to the literature^{8,7,13-15}.

A. Generalities on PP-waves spacetime

pp-waves (plane-fronted waves with parallel ray) are general class of spacetime admitting a covariantly constant null Killing vector field v^μ

$$\begin{aligned} \nabla_\mu v_\nu &= 0, \\ v^\mu v_\mu &= 0 \end{aligned} \tag{2.1}$$

In general relativity, they form simple solutions to Einstein's equations with many curious properties. The presence of the covariantly constant null Killing field implies that these spacetime have vanishing scalar curvature invariants, much the same as flat space. The most general form of the pp-waves metrics is given by

$$\begin{aligned} ds^2 &= -2dudv - F(u, x^i)du^2 \\ &+ 2A_i(u, x^j)dudx^i + g_{ij}(u, x^i)dx^i dx^j. \end{aligned} \tag{2.2}$$

To simplify the discussion, it is convenient to work with spacetime for whom $A_i(u, x^i) = 0$. The front factor $F(u, x^i)$ is shown to satisfy, by virtue of vacuum Einstein's equations of pure gravity, the transverse Laplace equation for each u and that the transverse space be Ricci flat. However, $F(u, x^i)$ can be considered as an arbitrary function of the longitudinal coordinate u . A useful simplification of the above metrics, consists in considering pp-wave spacetime with flat transverse part

$$\begin{aligned} ds^2 &= -2dudv - F(u, x^i)du^2 \\ &+ dx^i dx^i. \end{aligned} \tag{2.3}$$

To restrict much more these classes of pp-waves, one consider the plane waves spacetime which are those for which the harmonic function $F(u, x^i)$ is quadratic $F(u, x^i) = f_{ij}(u)x^ix^j$. Thus, the plane wave metrics takes the form

$$\begin{aligned} ds^2 &= -2dudv - f_{ij}(u)x^ix^j du^2 \\ &+ dx^i dx^i, \end{aligned} \tag{2.4}$$

where $f_{ij}(u)$ is any function of the longitudinal coordinate u , symmetric and traceless by virtue of vacuum Einstein's equations.

The homogeneous plane waves further restrict our general pp-wave metric by taking out f 's dependence on u . We have

$$\begin{aligned} ds^2 &= -2dudv - \mu_{ij}x^ix^j du^2 \\ &+ dx^i dx^i, \end{aligned} \tag{2.5}$$

where $f_{ij} \equiv \mu_{ij}$ defines a symmetric constant.

A particular example of the homogeneous plane waves is given by the BMN plane wave metrics^{3,16} for which $f_{ij} = \mu^2 \delta_{ij}$.

B. Parallelizability

Consider an n dimensional manifold M . Due to Cartan-Schouten¹⁵, this manifold is said to be parallelizable if there exists a torsion which "flattens" the manifold, i.e. makes the Riemann curvature tensor vanish. Explicitly, we can decompose the

connection into a Christoffel piece and a torsion contribution:

$$\begin{aligned} \hat{\Gamma}_{\mu\nu}^\lambda &= \Gamma_{\mu\nu}^\lambda \\ &+ T_{\mu\nu}^\lambda \end{aligned} \quad (2.6)$$

where $\Gamma_{\mu\nu}^\lambda$ is symmetric in $\mu\nu$ indices and $T_{\mu\nu}^\lambda$ (torsion) is anti-symmetric. The curvature $\hat{R}_{\mu\nu\rho\sigma}$ may be decomposed in a similar way, into a piece which comes only from the Christoffel connection and the torsional contributions:

$$\begin{aligned} \hat{R}_{\mu\nu\rho\sigma} &\equiv R_{\mu\nu\rho\sigma} + \nabla_\rho T_{\mu\nu\sigma} - \nabla_\sigma T_{\mu\nu\rho} \\ &+ T_{\mu\sigma\lambda} T_{\nu\rho}^\lambda - T_{\mu\rho\lambda} T_{\nu\sigma}^\lambda. \end{aligned} \quad (2.7)$$

The parallelizability condition is equivalent to set $\hat{R}_{\mu\nu\rho\sigma} = 0$. Furthermore, the manifold is said to be Ricci-parallelizable if the following Ricci tensor $\hat{R}_{\mu\nu} \equiv R_{\mu\nu} + \nabla_\lambda T_{\mu\nu}^\lambda - T_{\mu\sigma\lambda} T_{\nu}^{\sigma\lambda}$ is zero. Requesting the Ricci-parallelizability yields the vanishing of the symmetric and antisymmetric parts of $\hat{R}_{\mu\nu}$ namely

$$\begin{aligned} R_{\mu\nu} - T_{\mu\sigma\lambda} T_{\nu}^{\sigma\lambda} &= 0, \\ \nabla_\lambda T_{\mu\nu}^\lambda &= 0. \end{aligned} \quad (2.8)$$

In most string theories, where a torsion field arises naturally, parallelizability leads to important implications for supergravities and their solutions. The type II NS-NS part of supergravity action is shown to take the following form

$$\begin{aligned} S &= \frac{1}{l_p^8} \int d^{10}x \sqrt{-g} e^{-2\Phi} \times \\ &\times \left(R + 4(\nabla_\mu \Phi)^2 - \frac{1}{12} H_{\mu\nu\rho} H^{\mu\nu\rho} \right). \end{aligned} \quad (2.9)$$

The non dilatonic solutions lead to the supergravity equations of motion for the metric and $B_{\mu\nu}$ field, namely

$$\begin{aligned} R_{\mu\nu} - \frac{1}{4} H_{\mu\sigma\lambda} H_{\nu}^{\sigma\lambda} &= 0, \\ \nabla_\lambda H_{\mu\nu}^\lambda &= 0, \end{aligned} \quad (2.10)$$

with $H = dB$. Henceforth, $H_{\mu\sigma\lambda}$ is assumed to be closed $dH = 0$. Once we define the torsion $T_{\mu\sigma\lambda}$ to be $\frac{1}{2} H_{\mu\sigma\lambda}$, the previous supergravity equations are nothing but the Ricci-parallelizability condition (8).

C. Parallelizable pp-waves and homogeneous plane-waves

Referring to a theorem presented in⁸, in which the authors demonstrate that parallelizable pp-waves backgrounds are necessarily homogeneous

plane waves and that a large class of homogeneous plane waves are parallelizable stating the necessary conditions. We remind here bellow the main steps of this proof. The starting point is to consider the most general ten dimensional pp-wave geometry whose metric is given in (2) where the index i, j stands for the transverse coordinates $i, j = 1, 2, \dots, 8$. The functions F, A_i, g_{ij} are chosen to satisfy supergravity equations of motion. Next, the most general NS-NS H field compatible with the covariantly constant Killing vector v^μ is given by

$$H_{uij} = h_{ij}(u, x^k), \quad (2.11)$$

and all the remaining components zero. Therefore, by imposing the Ricci-parallelizability conditions, it's pointed out that the only nonvanishing component of the Ricci curvature is R_{uu} . Performing some transformations based on the parallelizability condition, the authors present the most general parallelizable pp-waves

$$ds^2 = -2dudv - \mu_{ij} x^i x^j du^2 + dx^i dx^i \quad (2.12)$$

$$H_{uij} = h_{ij} = \text{constant}. \quad (2.13)$$

with

$$\mu_{ij} = \frac{1}{4} h_{ik} h_{jk}, \quad (2.14)$$

which is of the form of an homogeneous plane wave, a fact which complete the proof. As claimed also in⁸, for the inverse case, not all homogeneous plane-wave geometries are parallelizable. The exception is made only for homogeneous plane wave for which μ_{ij} has doubly degenerate eigenvalues

$$\mu_{ij} = \text{diag}(a_1^2, a_1^2, a_2^2, a_2^2, a_3^2, a_3^2, a_4^2, a_4^2) \quad (2.15)$$

where the four real numbers a_i , determining completely the parallelizable pp-wave, are expressed in terms of the antisymmetric 8×8 matrix h_{ij} whose only non-zero components are

$$h_{12} = 2a_1, h_{34} = 2a_2, h_{56} = 2a_3, h_{78} = 2a_4. \quad (2.16)$$

III. TYPE IIB STRING BACKGROUNDS EQUATIONS ON PARALLELIZABLE PP-WAVES

A. Parallelizable pp-waves backgrounds:

The starting point is the lowest order effective action of type IIB supergravity in 10-dimensions with non-trivial R-R 5-form \mathcal{F}^5 .

$$\begin{aligned}
 S_{eff} = & \int d^{10}x \sqrt{-g} e^{-2\Phi} \times \\
 & \times (R + 4(\nabla_\mu \Phi)^2 - \frac{1}{12} H_{\mu\nu\rho} H^{\mu\nu\rho} \\
 & - \frac{e^{2\Phi}}{4.5!} \mathcal{F}_{\mu\nu\rho\sigma\tau} \mathcal{F}^{\mu\nu\rho\sigma\tau}), \quad (3.1)
 \end{aligned}$$

in the presence of a metric g , a non trivial dilaton Φ , an anti-symmetric NS-NS B field with field strength $H = dB$. Setting $C^{(0)} = 0 = C^{(2)}$ for the remaining R-R fields, we may write $\mathcal{F} = \partial C^{(4)}$ where $C^{(4)}$ is the corresponding R-R 4-form appropriately normalized. In 10 dimensions, the self duality of \mathcal{F} namely $*\mathcal{F} = \mathcal{F}$ does not admit a natural derivation from a covariant action principle and is imposed as an additional constraint at the level of the derived supergravity equations of motion in the σ -model frame

$$\begin{aligned}
 R_{\mu\nu} = & -2\nabla_\mu \nabla_\nu \Phi + \frac{1}{4} H_{\mu\rho\sigma} H_\nu{}^{\rho\sigma} \\
 & + \frac{e^{2\Phi}}{4.4!} \left(\mathcal{F}_{\mu\kappa\lambda\rho\sigma} \mathcal{F}_\nu{}^{\kappa\lambda\rho\sigma} - \frac{1}{10} g_{\mu\nu} \mathcal{F}_{\kappa\lambda\rho\sigma\tau} \mathcal{F}^{\kappa\lambda\rho\sigma\tau} \right), \\
 0 = & \nabla_\mu \nabla^\mu \Phi - 2(\nabla_\mu \Phi)(\nabla^\mu \Phi) + \frac{1}{12} H_{\mu\nu\rho} H^{\mu\nu\rho}, \\
 0 = & \nabla_\mu (e^{-2\Phi} H^{\mu\nu\rho}), \\
 0 = & \nabla_\mu \mathcal{F}^{\mu\nu\kappa\lambda\rho}, \quad (3.2)
 \end{aligned}$$

where ∇_μ is the covariant derivative with respect to the Levi-Civita connection $\Gamma_{\mu\rho}^\nu$. The Greek indices stands for the values $0, 1, \dots, 9$. The field strength $H_{\mu\nu\rho}$ of the NS-NS two-form $B_{\mu\nu}$ is assumed to be a closed 3-form, $dH = 0$.

Next, we focus to adapt the parallelizability framework reviewed in section 2, to the previous supergravity equations a fact which give rise to some unsuspected properties. Indeed, requesting the Ricci parallelizability conditions

$$\begin{aligned}
 R_{\mu\nu} = & \frac{1}{4} H_{\mu\rho\sigma} H_\nu{}^{\rho\sigma}, \\
 0 = & \nabla_\mu H^{\mu\nu\rho}, \quad (3.3)
 \end{aligned}$$

equations (18) reduce then to

$$\begin{aligned}
 0 = & -2\nabla_\mu \nabla_\nu \Phi \\
 & + \frac{e^{2\Phi}}{4.4!} \left(\mathcal{F}_{\mu\kappa\lambda\rho\sigma} \mathcal{F}_\nu{}^{\kappa\lambda\rho\sigma} - \frac{1}{10} g_{\mu\nu} \mathcal{F}_{\kappa\lambda\rho\sigma\tau} \mathcal{F}^{\kappa\lambda\rho\sigma\tau} \right), \\
 0 = & \nabla_\mu \nabla^\mu \Phi - 2(\nabla_\mu \Phi)(\nabla^\mu \Phi) + \frac{1}{12} H_{\mu\nu\rho} H^{\mu\nu\rho}, \\
 0 = & H^{\mu\nu\rho} \nabla_\mu e^{-2\Phi}, \\
 0 = & \nabla_\mu \mathcal{F}^{\mu\nu\kappa\lambda\rho}, \quad (3.4)
 \end{aligned}$$

Note by the way, that once $\mathcal{F} = 0$, the above equations reduce to

$$\begin{aligned}
 \nabla_\mu \nabla_\nu \Phi = & 0, \\
 \nabla_\mu \Phi \nabla^\mu \Phi = & \frac{1}{24} H_{\mu\nu\rho} H^{\mu\nu\rho}, \\
 H^{\mu\nu\rho} \nabla_\mu e^{-2\Phi} = & 0. \quad (3.5)
 \end{aligned}$$

Therefore, as mentioned in¹⁰, it follows from the above equations that non-dilatonic background solutions are viable if $H_{\mu\nu\rho} H^{\mu\nu\rho} = 0$, which is equivalent to set the scalar curvature R to be zero. This corresponds to the parallelizable pp-waves discussed in^{8,7}. However, for $H_{\mu\nu\rho} H^{\mu\nu\rho} \neq 0$, it's also shown that the set of equations (21) can be solved easily with linear dilaton field Φ in the direction transverse to $H_{\mu\nu\rho}$. Since our interest focuses on the $\mathcal{F} \neq 0$ case, we use the most general parallelizable pp-wave⁸.

$$ds^2 = -2dudv - \mu_{ij} x^i x^j du^2 + dx^i dx^i \quad (3.6)$$

$$H_{uij} = h_{ij} = constant. \quad (3.7)$$

in order to simplify the string background equations (20) where the constant μ_{ij} and h_{ij} are given by (14-16). As we will show later, the type IIB supergravity solutions in the presence of the R-R 5-form is associated to a non linear conformal Liouville realization of the dilaton field. This is an unusual property which goes beyond the special NS-NS case $\mathcal{F} = 0$ and which provides then a possibility to connect conformal models with string background solutions.

Thereafter, as we are interested to simplify more the string background equations, we assume for the moment that the dilaton Φ is arbitrary and consider the following realization for the 5-form⁵,

$$\mathcal{F} = du \wedge \varphi(x^i) \quad (3.8)$$

where $\varphi(x^i)$ is an anti-self dual closed 4-form in the transverse eight dimensional space with

$$\begin{aligned}
 \varphi(x^i) = & -*\varphi(x^i) \\
 d\varphi(x^i) = & 0 \quad (3.9)
 \end{aligned}$$

such that $\mathcal{F} = *\mathcal{F}$. Thus, taking the uu -component of the first equation of (20), we have to consider different cases,

A) $\Phi = \Phi(u)$ one obtains

$$0 = -2\nabla_u \nabla_u \Phi + \frac{e^{2\Phi}}{4.4!} \varphi_{ijkl} \varphi^{ijkl} \quad (3.10)$$

This case gives rise to non standard results as we will explicitly discuss in the next subsection. Later on we will use the following notation $\varphi_{ijkl} \varphi^{ijkl} \equiv \varphi^2$ for the square of the 4-form.

B) $\Phi = \Phi(x^i)$ one obtains

$$0 = \varphi_{ijkl} \varphi^{ijkl} \quad (3.11)$$

Before any comments about this equation, lets remind a different formula found by the authors of⁵ in the same context but without requesting the parallelizability

$$\nabla_i (e^{-2\Phi} \nabla^i F) = \frac{1}{2 \cdot 4!} \varphi_{ijkl} \varphi^{ijkl}. \quad (3.12)$$

This equation, obtained by taking the uu -component of the non parallelizable supergravity equation of motion, gives a relation between the front factor F and the anti-self dual closed 4-forme φ . Thus, it should undergo an important change while requiring the parallelizability. Indeed, once the parallelizability condition is applied to string background equations of motion (18) for $\Phi = \Phi(x^i)$, the previous (φ, F) -coupling is not anymore allowed, since the derivative term associated to the front F although non zero decouples from $\varphi_{ijkl} \varphi^{ijkl}$ which becomes zero as given in (27). For this case, we point out that the remaining string background equations

$$\begin{aligned} 0 &= \nabla_i \nabla^i \Phi - 2 (\nabla_i \Phi) (\nabla^i \Phi) + \frac{1}{12} h^2, \\ 0 &= H^{uij} \nabla_i e^{-2\Phi}, \end{aligned} \quad (3.13)$$

with $h^2 \equiv H_{uij} H^{uij}$; can be solved by a linear realization of the dilaton field $\Phi(x^i)$ which provides also a solution for the first equation of (20) once $\varphi^2 = 0$. However, in the transverse space, the equations of motion (20) reduce to

$$\begin{aligned} 0 &= -2 \nabla_i \nabla_j \Phi, \\ 0 &= \nabla_i \nabla^i \Phi - 2 (\nabla_i \Phi) (\nabla^i \Phi), \end{aligned} \quad (3.14)$$

since the only nonvanishing components of H -field are $H_{uij} = h_{ij}$ and therefore $H_{ijk} = 0$. The first equation shows that the dilaton field Φ should be linear in the transverse coordinates and based on this fact the second one corresponds to the vanishing of the dilaton gradient $\nabla_i \Phi$. As a results the dilaton is constrained to be a constant in the transverse coordinates.

B. Parallelizable PP waves and conformal Liouville field Theory

We focus in this subsection to give more details about the derived supergravity equation of motion presented in the first case (26) since it gives rise to important non standard results. The idea behind our analysis consists in complexifying the above equation, a fact which makes the link with $2d$ -conformal symmetry possible. Indeed, consider the differential equation (26), for some reasons that we

will explain later, let's suppose that one can temporarily introduce by hand the derivation with respect to the v -longitudinal direction so that the equation becomes[†]

$$0 = -2(\nabla_u^2 + \nabla_v^2)\Phi + \frac{e^{2\Phi}}{4 \cdot 4!} \varphi_{ijkl} \varphi^{ijkl} \quad (3.15)$$

This way to write the things made us recall the complex notation and $2d$ -conformal invariant equations. Indeed, adopting this analogy for which we put,

$$\nabla_u^2 + \nabla_v^2 \equiv \partial \bar{\partial}, \quad (3.16)$$

with $\partial = \frac{\partial}{\partial z} \equiv \nabla_u + i \nabla_v$ and $\bar{\partial} = \frac{\partial}{\partial \bar{z}} \equiv \nabla_u - i \nabla_v$. This drives us to rewrite the supergravity equation of motion (26) as a bidimensional conformal Liouville like equation, namely

$$\partial \bar{\partial} \Phi - \chi(\varphi) \exp(2\Phi) = 0, \quad (3.17)$$

or equivalently

$$\beta \partial \bar{\partial} \Phi - \chi(\varphi) \exp(2\beta\Phi) = 0, \quad (3.18)$$

modulo the following scaling $\Phi \rightarrow \beta\Phi$, where β is a real constant parameter and where the coefficient $\chi(\varphi)$ stands for the constant $\chi(\varphi) \equiv \frac{\varphi^2}{4! \cdot 8}$. By virtue of our knowledge on $2d$ -CFT and integrable models¹⁸⁻²⁰, this equation is shown to be solvable as we will show later. The fact to add the derivation with respect to the v -longitudinal coordinate is only a formal trick of calculation that is going to be ignored thereafter. Note also that we have omitted in the above complexification the fact that ∇_u designates really the covariant derivatives. This omission doesn't influence our analysis and the comparison with the conformal Liouville equation imposes itself by nature. Now before discussing the conformal properties and the solution of this new equation (34), for the meantime we are going to recall some known properties of the standard Liouville equation^{21,22}. The local equation of motion of the two dimensional Liouville field $\phi(z, \bar{z})$ is given by

$$\beta \partial \bar{\partial} \phi - \exp(2\beta\phi) = 0, \quad (3.19)$$

where β is a real coupling constant. $2d$ Liouville equation was the subject of several studies in the

[†]It's important to note at this level that the v -longitudinal direction ignored in the standard supergravity backgrounds formulation is now resuscitated to build a $2d$ -conformal field theory framework.

literature and in different contexts^{23,24}. It is a non linear differential equation that has the particularity to be conformally invariant. Because of the rich structure of bidimensional conformal models, the conformal symmetry represents in this setting a guarantee of the solvability (integrability) and therefore assures a means to overcome the non linearity. To study the integrability of (35), different techniques including the Lax method were developed. Here, we content ourselves to recall that the explicit solution of the nonlinear Liouville equation can be written as²²

$$\exp(2\beta\phi) = k \frac{f'(z) \cdot \bar{f}'(\bar{z})}{(1 - f(z) \cdot \bar{f}(\bar{z}))^2}, \quad (3.20)$$

where $f(z)$ and $\bar{f}(\bar{z})$ are arbitrary analytic and anti-analytic functions, $f'(z) = \partial f(z)$ and $\bar{f}'(\bar{z}) = \bar{\partial}(\bar{f}(\bar{z}))$ and k is an arbitrary constant for the moment. As it is well known, the integrability of the Liouville equation is due to conformal symmetry generated by the following classical energy momentum tensor

$$T(\varphi) = (\partial\varphi)^2 - \frac{1}{\beta}(\partial^2\varphi). \quad (3.21)$$

The conservation law of this conformal current, follows immediately by using the equation of motion as shown here below

$$\begin{aligned} \bar{\partial}T(\varphi) &= 2(\partial\bar{\partial}\varphi)(\partial\varphi) - \frac{1}{\beta}\partial(\partial\bar{\partial}\varphi) \\ &= -\frac{1}{\beta}(\partial - 2\beta\partial\varphi)\partial\bar{\partial}\varphi \\ &= 0. \end{aligned} \quad (3.22)$$

Now having given the necessary ingredients of the conformal Liouville equation, we pass now to study our equation (34). The non-trivial behavior that ensues from our analysis is based on the fact that the striking resemblance with conformal Liouville equation is going to allow us to adapt conformal symmetry and integrable models backgrounds to the present context of parallelizable pp-waves.

Another point to evoke concerns the limit to consider in order to recover the standard formalism where the supergravity equations are expressed according to the covariant derivative $\dagger \nabla_u$ and not of the complex ones ∂ and $\bar{\partial}$. Such a limit is given simply by performing the following transformations at the level of the derived complex formulas

$$\begin{aligned} z, \bar{z} &\equiv u \\ \partial, \bar{\partial} &\equiv \nabla_u^2 \\ \partial f(z) &\equiv \nabla_u f(u) \equiv \bar{\partial}(\bar{f}(\bar{z})) \end{aligned} \quad (3.23)$$

Now, consider the derived $2d$ conformal Liouville equation of motion (34). This equation describes the results of the parallelizable pp-waves constraint on the string backgrounds given by the supergravity effective action (17). The direct contact with the standard Liouville equation (35) consists in setting for instance as an ansatz

$$\chi(\varphi) \equiv \frac{\varphi^2}{4! \cdot 8} = 1. \quad (3.24)$$

For this simple choice, our derived Liouville equation (34) is solved and the solution is given by (36) for any arbitrary analytic and anti-analytic functions $f(z)$ and $\bar{f}(\bar{z})$ respectively. One should notice at this level the importance of the complex formulation since it gives rise to several solutions related the arbitrary character of the functions $f(z)$ and $\bar{f}(\bar{z})$. As an example, consider the linear realization $f(z) = az$ and $\bar{f}(\bar{z}) = b\bar{z}$. One easily check that this linear choice corresponds to a solution of the Liouville equation (34) and in the same way fixes the constant k to take the value $k = 1$. Such a richness in the solutions is a natural feature of $2d$ -conformal symmetry, a property which is no longer guaranteed if the conformal symmetry is lost.

Now, as we need to recover our derived string backgrounds equations (26), one have to perform the limit equations (39) given above. A remarkable fact here, is that the introduction of these limit equations break automatically the conformal symmetry and the previous analysis must be controlled with prudence. The loss of the conformal symmetry, can be traced to the fact that the contribution of the v -longitudinal coordinate is annulled. Indeed, the conformal splitting in two different, analytic and anti-analytic, sectors is not more valid since we will have only one sector described by the contribution of the u -coordinate.

To illustrate these ideas for the simple case $\chi(\varphi) = 1$, let's consider the above $2d$ -conformal approach in the limit (39). The Liouville equation of motion (34) is now reduced to the original string backgrounds equation (26) (the u -Liouville equation) whose formal solution is assumed for the moment to derive from the complex one (36), once the limit (39) are performed, we set

$$\exp(2\beta\Phi) = k \left(\frac{\nabla_u f(u)}{1 - f^2(u)} \right)^2. \quad (3.25)$$

As discussed above, the fact to ignore the contribution of the v -coordinate will certainly induces some

[†]This is because the v -direction is ignored in the standard supergravity computations.

constraint on the functions $f(u)$ that are not more arbitrary. Note that we preserved the shape (41) of the solutions to assure the compatibility with the above approach. Also, this equation shows how the dilaton field Φ is explicitly expressed in terms of the ∇_u -derivatives of some constrained u -dependent functions $f(u)$ that we will try to discuss.

Thereafter, we are interested to derive explicitly the functions $f(u)$ for which (41) defines a solution of (26). Simple formal computations show that the u -Liouville equation (26) has in fact two explicit solutions for the dilaton $\Phi(u)$ namely

$$\Phi_{\pm}(u) = \log \left[\pm a \sqrt{-1 + \tanh^2(au + b)} \right] \quad (3.26)$$

or equivalently

$$\Phi_{\pm}(u) = \log[\pm ia \operatorname{sech}(au + b)] \quad (3.27)$$

where a and b are two positive constant numbers with $\operatorname{sech}(\alpha) = \frac{1}{\cosh(\alpha)}$. Actually, the two explicit solutions of the equation of motion (26), can be extracted from our proposed solution (41). Indeed, performing straightforward by lengthy computations show that the two derived solutions are described by two doubly degenerate expressions of the the function $f(u)$ namely

$$f_{\pm}(u) = \left[-1 + \frac{2}{1 + \exp\left[\pm \frac{4i \arctan[\tanh(\alpha u + \beta)]}{\sqrt{k}} + \gamma\right]} \right]^{-1} \quad (3.28)$$

with $\alpha = \frac{1}{2}a$, $\beta = \frac{1}{2}b$ and γ are constant numbers. We note by the way that the u -Liouville equation (26) possesses also a simple solution namely [§]

$$\Phi(u) = -\log[\cos(au)], \quad (3.29)$$

such a solution is satisfied for the constant a such that $\chi(\varphi) \equiv \frac{\varphi^2}{41.8} = a^2$. Its also important to search, at the level of general formula (41), for the associated function $f(u)$. In fact the same analysis drives us to write for the simple case $a = 1$

$$f(u) = \frac{K_1}{\sin(u) - K_2} - K_2, \quad (3.30)$$

where K_1 and K_2 are two constants related as follows $K_1 = 1 - K_2^2$. One can signal at this level the

[§]We thank M. Blau for fruitful discussions and suggestions.

importance of the conformal approach described above and from which we have been able to derive the expression of the general solution (41). The previous results show clearly how this solution is more general as it incorporates most of the solutions discussed. This property, gives in some sense a possibility to classify all the existing solutions of the u -Liouville string backgrounds equation of motion (26) and describes in the same time an inverse problem based on the fact that finding $f(u)$ such that (26) is true is a guarantee of the integrability.

Next, using the same analysis described before, the conserved conformal current leads in the limit case to the following equation

$$T(\Phi) = (\nabla_u \Phi)^2 - \frac{1}{\beta} (\nabla_u^2 \Phi), \quad (3.31)$$

To understand the meaning of this conserved current in the string background supergravity equations, one recall that the remaining equations (20) behave with respect to the u -dependence of the dilaton field as

$$0 = \nabla_u^2 \Phi - 2 (\nabla_u \Phi)^2 + \frac{1}{12} h^2. \quad (3.32)$$

Identifying (47) and (48), one may extract then the following important relations

$$T(\Phi) \equiv \mu_{ii} = \frac{1}{4} h^2 = 2 \sum_{i=1}^4 a_i^2. \quad (3.33)$$

This is an unusual form of the stress energy momentum tensor whose conservation law follows naturally by using the Liouville equation of motion since

$$\mu_{ii} = (\nabla_u \Phi)^2 - \frac{1}{2} \nabla_u^2 \Phi, \quad (3.34)$$

is a constant (with respect to the u -direction). We have

$$\begin{aligned} \bar{\partial} T(\Phi) &\equiv \nabla_u T(\Phi) \\ &= \nabla_u \mu_{ii} \\ &= 0 \end{aligned} \quad (3.35)$$

Remark also that the previous formulas impose for the β -parameter to take the value $\beta = 2$.

Now having discussed a special case associated to the ansatz (40), we intend thereafter to present the general solution of the conformal Liouville equation $\beta \bar{\partial} \bar{\partial} \Phi - \chi(\varphi) \exp(2\beta \Phi) = 0$, for any arbitrary value of the constant $\chi(\varphi)$. As we can easily check, the conformal conserved current associated to this Liouville equation does not depend on $\chi(\varphi)$ and conserve then the same shape (47). Now, let's assume that the solution can be written as follows

$$\exp(2\beta\Phi) = \eta \frac{f'(z) \cdot \bar{f}'(\bar{z})}{(1 - f(z) \cdot \bar{f}(\bar{z}))^2}, \quad (3.36)$$

where we have to determine later the constant η in terms of $\chi(\varphi)$. Indeed, straightforward but lengthy calculations show that the quantities $\chi(\varphi)$ and η are forced to satisfy the following relation

$$\eta \cdot \chi(\varphi) = 2, \quad (3.37)$$

and then the final solution is given by

$$\exp(2\beta\Phi) = \frac{2}{\chi(\varphi)} \cdot \left(\frac{\nabla_u f(u)}{1 - f^2(u)} \right)^2, \quad (3.38)$$

once the limit equations are performed. Setting $\chi(\varphi) = 1$ one recover the special case discussed before and also fixes the constant k to be $k = 2$.

As a final point, from the third equation of (20) namely

$$0 = H^{uj} \nabla_u e^{-2\Phi} \quad (3.39)$$

we learn that the dilaton gradient introduced in the u -direction should, by virtue of (26), satisfy a non linear conformal Liouville equation. This results goes beyond the one presented in¹⁰ in which a linear solution for the dilaton field in the u -direction is shown to deal with the absence of the Ramond-Ramond 5-form.

IV. CONCLUSION

In this work, we present an adaptation of the parallelizability process to type IIB string backgrounds in $D = 10$ dimensions. Essentially, we show that the two dimensional conformal Liouville theory may be involved in this sense in a natural way. This is an unsuspected property based on the assumption that the R-R 5-form \mathcal{F} is nonvanishing a fact which leads to exact solutions of type IIB string backgrounds on parallelizable pp-wave.

The link with conformal field theories is made through the bidimensional Liouville model. It is a model based on a nonlinear differential equation and that possesses the particularity to be conformally invariant and integrable. This conformal Liouville equation that appears naturally, at the level of the string background solutions, can be traced to the fact that the dilaton field depends on the longitudinal coordinate u . This hypothesis is also reinforced with the fact that the R-R 5-form \mathcal{F} is non trivial ($\varphi_{ijkl} \varphi^{ijkl} \neq 0$) since at the level of the equation (31), the Liouville potential $exp(2\Phi)$ is coupled to $\varphi_{ijkl} \varphi^{ijkl}$.

We show also that the corresponding conserved conformal current $T(\Phi)$ coincides exactly with the

trace of the constant symmetric matrix μ_{ij} appearing in the quadratic front factor of the parallelizable pp-wave namely $F = \mu_{ij} x^i x^j$. Furthermore, we consider the transverse space dependence of the dilaton Φ and show that the supergravity equations are easily solved with the linear dilaton field. Other remarkable features are also discussed.

Finally, it might be of particular interest to study string theory on the background developed in this work. Also, in the same context, it would be nice to see if $2d$ -sigma model is solvable at least for some special cases and to look for the supersymmetric behavior of the present analysis**.

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