African Journal Of Mathematical Physics Vol **1** *No 2 (2004)177-183*

Some spectral properties of the Pauli-Hamiltonian and of its square root; the Dirac-Hamiltonian

Allal Ghanmi*

EDP & Spectral Geometry, Department of Mathematics, Faculty of Science,
Avenue Ibn Battouta, P.O. Box 1014, Agdal Rabat, Morocco.

Abstract

Abstract: The so-called Dirac-Hamiltonian is realized as the square root of the Pauli-Hamiltonian with uniform magnetic field. Some L^2-spectral properties of both operators are discussed.

Keywords: *Pauli and Dirac Hamiltonians; Spectrum;* L^2*-eigenforms.*

* Email: aghanmi@Math.net

I. INTRODUCTION

It is well known that Pauli-Hamiltonian H_ν^{Pauli} for the plane $\mathbb{R}^2 = \mathbb{C}$ is given in the matrix form by ,

$$H_\nu^{Pauli} = \begin{pmatrix} H_\nu & 0 \\ 0 & H_\nu \end{pmatrix} - 4\nu \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \nu \in \mathbb{R}, \quad (1.1)$$

where H_ν is the usual Landau Hamiltonian that given by

$$H_\nu = -(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}) - 4i\nu(y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}) + 4\nu^2(x^2 + y^2)$$

and which can be rewritten in the complex coordinate $z = x + iy$ as

$$H_\nu = -4\left\{\frac{\partial^2}{\partial z \partial \bar{z}} + \nu(z\frac{\partial}{\partial z} - \bar{z}\frac{\partial}{\partial \bar{z}}) - \nu^2|z|^2\right\}.$$

The above Pauli-Hamiltonian describes the motion of a non-relativistic electron with spin $\frac{1}{2}$ on the plane $\mathbb{R}^2$ in the presence of the external uniform magnetic field of strength "4ν"

$$\mathcal{B} = 4\nu dx \wedge dy$$

associated to the gauge vector potential

$$a_\nu = -2\nu(ydx - xdy); \qquad (\mathcal{B} = da)$$

and applied perpendicularly to the plane $\mathbb{R}^2 = \mathbb{C}$. Such Hamiltonians H_ν^{Pauli} have been extensively studied by many authors in different contexts of both mathematics and physics. There has been much interest works on them (see, e.g.,[5],[6],[3],[11] and the references therein). Some needed L^2-spectral properties of H_ν^{Pauli} are recalled in Section 2. Further they can be realized as differential operator[8] by considering

$$(d + i\nu\text{ext}\theta)^*(d + i\nu\text{ext}\theta) + (d + i\nu\text{ext}\theta)(d + i\nu\text{ext}\theta)^*$$

acting on differential 1-forms, where d denotes the usual exterior derivative and extθ is the exterior multiplication by the real 1-form

$$\theta(z) = i(\bar{z}dz - zd\bar{z}) \qquad (-\nu\theta = a).$$

Here $*$ stands for the formal adjoint of the operator $d + i\nu\text{ext}\theta$ with respect to the natural Hermitian scalar product $(.,.)$ induced from the Eu-

---

clidean metric $|dz|$ on the space $\mathcal{C}_c^\infty(\mathbb{C};\Lambda^1\mathbb{C})$ of $\mathcal{C}^\infty$-differential 1-forms that are compactly supported. That is

$$(\alpha,\beta) := \int_{\mathbb{C}} \alpha\wedge\star\beta, \quad \alpha,\beta\in\mathcal{C}_c^\infty(\mathbb{C};\Lambda^1\mathbb{C}). \tag{1.2}$$

In the integrand $\alpha\wedge\star\beta$, $\star$ denotes the Hodge star operator canonically associated to ds^2 and which acts on differential forms of $\mathbb{C}$ as follows

$$\begin{aligned} \star 1 &= \frac{i}{2} dz\wedge d\bar{z};\\ \star dz &= i d\bar{z};\\ \star d\bar{z} &= -i dz;\\ \star(dz\wedge d\bar{z}) &= 2i. \end{aligned} \tag{1.3}$$

Then it is easy to see that the Hilbert space $L^2(\mathbb{C};\Lambda^1)$, of L^2-differential 1-forms $\omega = f dz + g d\bar{z}$ w.r.t. (1.2) and which is obtained as the completion of $\mathcal{C}_c^\infty(\mathbb{C};\Lambda^1\mathbb{C})$, consists of the couple of functions (f,g) on $\mathbb{C}$ that are square integrable with respect to the Lebesgue measure dm. That is,

$$L^2(\mathbb{C};\Lambda^1) = L^2(\mathbb{C};dm)dz \oplus L^2(\mathbb{C};dm)d\bar{z}.$$

Furthermore, it is well known that the above Pauli-Hamiltonians H_ν^{Pauli} on the plane can be realized as the square of a first-order differential operator $\mathbb{D}_\nu^{Dirac}$ called Dirac-Hamiltonian (or Dirac operator). They go back to the physicist P.A.M. Dirac[4] and appear in the framework of the construction of a relativistic electron theory. Further they had received a considerable amount of attention. See for example[11],[13],[1],[12],[2].
Such Dirac-Hamiltonian on the plane $\mathbb{R}^2$ associated to a gauge field

$$a = a_1 dx + a_2 dy; \qquad \mathrm{div} a = 0,$$

is defined as a selfadjoint operator by

$$\mathbb{D}_a^{Dirac} = \sigma_1(-i\partial_x - a_1) + \sigma_2(-i\partial_y - a_2), \tag{1.4}$$

where σ_1 and σ_2 are the selfadjoints 2 by 2 Pauli spin matrices that obey to the anti-commutation rule

$$\sigma_j\sigma_k + \sigma_k\sigma_j = 2\delta_{jk}\mathbf{I}.$$

As usual, δ_{jk} is the Krönecker symbol and $\mathbf{I}$ is the 2×2 identity matrix. A complex formulation of such operator will be presented in Section 3.

In the present work, we will derive some elementary spectral properties for the Pauli-Diarc operator $\mathbb{D}_\nu^{Dirac}$ realized as the square root of the Landau Hamiltonian H_ν^{Pauli} and so associated to the potential vector a_ν (Section 4). For this we need to recall those of its square, i.e., the Pauli-Hamiltonian H_ν^{Pauli}.

II. SPECTRAL PROPERTIES OF THE PAULI-HAMILTONIAN H_ν^{PAULI}.

In this section we consider for fixed $\nu > 0$ and $\lambda\in\mathbb{C}$ the following eigenvalue problem

$$H_\nu^{Pauli}\omega = \lambda\omega, \quad \omega\in L^2(\mathbb{C};\Lambda^1). \tag{2.1}$$

Then, since (2.1) leads to the scalar case, it is well established that such eigenvalue problem can be described concretely. Namely we have

Proposition 1 *Let $\nu > 0$ be fixed and H_ν^{Pauli} be the Pauli-Hamiltonian acting on 1-forms. Then we have*

i) The spectrum of H_ν^{Pauli} in $L^2(\mathbb{C};\Lambda^1)$ reduces only to the Landau levels, which occurs with infinite degeneracy. More explicitly we have

$$Spec(H_\nu^{Pauli}) = \{\lambda_l = 8\nu l;\ \ l = 0,1,2,\cdots\}. \tag{2.2}$$

ii) Any smooth differential 1-form $\omega = f dz + g d\bar{z}$ solution of $H_\nu^{Pauli}\omega = \lambda\omega$ in $L^2(\mathbb{C};\Lambda^1)$ can be expanded in as follows

$$\omega(z) = \phi_l^\nu(z)dz + \phi_{l-1}^\nu(z)d\bar{z} = \begin{pmatrix} \phi_l^\nu(z) \\ \phi_{l-1}^\nu(z) \end{pmatrix}$$

for some $l\in\mathbb{Z}^+$ with the convention that $\phi_{-1}^\nu = 0$. Here the function ϕ_l^ν, $l\in\mathbb{Z}^+$, is defined by

$$\begin{aligned} \phi_l^\nu(z) &= e^{-\nu|z|^2} A\\ A &= \sum_{j=0}^{+\infty} a_j {}_1F_1(-l;j+1;2\nu|z|^2)z^j\\ &+ \sum_{k=0}^{l} b_k {}_1F_1(k-l;k+1;2\nu|z|^2)\bar{z}^k \end{aligned} \tag{2.3}$$

such that the complex numbers a_j satisfy the following growth condition

$$\sum_{j=0}^{+\infty} \frac{(j!)^2}{(2\nu)^j(j+l)!}|a_j|^2 < +\infty.$$

Therefore in contract of the scalar case (i.e., the Landau Hamiltonian on functions), one can see, trough Proposition 1, that the spectrum of the Pauli-Hamiltonian H_ν^{Pauli} starts at zero. And so the L^2-harmonic eigenforms of degree 1 of such Pauli-Hamiltonian is constituted essentially of the $(1,0)$-eigenforms $f dz$ where the component function f is given by

$$f(z) = e^{-\nu|z|^2}\sum_{j=0}^{+\infty} a_j z^j,$$

where the coefficients a_j in the holomorphic function $\sum_{j=0}^{+\infty} a_j z^j$ satisfy the following growth condition

$$\sum_{j=0}^{+\infty} \frac{j!}{(2\nu)^j} |a_j|^2 < +\infty.$$

Now, for every fixed integer $l \in \mathbb{Z}^+$ let denote by $\mathcal{A}^{2,\nu}_{8\nu l,Pauli}(\mathbb{C})$ the L^2-eigenspace of H^{Pauli}_ν associated to the eigenvalue $8\nu l$ and defined by

$$\mathcal{A}^{2,\nu}_{8\nu l,Pauli}(\mathbb{C}) = \Big\{\omega; \quad \omega \in L^2(\mathbb{C};\Lambda^1),\ H^{Pauli}_\nu \omega = 8\nu l \omega\Big\}$$

$$l = 0, 1, 2, \cdots.$$

Such Hilbert space is invariant by the action T^ν_γ of the semi-direct group $G = U(1) \ltimes \mathbb{C}$. That is

$$T^\nu_\gamma \left(\mathcal{A}^{2,\nu}_{8\nu l,Pauli}(\mathbb{C})\right) \subset \mathcal{A}^{2,\nu}_{8\nu l,Pauli}(\mathbb{C}). \tag{2.4}$$

This holds since the Pauli-Hamiltonian is invariant by the action T^ν_γ, i.e.,

$$T^\nu_\gamma H^{Pauli}_\nu = H^{Pauli}_\nu T^\nu_\gamma.$$

Above the action T^ν_γ, $\gamma \in G$, of the group $G = U(1) \ltimes \mathbb{C}$ on smooth differential 1-forms ω of the plane $\mathbb{C}$ is defined by

$$[T^\nu_\gamma \omega](z) := j_\nu(\gamma; z)[\gamma^*\omega](z), \qquad z \in \mathbb{C}, \tag{2.5}$$

where the automorphic factor $j_\nu(\gamma; z)$ is given by

$$j_\nu(\gamma; z) := e^{2i\nu \Im m(z\overline{\gamma^{-1}.0})}, \qquad \text{for } \gamma \in G, \quad z \in \mathbb{C} \tag{2.6}$$

and where $\gamma^*\omega$ is the pull-back of the 1-form ω by the mapping $z \longmapsto \gamma.z$ with

$$\gamma.z = az + b, \qquad \gamma = \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \in G.$$

Reproducing kernel of the Pauli Hamiltonian H^{Pauli}_ν. Added to the invariance property (2.4), such Hilbert spaces $\mathcal{A}^{2,\nu}_{8\nu l,Pauli}(\mathbb{C})$ can be spanned as

$$\mathcal{A}^{2,\nu}_{8\nu l,Pauli}(\mathbb{C}) = \mathcal{A}^{2,\nu}_{4\nu(2l+1)}(\mathbb{C})dz \oplus \mathcal{A}^{2,\nu}_{4\nu(2l-1)}(\mathbb{C})d\bar z \tag{2.7}$$

with the convention that $\mathcal{A}^{2,\nu}_{-1}(\mathbb{C}) = \{0\}$ and where $\mathcal{A}^{2,\nu}_{\mu_l}(\mathbb{C})$ is the L^2-eigenspace of the (scalar) Landau Hamiltonian H_ν associated to the Landau level $\mu_l = 4\nu(2l+1)$. Therefore, it follows that the above Hilbert space $\mathcal{A}^{2,\nu}_{8\nu l,Pauli}(\mathbb{C})$ admits reproducing kernel. More exactly, we have

Proposition 2 *The reproducing kernel of the L^2-eigenspace $\mathcal{A}^{2,\nu}_{8\nu l,Pauli}(\mathbb{C})$, where $l \neq 0$, is given by*

$$\mathcal{K}^{Pauli}_{\nu,l}(z;w) = \frac{2\nu}{\pi} e^{\nu(z\bar w - \bar z w)} e^{-\nu|z-w|^2} \times \begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix}. \tag{2.8}$$

with

$$p = {}_1F_1(-l; 1; 2\nu|z-w|^2)$$
$$q = {}_1F_1(1-l; 1; 2\nu|z-w|^2)$$

For $l = 0$ the reproducing kernel of the L^2-harmonic eigenspace $\mathcal{A}^{2,\nu}_{0,Pauli}(\mathbb{C})$ is reduced further to be equal to

$$\mathcal{K}^{Pauli}_{\nu,l}(z;w) = \frac{2\nu}{\pi} e^{\nu(z\bar w - \bar z w)} e^{-\nu|z-w|^2} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

The above proposition holds by the use of the explicit expression of the reproducing kernel of the Landau Hamiltonian H_ν on functions[9] keeping in mind the fact (2.7). Here the reproducing kernel is taken in the vectorial sense. That is for any L^2-eigenform $\omega = f dz + g d\bar z \in \mathcal{A}^{2,\nu}_{8\nu l,Pauli}(\mathbb{C})$ we have

$$\begin{aligned} \begin{pmatrix} f \\ g \end{pmatrix}(z) &= \int_{\mathbb{C}} K^{Pauli}_{\nu,l}(z;w) \begin{pmatrix} f \\ g \end{pmatrix}(w) dm(w) \\ &= \int_{\mathbb{C}} \begin{pmatrix} K^+_{\nu,l}(z,w) & 0 \\ 0 & K^-_{\nu,l} \end{pmatrix} \begin{pmatrix} f \\ g \end{pmatrix}(w) dm(w) \\ &= \begin{pmatrix} \int_{\mathbb{C}} K^+_{\nu,l} f(w) dm(w) \\ \int_{\mathbb{C}} K^-_{\nu,l} g(w) dm(w) \end{pmatrix}, \end{aligned}$$

where we have use the notation

$$K^{Pauli}_{\nu,l}(z;w) = \begin{pmatrix} K^+_{\nu,l} & 0 \\ 0 & K^-_{\nu,l} \end{pmatrix}.$$

III. COMPLEX FORMULATION OF THE DIRAC-HAMILTONIAN.

In this section, we make use complex formalism to introduce the so-called Dirac-Hamiltonian. For this one consider instead of the σ-matrices the matrices τ_j defined by

$$\tau_1 = \frac{1}{2}(\sigma_1 - i\sigma_2), \qquad \tau_2 = \frac{1}{2}(\sigma_1 + i\sigma_2)$$

that given explicitly by

$$\tau_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \tau_2 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Such matrices generated the same Clifford algebra as σ_1 and σ_2, and satisfy the following useful identities

$$\begin{aligned} &\tau_1^* = \tau_2; \\ &\tau_k^2 = \mathbf{0}; \\ &\tau_1\tau_2 + \tau_2\tau_1 = \mathbf{I}; \\ &\tau_1\tau_2 - \tau_2\tau_1 = \sigma_3 = \mathrm{diag}[1, -1]; \\ &\tau_1\tau_2\tau_1 = \tau_1, \quad \tau_2\tau_1\tau_2 = \tau_2; \\ &(\tau_1 + \tau_2)^{-1} = \tau_1 + \tau_2. \end{aligned}$$

The gauge vector field $a = a_1 dx + a_2 dy$ can be written in the form $a = A_1 dz + A_2 d\bar z =: Q$ with

$$A_1 = \frac{1}{2}(a_1 - ia_2), \quad A_2 = \frac{1}{2}(a_1 + ia_2).$$

Finally, let consider the Wirtinger operators

$$\partial = \frac{\partial}{\partial z} = \frac{1}{2}(\partial_x - i\partial_y), \quad \bar{\partial} = \frac{\partial}{\partial \bar{z}} = \frac{1}{2}(\partial_x + i\partial_y)$$

and then set

$$\nabla_A = i\partial + A_1; \qquad \widetilde{\nabla}_A = i\bar{\partial} + A_2.$$

With these data and since

$$\begin{aligned} -i\partial_x - a_1 &= -\{(i\partial + A_1) + (i\bar{\partial} + A_2)\} \\ &= -\{\nabla_A + \widetilde{\nabla}_A\} \end{aligned}$$

and

$$\begin{aligned} -i\partial_y - a_2 &= -\{(i\partial + A_1) - (i\bar{\partial} + A_2)\} \\ &= -\{\nabla_A - \widetilde{\nabla}_A\}, \end{aligned}$$

one can note that the usual scalar Schrödinger operator H,

$$H = H_a = (-i\partial_x - a_1)^2 + (-i\partial_y - a_2)^2$$

can be rewritten as follows

$$\begin{aligned} H = H_A &= 2\{(i\partial + A_1)(i\bar{\partial} + A_2) + (i\bar{\partial} + A_2)(i\partial + A_1)\} \\ &= 2\{\nabla_A \widetilde{\nabla}_A + \widetilde{\nabla}_A \nabla_A\} \\ &= 2[\nabla, \widetilde{\nabla}_A]_+. \end{aligned}$$

Furthermore, we can note also that we have

$$\begin{aligned} \nabla_A \widetilde{\nabla}_A &= (i\partial + A_1)(i\bar{\partial} + A_2) = \Delta_A + iA_{2z} \\ \widetilde{\nabla}_A \nabla_A &= (i\bar{\partial} + A_2)(i\partial + A_1) = \Delta_A + iA_{1\bar{z}}, \end{aligned}$$

where

$$\Delta_A = -\partial\bar{\partial} + i(A_2\partial + A_1\bar{\partial}) + A_1A_2$$

and where $A_{2z} := \partial A_2$ and $A_{1\bar{z}} := \bar{\partial} A_1$. Hence it follows

$$\begin{aligned} [\nabla_A, \widetilde{\nabla}_A]_- &= \nabla_A \widetilde{\nabla}_A - \widetilde{\nabla}_A \nabla_A = i(A_{2z} - A_{1\bar{z}}) \\ [\nabla_A, \widetilde{\nabla}_A]_+ &= \nabla_A \widetilde{\nabla}_A + \widetilde{\nabla}_A \nabla_A = 2\Delta_A + i(A_{2z} + A_{1\bar{z}}). \end{aligned}$$

Here we note that $2i(A_{2z} - A_{1\bar{z}}) =: \mathcal{B}_{complex}$ is the magnetic field, written in complex notation, derived from the vector potential $a = A$. While $2i(A_{2z} + A_{1\bar{z}}) =: \mathrm{div}_{complex}(A)$ is the divergence of $a = A$. Therefore with the gauge condition on the vector potential A, i.e., $\mathrm{div}_{complex}(A) = 0$, we get again

$$H = 4\Delta_A = -4\{\partial\bar{\partial} - i(A_2\partial + A_1\bar{\partial}) - A_1A_2\}.$$

Above $i(A_2\partial + A_1\bar{\partial})$ represents the angular momenta.

At this level one can write the Dirac operator $\mathbb{D}_\nu^{Dirac}$, (1.4), in the complex notation as follows

$$\begin{aligned} \mathbb{D} &= -2\{\tau_2(i\partial + A_1) + \tau_1(i\bar{\partial} + A_2)\} \\ &= -2\begin{pmatrix} 0 & i\bar{\partial} + A_2 \\ i\partial + A_1 & 0 \end{pmatrix} \end{aligned}$$

and then

$$\begin{aligned} \mathbb{D}^2 &= 4\{\tau_1(i\bar{\partial} + A_2)\tau_2(i\partial + A_1) + \tau_2(i\partial + A_1)\tau_1(i\bar{\partial} + A_2)\} \\ &= 4\{\tau_1\tau_2(i\bar{\partial} + A_2)(i\partial + A_1) + \tau_2\tau_1(i\partial + A_1)(i\bar{\partial} + A_2)\} \\ &= 4\{(\tau_1\tau_2 + \tau_2\tau_1)\Delta_A + i(\tau_1\tau_2A_{2z} + \tau_2\tau_1A_{1\bar{z}})\} \\ &= 4\{\begin{pmatrix} \Delta_A & 0 \\ 0 & \Delta_A \end{pmatrix} + i\begin{pmatrix} A_{2z} & 0 \\ 0 & A_{1\bar{z}} \end{pmatrix}\}. \end{aligned}$$

But since $A_{1\bar{z}} = -A_{2z}$, we conclude that

$$\mathbb{D}^2 = 4\{\begin{pmatrix} \Delta_A & 0 \\ 0 & \Delta_A \end{pmatrix} + iA_{2z}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\}$$

which can be written out for the particular gauge vector potential

$$A_\nu(z) = -i\nu(\bar{z}dz - zd\bar{z})$$

as follows

$$\mathbb{D}^2 = \begin{pmatrix} H_\nu & 0 \\ 0 & H_\nu \end{pmatrix} - 4\nu\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = H_\nu^{Pauli}, \quad \nu \in \mathbb{R},$$

where $H_\nu = 4\Delta_{A_\nu}$ is written in the complex coordinates $z = x + iy$ as

$$H_\nu = -4\left\{\frac{\partial^2}{\partial z \partial \bar{z}} + \nu(z\frac{\partial}{\partial z} - \bar{z}\frac{\partial}{\partial \bar{z}}) - \nu^2|z|^2\right\}.$$

Thus one can note that instead of the Dirac operator $\mathbb{D}$,

$$\mathbb{D} = -2\{\tau_2(i\partial + A_1) + \tau_1(i\bar{\partial} + A_2)\},$$

one can work with its equivalent, the Dirac operator $\tilde{\mathbb{D}}$ defined by

$$\tilde{\mathbb{D}} = -2\{\tau_1(i\partial + A_1) + \tau_2(i\bar{\partial} + A_2)\}, \tag{3.1}$$

$$= -2\begin{pmatrix} 0 & i\partial + A_1 \\ i\bar{\partial} + A_2 & 0 \end{pmatrix}.$$

Indeed we have $\tilde{\mathbb{D}} = (\tau_1 + \tau_2)\mathbb{D}(\tau_1 + \tau_2)$ and $\tilde{\mathbb{D}}^2 = (\tau_1 + \tau_2)\mathbb{D}^2(\tau_1 + \tau_2)$. Thus the eigenvalue problem $\mathbb{D}\psi = \mu\psi$ for $\tilde{\mathbb{D}}$ with $\psi = \begin{pmatrix} \psi^+ \\ \psi_- \end{pmatrix}$ as eigenspinor is equivalent to the eigenvalue problem

$$\tilde{\mathbb{D}}\tilde{\psi} = \mu\tilde{\psi} \tag{3.2}$$

with the same eigenvalue μ and associated eigenspinor

$$\tilde{\psi} = (\tau_1 + \tau_2)\psi = \begin{pmatrix} \psi^- \\ \psi_+ \end{pmatrix}.$$

In the next section we will mainly be interested by related spectral properties to the above eigenvalue problem (3.2).

IV. SOME SPECTRAL PROPERTIES OF THE DIRAC-HAMILTONIAN

The Dirac-Hamiltonians $\mathbb{D}_\nu^{Dirac} := \tilde{\mathbb{D}}$, as defined above by (3.1), are elliptic first-order differential operators. Also, they are densely defined on the Hilbert space $L^2(\mathbb{C};\Lambda^1)$ and admit a unique selfadjoint realization on $L^2(\mathbb{C};\Lambda^1)$ that we will denote also $\mathbb{D}_\nu^{Dirac}$. Furthermore, they are invariant by the action of the group of motions G of $\mathbb{C}$. Indeed we have

Proposition 3 *(Invariance Property).*
The Dirac-Hamiltonian $\mathbb{D}_\nu^{Dirac}$ is a T_γ^ν-invariant operator. More exactly we have

$$\mathbb{D}_\nu^{Dirac} T_\gamma^\nu = \begin{pmatrix} \bar{a} & 0 \\ 0 & a \end{pmatrix} T_\gamma^\nu \mathbb{D}_\nu^{Dirac}$$

for all $\nu > 0$ and $\gamma = \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \in G$.

Proof. Firstly let recall that the action T_γ^ν, $\gamma \in G$, of the group $G = U(1) \ltimes \mathbb{C}$ on differential 1-forms, is reduced to the scalar case as follows:

$$\begin{aligned} T_\gamma^\nu(\omega) &= \Big[T_\gamma^\nu(fdz + gd\bar{z})\Big] \\ &= (\partial\gamma)\Big[T_\gamma^\nu(f)\Big]dz + \overline{(\partial\gamma)}\Big[T_\gamma^\nu(g)\Big]d\bar{z} \\ &= a\Big[T_\gamma^\nu(f)\Big]dz + \bar{a}\Big[T_\gamma^\nu(g)\Big]d\bar{z} \end{aligned}$$

which can be rewritten in the following matricial form

$$\Big[T_\gamma^\nu(fdz + gd\bar{z})\Big] = \begin{pmatrix} aT_\gamma^\nu & 0 \\ 0 & \bar{a}T_\gamma^\nu \end{pmatrix} \begin{pmatrix} f \\ g \end{pmatrix}$$

where we have identified the differential 1-form $fdz + gd\bar{z}$ to the vector column $\begin{pmatrix} f \\ g \end{pmatrix}$ and where $\gamma = \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \in G$.

Secondly, noting that

$$\begin{aligned} &\mathbb{D}_\nu^{Dirac}\Big[T_\gamma^\nu(fdz + gd\bar{z})\Big] \\ &= \begin{pmatrix} 0 & \partial - \nu\bar{z} \\ \bar{\partial} + \nu z & 0 \end{pmatrix} \begin{pmatrix} aT_\gamma^\nu & 0 \\ 0 & \bar{a}T_\gamma^\nu \end{pmatrix} \begin{pmatrix} f \\ g \end{pmatrix} && (4.1) \\ &= \begin{pmatrix} \bar{a}(\partial - \nu\bar{z})\Big[T_\gamma^\nu(g)\Big] \\ \\ a(\bar{\partial} + \nu z)\Big[T_\gamma^\nu(f)\Big] \end{pmatrix} && (4.2) \end{aligned}$$

and using direct computation we get

$$\partial\Big[T_\gamma^\nu(g)\Big] = a^2\Big[T_\gamma^\nu\Big(\partial g - \nu\overline{ab}g\Big)\Big] \qquad (a_1)$$

$$\nu\bar{z}\Big[T_\gamma^\nu(g)\Big] = a^2\Big[T_\gamma^\nu\Big(\nu\bar{z}g - \nu\overline{ab}g\Big)\Big] \qquad (a_2)$$

$$\bar{\partial}\Big[T_\gamma^\nu(f)\Big] = \bar{a}^2\Big[T_\gamma^\nu\Big(\bar{\partial}f + \nu abf\Big)\Big] \qquad (b_1)$$

$$\nu z\Big[T_\gamma^\nu(f)\Big] = \bar{a}^2\Big[T_\gamma^\nu\Big(\nu zf - \nu abf\Big)\Big]. \qquad (b_2)$$

Hence by combining (a_1) with (a_2) and (b_1) with (b_2), we see that the result in the above proposition holds. Indeed

$$\begin{aligned} \mathbb{D}_\nu^{Dirac} T_\gamma^\nu &= \begin{pmatrix} T_\gamma^\nu & 0 \\ 0 & T_\gamma^\nu \end{pmatrix} \begin{pmatrix} 0 & \partial - \nu\bar{z} \\ \bar{\partial} + \nu z & 0 \end{pmatrix} \\ &= \begin{pmatrix} \bar{a} & 0 \\ 0 & a \end{pmatrix} T_\gamma^\nu \mathbb{D}_\nu^{Dirac}. \end{aligned} \qquad (4.3)$$

Remark 1 *It is clear that, when we have restricted to the particular subgroup Γ_0,*

$$\Gamma_0 = \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix};\ b \in \mathbb{C} \right\},$$

the invariance property given in the above proposition reads simply

$$\mathbb{D}_\nu^{Dirac} T_\gamma^\nu = T_\gamma^\nu \mathbb{D}_\nu^{Dirac}, \quad \forall \gamma \in \Gamma_0. \qquad (4.4)$$

Now let $\mu \in \mathbb{R}$ and consider the following eigenvalue problem in $L^2(\mathbb{C};\Lambda^1)$ for the Dirac-Hamiltonian $\mathbb{D}_\nu^{Dirac}$, i.e.,

$$\mathbb{D}_\nu^{Dirac}\psi = \mu\psi, \qquad \psi \in L^2(\mathbb{C};\Lambda^1). \qquad (4.5)$$

By $\mathcal{A}_{\mu,Dirac}^{2,\nu}(\mathbb{C})$, we will denote the space of such solutions. That is

$$\mathcal{A}_{\mu,Dirac}^{2,\nu}(\mathbb{C}) = \Big\{\psi;\quad \psi \in L^2(\mathbb{C};\Lambda^1),\ \ \mathbb{D}_\nu^{Dirac}\psi = \mu\psi\Big\}.$$

The equation in (4.5) is written out for $\psi = \begin{pmatrix} f \\ g \end{pmatrix}$ as

$$\begin{cases} (\partial - \nu\bar{z})g = \mu f \\ \\ (\bar{\partial} + \nu z)f = \mu g \end{cases} \qquad \text{with } f, g \in L^2(\mathbb{C}; dm). \quad (4.6)$$

For instance, one can point out some immediate facts.

Lemma IV.1
1*) 0 is an eigenvalue of $\mathbb{D}_\nu^{Dirac}$ with infinite multiplicity.*
2*) The L^2-eigenforms of the Dirac-Hamiltonian $\mathbb{D}_\nu^{Dirac}$ of the form $\begin{pmatrix} f \\ 0 \end{pmatrix}$ are necessarily in $Ker(\mathbb{D}_\nu^{Dirac}) := \{\psi \in L^2(\mathbb{C};\Lambda^1);\ \mathbb{D}_\nu^{Dirac}\psi = 0\}$.*
3*) $Ker(\mathbb{D}_\nu^{Dirac}) = Ker(H_\nu^{Pauli})$. More precisely, we have*

$$Ker(\mathbb{D}_\nu^{Dirac}) = \begin{pmatrix} f \\ 0 \end{pmatrix}; \quad f(z) = e^{-\nu|z|^2}\sum_{j=0}^{+\infty} a_j z^j$$

$$\textit{with} \qquad \sum_{j=0}^{+\infty} \frac{j!|a_j|^2}{(2\nu)^j} < +\infty.$$

4*) There is non L^2-eigenform for the Dirac-Hamiltonian operator $\mathbb{D}_\nu^{Dirac}$ of the form $\begin{pmatrix} 0 \\ g \end{pmatrix}$.*

4'*) If $\begin{pmatrix} f \\ g \end{pmatrix}$ is a L^2-eigenform of the Dirac-Hamiltonian $\mathbb{D}_\nu^{Dirac}$ associated to the eigenvalue μ with $\mu \neq 0$, then we have necessarily $f \neq 0$ and $g \neq 0$.*

The assertions 1) and 3) holds since

$$\left\|\mathbb{D}_\nu^{Dirac}\psi\right\|^2 = \left(\mathbb{D}_\nu^{Dirac}\psi, \mathbb{D}_\nu^{Dirac}\psi\right) = \left(H_\nu^{Pauli}\psi, \psi\right).$$

While 2) is immediate by substitution in (4.6). 4) and $4'$) can be deduced from (4.6)and 3) easily.

Furthermore, it is easy to see that if $\begin{pmatrix} f \\ g \end{pmatrix}$ is a L^2-eigenform of $\mathbb{D}_\nu^{Dirac}$ with μ as eigenvalue, then $\begin{pmatrix} f \\ g \end{pmatrix}$ is also a L^2-eigenform of H_ν^{Pauli} with, this time, μ^2 as eigenvalue. In spite of, one can not decide for the sign of μ. The following lemma shows that both signs are possible[*]. In fact, we have

Lemma IV.2 *Let λ be in $Spec(H_\nu^{Pauli})$, $\lambda > 0$, and ψ a corresponding L^2-eigenform of H_ν^{Pauli}. Then*

i) The form $\mathbb{D}_\nu^{Dirac}\psi + \sqrt{\lambda}\psi$ is a L^2-eigenform of $\mathbb{D}_\nu^{Dirac}$ with $+\sqrt{\lambda}$ as eigenvalue.

ii) The form $\mathbb{D}_\nu^{Dirac}\psi - \sqrt{\lambda}\psi$ is a L^2-eigenform of $\mathbb{D}_\nu^{Dirac}$ with $-\sqrt{\lambda}$ as eigenvalue.

Whose the proof follows immediately from the facts that

$$(\mathbb{D}_\nu^{Dirac})^2 - \lambda = (\mathbb{D}_\nu^{Dirac} + \lambda)(\mathbb{D}_\nu^{Dirac} - \lambda)$$
$$(\mathbb{D}_\nu^{Dirac})^2 - \lambda = (\mathbb{D}_\nu^{Dirac} - \lambda)(\mathbb{D}_\nu^{Dirac} + \lambda).$$

As first comment, the above lemma gives a tool of constructing the L^2-eigenforms of the Dirac-Hamiltonian $\mathbb{D}_\nu^{Dirac}$ from those of the Pauli-Hamiltonian H_ν^{Pauli}.

Hence, according to the assertion i) in Proposition 1 and the above lemma, we get the following proposition given the spectrum of the Dirac-Hamiltonian $\mathbb{D}_\nu^{Dirac}$. Namely, we have

Proposition 4 *(Spectrum).*

The point spectrum of the Dirac-Hamiltonian $\mathbb{D}_\nu^{Dirac}$ in $L^2(\mathbb{C}; \Lambda^1)$, with uniform magnetic field on the plane $\mathbb{C}$, is given explicitly by

$$Spec(\mathbb{D}_\nu^{Dirac}) = \{\pm\sqrt{8\nu l}; \quad l = 0, 1, 2, \cdots\}.$$

At this level and according to the above remarks, one can note that for $l \neq 0$ we have the following decomposition

$$\mathcal{A}^{2,\nu}_{8\nu l, Pauli}(\mathbb{C}) = \mathcal{A}^{2,\nu}_{-\sqrt{8\nu l}, Dirac}(\mathbb{C}) \bigoplus \mathcal{A}^{2,\nu}_{-\sqrt{8\nu l}, Dirac}(\mathbb{C}) \bigoplus E, \qquad (4.7)$$

where E is a non trivial space of infinite dimension. Indeed, it contains at least the eigenforms of type $\begin{pmatrix} 0 \\ \phi_l^\nu \end{pmatrix}$, where ϕ_l^ν is as given in Proposition 1.

Furthermore, a part of the L^2-harmonic eigeforms of $\mathbb{D}_\nu^{Dirac}$ for which we have $Ker(\mathbb{D}_\nu^{Dirac}) = Ker(H_\nu^{Pauli})$, we can show that for a L^2-eigeform $\begin{pmatrix} f \\ g \end{pmatrix}$ of the Dirac Hamiltonian $\mathbb{D}_\nu^{Dirac}$ associated to the eigenvalue $\mu_l = \pm\sqrt{8\nu l}$, $l \neq 0$, we have $f = \phi_l^\nu$ and $g = \phi_{l-1}^\nu$, where ϕ_l^ν is as given in Proposition 1. Thus characterizing $\begin{pmatrix} f \\ g \end{pmatrix}$ turn out to find the relationship between the Fourier coefficients of f and g this will be reads in $g = \mu_l^{-1}(\bar{\partial} + \nu z)f$. Keeping in mind this relation between f and g and using the harmonic oscillator approach, instead of the resolution of the resulting hypergeometric differential equation, one can give a concrete description of such L^2-eigenforms involving Hermite polynomials. Indeed we get (see[7] for details):

Proposition 5 *(L^2-eigenforms)*

Let $\mu_l = \pm\sqrt{8\nu l}$ for $l \neq 0$. Then $\begin{pmatrix} f \\ g \end{pmatrix}$ is in $A^{2,\nu}_{\mu_l}(\mathbb{D}_\nu)$ if and only if $g = \mu_l^{-1}(\bar{\partial} + \nu z)f$ and f can be expanded explicitly in $L^2(\mathbb{C}; dm)$ in terms of the Hermite polynomials $H^\nu_{l,p}(z)$,

$$H^\nu_{l,p}(z) := \frac{(-1)^p}{(2\nu)^p} e^{2\nu|z|^2} \frac{\partial^{l+p}}{\partial z^l \partial \bar{z}^p}(e^{-2\nu|z|^2}), \quad l, p \in \mathbb{Z}^+,$$

as follows

$$f(z) = e^{-\nu|z|^2} \sum_{p=0}^{+\infty} a_{lp} H^\nu_{l,p}(z), \qquad a_{lp} \in \mathbb{C},$$

where the coefficients a_{lp} satisfy the following growth condition

$$\sum_{p \geq l}^{+\infty} \frac{l!p!}{(2\nu)^p} |a_{lp}|^2 < +\infty.$$

CONCLUSION

In this paper we have presented some spectral properties of the Pauli Hamiltonian H_ν^{Pauli}. We have seen that 0 is the bottom eigenvalue of H_ν^{Pauli}. Further we have described concretely their L^2-eigenforms and we have given the explicit expression of the corresponding L^2-eigenprojector kernel. Some other spectral quantities such as the the resolvent kernel or Heat kernel can also be given.

Furthermore we have determinate the L^2-spectrum of the associated Dirac Hamiltonian $\mathbb{D}_\nu^{Dirac}$, realized as the square root of H_ν^{Pauli}. And we have characterized the associated L^2-eigenforms. Further concrete results on the Dirac operator, like reproducing and resolvent kernels can be found in[7]

Finally let note that all previous results can be generalized to higher dimensional space easily. Also let mention that we can proceed similarly for the case of

---

[*] In[11] Shigekawa had considered a general operator $\mathbb{D}_{p,m} = \mathbb{D}_\nu^{Dirac} + m\sigma_3$, $m \in \mathbb{R}$.

the hyperbolic disc for a square root of the Pauli Hamiltonian "if it exists". But this time one must take in account the conformal metric of the disc $h^{-2}(z)|dz|$.

Acknowledgements Special thanks are addressed to the organizers of the Workshop especially professor E.H. Saidi. Also I would like to thank professor A. Intissar and H. Sami for help, encouragement and support.

---

[1] BÄR, C.: The Dirac Operator on Hyperbolic Manifolds of finite volume.

[2] CAMPORESI, R.; HUGUCHI, A.: On the eigenfunctions of the Dirac operator on sphers and real hyperbolic spaces. Journal of Geometry and Physics 20, 1-18 (1996).

[3] CYCON, H.L.; FROESE, R.G.; KIRSCH, W.; SIMON, B.: *Schrödinger operators with applications to quantum mechanics and global geometry.* Springer-Verlag. Berlin, Heidelberg, New York, London, Paris, Tokyo 1987.

[4] 13. DIRAC; P.A.M.: The Quantum Theory of the electron. Proc. Roy. Soc. of London A, V 117, 610-624 (1928).

[5] GAVEAU, B.; VAUTHIER, J. Intégrales oscillantes stochastiques: l'équation de Pauli. *J. Funct. Anal.* **44** (1981), **no. 3**, 388–400.

[6] GESZTESY, F.; GROSSE, H.; THALLER, B.A.:, A rigourous approach to relativistic corrections of bound state energies for spin-$\frac{1}{2}$ particles. *Ann. Inst. H. Poincaré.* **40** (1984), 159-174.

[7] A. GHANMI, Eigenprojector and resolvent kernels of the 2D Pauli-Dirac operator with constant magnetic field. To appear.

[8] A. GHANMI, Euclidean limit of L^2-spectral properties of the Pauli Hamiltonians on constant curvature Riemann sufaces. Submitted for publication

[9] A. GHANMI, A. INTISSAR, Asymptotic behaviour of reproducing kernels of the L^2-eigenspaces associated to Magnetic Laplacians on the Bergman ball of radius $\rho > 0$. To appear in the Proceedings of the Third European-Maghreb Workshop. Marrakech 2002.

[10] NIKIVAROV, A.; OUVAROV, V.: *Fonctions spéciales de la physique mathématique.* Editions Mir. Moscou. (1978).

[11] SHIGEKAWA, I.: Spectral properties of Schrödinger operators with magnetic fields for a spin $\frac{1}{2}$ particle. *J. Funct. Anal.* **101** (1991), **no. 2**, 255–285.

[12] THALLER, B.: The Dirac equation. Springer Verlag, Berlin, 1992.

[13] TRAUTMAN, A.: The Dirac operator on hypersurfaces. Acta Physica Polinica B, Vol 26 (1995).