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The Hierarchy of Hamiltonians for Spherical Woods-saxon potential

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Abstract

The three dimensional Woods-Saxon potential is studied within the context of Supersymmetry Quantum Mechanics. We obtain hierarchy of hamiltonians related to Woods-Saxon superpotentials that have not been determined yet. In the basis of supersymmetry method[17], we obtain the shape invariant superfamily related to original woods-Saxon potential. Also we derive n-th ground state energy, ϵ_0^n corresponding to n-th superpotential associated to generalized Woods-Saxon potential. The results obtained here are consistent with results obtained from another ways[24]. The generalized Woods-Saxon potential obtained here completely shape invariant. By application of this method,it is possible to solve the schrödinger equation for this potential.

Keywords: *Supersymmetry; Woods-Saxon; Jacobi equation;riccati condition; Bound State; Superpotential; Hamiltonian*

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I. INTRODUCTION

The idea of supersymmetry in context of quantum mechanics was studied first by Nicolai and Witten[1,3] and later by Cooper and Freedman[4]. Two decades ago supersymmetric Quantum Mechanics, SQM, was born to study SUSY breaking of higher dimensional quantum field theory. A developed structure of this method applied to unify the four fundamental interactions in nature, namely the electroweak , strong and gravitational interactions[5] . So far , SQM has been extensively used to explore different aspects of non-relativistic quantum mechanical systems[5]. This algebric method was successful to study analytically solvable[5,9], the partially solvable,[10,12], the isospectral,[13] and periodic potentials[14]. The SQM formalism along with the variational method has been introduced to obtain the approximate energy spectra of non-exactly solvable potentials[15,17]. Recently a new methodology based on an Ansate for the superpotential which is related to a trial wave function has been proposed[18]. By using this approach, it is possible to make an Ansate for the superpotential which satisfy the Riccati equation.

In this paper, we try to examine this method to solve schrödinger equation for Nuclear Woods-Saxon potential. We show that the application of this method able us to derive the n-th ground state energy. The generalized Woods-Saxon superpotential also obtained. Recently,by using of factorization method, the bound sates associated to Woods-

Saxon potential have been obtained analytically[24].

II. SUPERSYMMETRIC QUANTUM MECHANICS

In $N = 2$ supersymmetry quantum mechanic it is necessary to define two nilpotent operators[17] namely, Q and Q_+ that satisfy the well known algebric relations

$$\{Q, Q_+\} = H \qquad ; Q^2 = Q_+^2 = 0, \tag{2.1}$$

where H is the supersymmetry hamiltonian . These operators can be realized as

$$Q = \begin{bmatrix} 0 & 0 \\ A^- & 0 \end{bmatrix}, \qquad Q_+ = \begin{bmatrix} 0 & A^+ \\ 0 & 0 \end{bmatrix} \tag{2.2}$$

where A^+ and A^- are bosonic operators. The hamiltonian, H in terms of these operators is given by

$$H = \begin{bmatrix} A^+A^- & 0 \\ 0 & A^-A^+ \end{bmatrix} = \begin{bmatrix} H_+ & 0 \\ 0 & H_- \end{bmatrix}, \tag{2.3}$$

where $H_\pm$ are partner hamiltonian that share the spectra, apart from the non-degenerate ground state. By using of the superalgebra , the hamiltonian in terms of bosonic operators (in units of $\hbar = c = 1$) can be factorized as follow

$$H_1 = -\frac{1}{2}\frac{d^2}{dr^2} + V_1(r) = A_1^+ A_1^- + \epsilon_0^{(1)}, \tag{2.4}$$

where $\epsilon_0^{(1)}$ is the ground state eigenvalue of $V_1(r)$. The bosonic operators are defined by

$$A_1^\pm = (\mp\frac{d}{dr} + W_1(r)), \tag{2.5}$$

$W_1(r)$ is the superpotential that satisfies the following Riccati equation

$$W_1^2(r) - W_1'(r) = V_1(r) - \epsilon_0^{(1)}, \tag{2.6}$$

and the lowest eigenfunction associated to $V_1(r)$ can be obtained as

$$\Psi_0^{(1)}(r) = Nexp(-\int_0^r W_1(r)dr) \tag{2.7}$$

Or conversely the supersymmetry partner Hamiltonian is define as

$$\begin{aligned} H_2 &= A_1^- A_1^+ + \epsilon_0^{(1)} \\ &= -\frac{1}{2}\frac{d^2}{dr^2} + \frac{1}{2}(W_1^2(r) + W_1'(r)) + \epsilon_0^{(1)} \end{aligned} \tag{2.8}$$

Now we can redefine H_2 in terms of even operators,

$$\begin{aligned} H_2 &= A_2^+ A_2^- + \epsilon_0^{(2)} \\ &= -\frac{1}{2}\frac{d^2}{dr^2} + \frac{1}{2}(W_2^2(r) - W_2'(r)) + \epsilon_0^{(2)} \end{aligned} \tag{2.9}$$

where $\epsilon_0^{(2)}$ is the lowest eigenvalue of $H_2(r)$ and its corresponding superpotential $W_2(r)$, satisfy the following Ricati equation

$$W_2^2(r) - W_2'(r) = V_2(r) - \epsilon_0^{(2)}. \tag{2.10}$$

Since by repeating this method n times, a whole hierarchy of hamiltonians can be obtain. The results have shown below

$$\begin{aligned} H_n &= A_n^+ A_n^- + \epsilon_0^{(n)} \\ A_n^\pm &= \frac{1}{\sqrt{2}}(\mp\frac{d}{r} + W_n(r)) \end{aligned} \tag{2.11}$$

$$\begin{aligned} \Psi_n^{(1)}(r) &= A_1^+ A_2^+ ... \Psi_0^{(n+1)}(r) \\ \epsilon_n^{(1)} &= \epsilon_0^{(n+1)}. \end{aligned} \tag{2.12}$$

III. THE WOODS-SAXON POTENTIAL

The spherical Woods-Saxon potential that was used as a major part of nuclear shell model, has received a lot of attention in nuclear mean field model. It can be used as central part of the interaction potential of neutron with heavy nucleus[18]. With the help of the axially-deformed Woods-Saxon potential, we may construct the structure of single-particle shell model[19]. The Woods-Saxon potential was used as a part of optical model in elastic scattering of some ions in a range of energies[20]. Generally, the Woods-Saxon potential and it's various modified shapes was successful to describe the metallic clusters[21]. Recently the Dirac equation has been solved using two component spinors for Woods-Saxon potential in a special case[22]. Furthermore, the pseudo spin symmetry in nuclei by considering the Dirac equation with central Woods-Saxon potential, that having Lorentz structure, has been investigated. It has been shown that the isospin asymmetry of the nuclear pseudo spin interaction, which has quasi-isospin symmetry, is opposed to the nuclear spin-orbit interaction[23].

The Woods-Saxon potential in atomic units can be given by

$$V(r) = \frac{V_0}{1 + e^{\frac{(r-r_0)}{R}}}; \qquad \frac{(r - r_0)}{R} \longrightarrow (r - r_0) \tag{3.1}$$

with

$$\frac{2MV_0R^2}{\hbar^2} = \lambda^2$$

and the corresponding eigenvalues can be define as

$$\frac{2MR^2E_n}{\hbar^2} = \epsilon_n$$

The time-independent zero angular momentum radial part of schrödinger equation for the spherical Woods-Saxon potential can be given by

$$\left(\frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr}\right)\Psi(r) + (\epsilon_n - V(r))\,\Psi(r) = 0. \quad (3.2)$$

To construct the shape invariant generalized Woods-Saxon potential we need to replace the Woods-saxon potential, $V(r)$ with the factorized potential, $V_1(r)$ in the schrödinger equation which define as follow

$$V_1(r) = \frac{-\lambda^2}{1+e^{(r-r_0)}} \quad (3.3)$$

the first member of superpotential family, $W_1(r)$ which is the lowest eigenfunction of $V_1(r)$, can be obtain using $W_1(r) = -\frac{d}{dr}log(\Psi_0^{(1)}(r))$, with $\Psi_n^{(1)}(r) = \Psi_n(r)$ and it also satisfies the following associated Riccati equation,

$$W_1^2(r) - W_1^{'}(r) = V_1(r) - \epsilon_0^{(1)} \quad (3.4)$$

where $\epsilon_n^{(1)} = \epsilon_n$. From condition (16) we make following Ansate for the superpotential

$$W_1(r) = \frac{a_1}{1+e^{(r-r_0)}} + b_1 \quad (3.5)$$

by substituting $W_1(r)$ defined in (17) and $V_1(r)$ in (15) in to equation (16) and equating equal parts in both sides we have,

$$\begin{aligned} a_1^2 + 2a_1b_1 + b_1^2 &= -\lambda^2 - \epsilon_0^{(1)} \\ 2a_1b_1 + 2b_1^2 + a_1 &= -\lambda^2 - 2\epsilon_0^{(1)} \\ b_1^2 &= -\epsilon_0^{(1)} \end{aligned} \quad (3.6)$$

Therefore by solving these simple equations we could get a_1, b_1 and $\epsilon_0^{(1)}$ as follow,

$$\begin{aligned} a_1 &= 1, \\ b_1 &= -\frac{1}{2}(\lambda^2+1), \\ \epsilon_0^{(1)} &= -\frac{1}{4}(\lambda^2+1)^2. \end{aligned} \quad (3.7)$$

finally by substituting for a_1 and b_1 in (17) the superpotential is given by

$$W_1(r) = \frac{1}{1+e^{(r-r_0)}} - \frac{1}{2}(1+\lambda^2). \quad (3.8)$$

Then the superpartner hamiltonian satisfies the equation

$$H_2 - \epsilon_0^{(1)} = A_1^- A_1^+, \quad (3.9)$$

where $\epsilon_0^{(1)}$ is the ground state eigenvalue of $V_1(r)$ which along with $V_2(r)$ satisfy following expression

$$V_2(r) = W_1^2(r) + W_1'(r) + \epsilon_0^{(1)} \quad (3.10)$$

with $V_2(r)$ that define as follow,

$$V_2(r) = -\frac{\lambda^2}{1+e^{(r-r_0)}} - \frac{2e^{(r-r_0)}}{(1+e^{(r-r_0)})^2} \quad (3.11)$$

To construct the next member of the superfamily, we need to factorize the schrödinger equation for $V_2(r)$, which depends on the superpotential $W_2(r)$ by following relation

$$H_2 - \epsilon_0^{(2)} = A_2^+ A_2^-, \quad (3.12)$$

where

$$A_2^{\pm} = \mp\frac{d}{dr} + W_2(r)$$

and $W_2(r)$ also satisfies the associated Riccati equation,

$$W_2^2(r) - W_2^{'}(r) = V_2(r) - \epsilon_0^{(2)} \quad (3.13)$$

$\epsilon_0^{(2)}$ is the ground state of $V_2(r)$ and it is such that $\epsilon_0^{(2)} = \epsilon_1^{(1)}$. In analogy with the $W_1(r)$,we may define $W_2(r)$ as follow

$$W_2(r) = \frac{a_2}{1+e^{(r-r_0)}} + b_2 \quad (3.14)$$

By using associated reccati equation (25) we obtain

$$\begin{aligned} a_2 &= 2, \\ b_2 &= -\frac{1}{4}(\lambda^2+4), \\ \epsilon_0^{(2)} &= -\frac{1}{16}(\lambda^2+4)^2. \end{aligned} \quad (3.15)$$

where $\epsilon_0^{(2)}$ is ground state eigenvalue of $V_2(r)$. Also We obtain the following expression for $W_2(r)$

$$W_2(r) = \frac{2}{1+e^{(r-r_0)}} - \frac{1}{4}(4+\lambda^2). \quad (3.16)$$

The next superpartner hamiltonian ,H_2 is defined by

$$W_2^2(r) + W_2^{'}(r) = V_3(r) - \epsilon_0^{(2)} \quad (3.17)$$

where

$$V_3(r) = -\frac{\lambda^2}{1+e^{(r-r_0)}} - \frac{6e^{(r-r_0)}}{(1+e^{(r-r_0)})^2} \qquad (3.18)$$

By repeating the procedure used for H_1, one can obtain $W_3(r)$ and $V_4(r)$

$$H_3 - \epsilon_0^{(3)} = A_3^+ A_3^-; \qquad A_3^\pm = \mp\frac{d}{dr} + W_3(r) \qquad (3.19)$$

where $V_3(r)$ satisfies the Riccati equation,

$$W_3^2(r) - W_3^{'}(r) = V_3(r) - \epsilon_0^{(3)} \qquad (3.20)$$

and

$$W_3(r) = \frac{a_3}{1+e^{(r-r_0)}} + b_3 \qquad (3.21)$$

where

$$a_3 = 3,$$
$$b_3 = -\frac{1}{6}(\lambda^2+9),$$
$$\epsilon_0^{(3)} = -\frac{1}{36}(\lambda^2+9)^2. \qquad (3.22)$$

then

$$W_3(r) = \frac{3}{1+e^{(r-r_0)}} - \frac{1}{6}(9+\lambda^2). \qquad (3.23)$$

and

$$V_4(r) = -\frac{\lambda^2}{1+e^{(r-r_0)}} - \frac{12e^{(r-r_0)}}{(1+e^{(r-r_0)})^2} \qquad (3.24)$$

Where $\epsilon_0^{(3)}$ is the ground state energy of the potential $V_3(r)$, with $\epsilon_0^{(3)} = \epsilon_2^{(1)}$.

In the $n+1$-th order $W_{n+1}(r)$ is related to associated potential $V_{n+1}(r)$ and ground state energy, $\epsilon_0^{(n+1)}$ by the following conditions,

$$W_{n+1}^2(r) - W_{n+1}^{'}(r) = V_{n+1}(r) - \epsilon_0^{(n+1)} \qquad (3.25)$$

and

$$W_n^2(r) + W_n^{'}(r) = V_{n+1}(r) - \epsilon_0^{(n)} \qquad (3.26)$$

By replacing for $V_{n+1}(r)$,one can obtain following relationship between $W_{n+1}(r)$ and $W_n(r)$,

$$W_{n+1}^2(r) - W_{n+1}^{'}(r) = W_n^2(r) + \epsilon_0^{(n)} - \epsilon_0^{(n+1)} \qquad (3.27)$$

where

$$\epsilon_0^{(n)} = -\frac{1}{4n^2}\left(n^2+\lambda^2\right)^2;$$
$$\epsilon_0^{(n+1)} = -\frac{1}{4(n+1)^2}\left((n+1)^2+\lambda^2\right)^2 \qquad (3.28)$$

where $\epsilon_0^{(n+1)}$ and $\epsilon_0^{(n)}$ are ground states eigenvalue of $V_{n+1}(r)$ and $V_n(r)$ respectively. We may characterize the $n+1$-th potential, $V_{n+1}(r)$ as

$$V_{n+1}(r) = -\frac{\lambda^2}{1+e^{(r-r_0)}} - \frac{C(n)e^{(r-r_0)}}{(1+e^{(r-r_0)})^2} \qquad (3.29)$$

where

$$C(n) = -\frac{\hbar^2}{MR^2}n(n+1) \qquad (3.30)$$

and also $n+1$-th super potential can be given as

$$W_{n+1} = \frac{n+1}{1+e^{(r-r_0)}} - \frac{(n+1)^2+\lambda^2}{2(n+1)} \qquad (3.31)$$

The mentioned above superpartner potential is the generalized form of original Woods-Saxon potential and must be shape invariant respect to parameter n. The radial part of the schrödinger equation for $V_{n+1}(r)$ in zero angular momentum case can be shown as

$$0 = \left(\frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr}\right)\Psi(r) + \frac{2M}{\hbar^2}\left(E - \frac{V_0}{1+e^{(r-r_0)}} - \frac{C(n)}{(1+e^{(r-r_0)})^2}\right)\Psi(r) \qquad (3.32)$$

This equation have been solved exactly[24] using associated Jacoobi polynomial. The solution have been written respect to variable x that relate to original variable,r by following relation

$$x = \tanh(r-r_0) \qquad (3.33)$$

and the final solution can be shown as

$$\Psi(r(x)) = U(x)P_{(n,m)}^{(\alpha,\beta)}(x) \qquad (3.34)$$

where $P_{n,m}^{(\alpha,\beta)}(x)$ is the Jacoobi polynomial and $U(x)$ may be defined as,

$$U(x) = \frac{(1-x^2)^{\frac{\alpha+\beta}{4}}}{\tanh^{-1}(x)}e^{\frac{1}{2}(\beta-\alpha)\coth^{-1}(x)} \qquad (3.35)$$

$$C(\alpha,\beta;n) = -\frac{\hbar^2}{4MR^2A}$$
$$A = [4n(\alpha+\beta+n+1) + (\alpha+\beta)(\alpha+\beta+2)] \qquad (3.36)$$

In the case when we calculate the $C(\alpha,-\alpha;n)$ we can get,

$$C(\alpha, -\alpha; n) = -\frac{\hbar^2}{MR^2}\left[n(n+1)\right] \qquad (3.37)$$

therefore by using of equations (46)to(49) we can obtain the $n-th$ ground state eigen-function,

$$\Psi_0^n(x) = \frac{e^{(-\alpha \coth^{-1}(x))}}{\tanh^{-1}(x)} P_{(n,m)}^{(\alpha,-\alpha)}(x). \qquad (3.38)$$

IV. CONCLUSION

In this paper we have solved the radial part of schrödinger equation in the case of zero angular momentum, and also obtained hierarchy of hamiltonian related to Woods-Saxon superpotential. The superfamily hamiltonian found here are related to original Woods-Saxon potential.The generalized wood-saxon potential maybe solved by Darboux method.The generalization parameter $C(\alpha,\beta)$ found in our solution (relation (42)) is consistent with result found in ref[24] for special case $\alpha = -\beta$(relation (49)) also when $\alpha = \beta$ the first term of potential is cancelled. It is also necessary to mention that in usual spherical Woods-Saxon potential, bound states are expressed in terms of the hypergeometric functions. One may investigate the description of the associated hypergeometric functions and associated jacobi functions in terms of each other.

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