African Journal Of Mathematical Physics Vol **1** *No 2 (2004)201-207*

Generalized KdV- and mKdV-type bilinear equations

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Abstract

Generalized KdV-type and mKdV-type bilinear equations are considered. Under certain conditions, we show that the equations have one- or two-soliton solutions. Furthermore, some special forms of mKdV-type equations are studied in some detail. The conditions under which 3-soliton-like solutions exist are given. Some examples are illustrated.

PACS: *02.30, 03.40K.*

I. INTRODUCTION

During the past twenty years or so, Hirota bilinear method has become a powerful tool to find exact solutions of nonlinear evolution equations. In literature, there are at least different five types of bilinear equations which arise [1-6]. For example, the so-called KdV-type bilinear equation is as follows

$$F(D_x, D_y, D_t) f \bullet f = 0 \tag{1.1}$$

where bilinear operator $D_x^l D_y^m D_t^n$ is defined by [7-9]

$$D_x^l D_y^m D_t^n \ a \bullet b \equiv \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'}\right)^l \left(\frac{\partial}{\partial y} - \frac{\partial}{\partial y'}\right)^m \times \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial t'}\right)^n a(x,y,t) b(x',y',t')|_{x'=x, y'=y, t'=t} \tag{1.2}$$

and F is an even polynomial of D_x, D_y and D_t. Hirota proved [7] that if $F(0,0,0) = 0$, then equation (1) always possesses the following 2-soliton solution

$$f = 1 + e^{\eta_1} + e^{\eta_2} + A_{12} e^{\eta_1 + \eta_2}$$

where $\eta_i = k_i x + p_i y + q_i t + \eta_i^0$, $F(k_i, p_i, q_i) = 0$ $(i = 1, 2)$ and phase shift A_{12} is given by

$$A_{12} = -\frac{F(k_1 - k_2, p_1 - p_2, q_1 - q_2)}{F(k_1 + k_2, p_1 + p_2, q_1 + q_2)}$$

In [10], one of authors generalized (1) to the following form

$$\sum_{k=1}^{l} H_k(D_x, D_y, D_t)[F_k(\partial_x, \partial_y, \partial_t) f] \bullet [G_k(\partial_x, \partial_y, \partial_t) f] = 0 \tag{1.3}$$

and showed that some special forms of this equation have two-soliton solutions. In [11], Liu and Fokas consider the following generalized form

$$D_x D_t f \bullet f = \sum_{i,j} \alpha_{ij} (\partial_x^i f)(\partial_x^j f) \tag{1.4}$$

or alternatively

$$D_xD_tf\bullet f=P(\partial_x,\partial'_x)f(x,t)f(x',t')|_{x'=x,t'=t},$$
$$P(x,y)=\sum_{i,j}\alpha_{ij}x^iy^j \quad (1.5)$$

In the following, we will consider the special form of (2) given by

$$F(D_x,D_t,\partial_x,\partial_t)f\bullet f=0 \quad (1.6)$$

where F is a polynomial of $D_x,D_t,\partial_x,\partial_t$ such that

$$F\ (0,0,\partial_x,\partial_t)=0,$$
$$F(-D_x,-D_t,\partial_x,\partial_t)=F(D_x,D_t,\partial_x,\partial_t) \quad (1.7)$$

and

$$D_x^mD_t^n\partial_x^k\partial_t^l\ a\bullet b\equiv(\frac{\partial}{\partial x}-\frac{\partial}{\partial x'})^m(\frac{\partial}{\partial t}-\frac{\partial}{\partial t'})^n$$
$$\times\ (\frac{\partial}{\partial x}+\frac{\partial}{\partial x'})^k(\frac{\partial}{\partial t}+\frac{\partial}{\partial t'})^la(x,t)b(x',t')|_{x'=x,t'=t} \quad (1.8)$$

This paper is organized as follows. In section 2, the generalized bilinear equation (4) is considered. The conditions under which 2-soliton solutions exist are given. Some examples are illustrated. In section 3, a generalized form of the mKdV-type bilinear equations is presented. Furthermore, some conditions are obtained which guarantee the equations under consideration pass 2-, or 3-soliton solution test.

II. GENERALIZED KDV-TYPE BILINEAR EQUATIONS

Before we discuss soliton solutions of (4), we give some examples of equations which can be written in form (4). For example, the Burgers' equation

$$w_t+ww_x+w_{xx}=0$$

can be written as $(D_xD_t+D_x^2\partial_x)f\bullet f=0$ through the dependent variable transformation $w=(\ln f)_x$. Also the Thomas equation $u_{xt}+u_xu_t+\alpha u_x+\beta u_t=0$ can be written as

$$[\frac{1}{2}D_x^2\partial_t+\frac{1}{2}D_xD_t\partial+\alpha D_x^2+\beta D_xD_t]f\bullet f=0$$

by making the transformation $u=\ln\ f$ and the The equation $u_t=u_x+2v,\quad v_t=-2uv$ can be written as

$$(D_t^2\partial_t-D_xD_t\partial_t)f\bullet f=0$$

by setting $u=(\ln\ f)_t$. It is well known that the Burgers' and Thomas equations can be transformed into linear equations [12-14].

Now we consider the conditions under which (4) has the 2-soliton solution

$$f=1+e^{\eta_1}+e^{\eta_2}+A_{12}e^{\eta_1+\eta_2} \quad (2.1)$$

A direct calculation shows that (4) has a 2-soliton solution (5) provided that

$$F(k_1,\Omega_1,k_1+2k_2,\Omega_1+2\Omega_2)=0 \quad (2.2)$$
$$F(k_2,\Omega_2,k_2+2k_1,\Omega_2+2\Omega_1)=0 \quad (2.3)$$

where $\eta_i=k_ix+\Omega_it+\eta_i^0, F(k_i,\Omega_i,k_i,\Omega_i)=0$ and

$$A_{12}=-\frac{F(k_1-k_2,\Omega_1-\Omega_2,k_1+k_2,\Omega_1+\Omega_2)}{F(k_1+k_2,\Omega_1+\Omega_2,k_1+k_2,\Omega_1+\Omega_2)} \quad (2.4)$$

To illustrate this, consider the following examples.

Example 1.

$$[D_t^2-D_x^4+A(3D_x^4+D_x^2\partial_x^2-2D_x^2\partial_t-2D_xD_t\partial_x)]f\bullet f=0 \quad (2.5)$$

It is easily verified that (9) has the following 2-soliton solution (5) with $\eta_i=k_ix+k_i^2t+\eta_i^0$ and

$$A_{12}=\frac{(1-A)(k_1-k_2)^2}{k_1^2+k_1k_2+k_2^2-2A(k_1+k_2)^2}$$

In particular, when $A\longrightarrow\infty$, (9) becomes

$$[3D_x^4+D_x^2\partial_x^2-2D_x^2\partial_t-2D_xD_t\partial_x]f\bullet f=0 \quad (2.6)$$

which has a 2-soliton solution with phase shift

$$A_{12}=\left(\frac{k_1-k_2}{k_1+k_2}\right)^2$$

which is the same as that of the KdV equation $D_x(D_t-D_x^3)f\bullet f=0$ with $\eta_i=k_ix+k_i^3t+\eta_i^0$. By the transformation $w=2(\ln f)_x$, (10) becomes

$$w_{xt}+\frac{1}{2}ww_t+\frac{1}{2}w_x\partial_x^{-1}w_t-w_{xxx}$$
$$-\frac{5}{2}w_x^2-\frac{1}{2}ww_{xx}-\frac{1}{4}w^2w_x=0 \quad (2.7)$$

A further calculation shows (10) does not exist 3-soliton solution. Thus (11) is thought not to be integrable in the sense of having N-soliton solution.

Example 2.

$$[D_t(D_t-D_x^3)+\alpha_1D_x^3(D_x^3-D_t)$$
$$+\alpha_2D_x(D_x^3-D_t)\partial_x^2]f\bullet f=0 \quad (2.8)$$

By the transformation $u=2\ln f$, (12) becomes

$$u_{tt}-(1+\alpha_1+\alpha_2)u_{xxxt}-(3+3\alpha_1+\alpha_2)u_{xx}u_{xt}$$
$$+(\alpha_1+\alpha_2)u_{6x}$$
$$+(15\alpha_1+7\alpha_2)u_{xx}u_{4x}+(15\alpha_1+3\alpha_2)u_{xx}^3$$
$$+\alpha_2(-2u_xu_{xxt}-u_x^2u_{xt}+6u_{xxx}^2+2u_xu_{5x}$$
$$+12u_xu_{xx}u_{xxx}+u_x^2u_{4x}+3u_x^2u_{xx}^2)=0 \quad (2.9)$$

It is easily verified that (12) has the following 2-soliton solution

$$f = 1 + e^{\eta_1} + e^{\eta_2} + A_{12}e^{\eta_1+\eta_2} \tag{2.10}$$

with $\eta_i = k_i x + k_i^3 t + \eta_i^0$ and

$$A_{12} = \left(\frac{k_1 - k_2}{k_1 + k_2}\right)^2 \frac{k_1^2 + \frac{1+2\alpha_1-2\alpha_2}{1-\alpha_1-\alpha_2}k_1k_2 + k_2^2}{k_1^2 + \frac{-1-2\alpha_1-2\alpha_2}{1-\alpha_1-\alpha_2}k_1k_2 + k_2^2} \tag{2.11}$$

In particular, we consider the following three special cases:

(1) $\alpha_1 = \frac{1}{4}, \alpha_2 = -\frac{9}{4}$. In this case,

$$A_{12} = \frac{(k_1 - k_2)^2}{k_1^2 + k_1k_2 + k_2^2} \equiv A_{12}^{Boussinesq}$$

Here by $A_{12}^{Boussinesq}$ we mean the phase shift of the 2-soliton solution

$$f = 1 + e^{\eta_1} + e^{\eta_2} + A_{12}^{Boussinesq}e^{\eta_1+\eta_2}$$

for the Boussinesq equation $(D_t^2 - D_x^4)f \bullet f = 0$, where $\eta_i = k_i x + k_i^2 t + \eta_i^0$.

(2) $\alpha_1 = -2, \alpha_2 = 0$. In this case,

$$A_{12} = \left(\frac{k_1 - k_2}{k_1 + k_2}\right)^2 \frac{k_1^2 - k_1k_2 + k_2^2}{k_1^2 + k_1k_2 + k_2^2} \equiv A_{12}^{CDGKS}$$

Here by A_{12}^{CDGKS} we mean the phase shift of the 2-soliton solution

$$f = 1 + e^{\eta_1} + e^{\eta_2} + A_{12}^{CDGKS}e^{\eta_1+\eta_2}$$

for the CDGKS equation $(D_xD_t - D_x^6)f \bullet f = 0$, where $\eta_i = k_i x + k_i^3 t + \eta_i^0$.

(3) $\alpha_1 = -\frac{1}{2}, \alpha_2 = -\frac{3}{2}$. In this case,

$$A_{12} = \left(\frac{k_1 - k_2}{k_1 + k_2}\right)^2 \equiv A_{12}^{KdV}$$

Similarly, by A_{12}^{KdV} we mean the phase shift of the 2-soliton solution

$$f = 1 + e^{\eta_1} + e^{\eta_2} + A_{12}^{KdV}e^{\eta_1+\eta_2}$$

for the KdV equation $(D_xD_t - D_x^4)f \bullet f = 0$, where $\eta_i = k_i x + k_i^3 t + \eta_i^0$. Furthermore, a direct calculation shows that the equation (12) in these three special cases does not pass 3-soliton test.

Example 3.

$$(D_x^2\partial_t - D_t^2\partial_t + \alpha D_x^2 + \beta D_xD_t + \gamma D_t^2)f \bullet f = 0 \tag{2.12}$$

It is easily verified that if $\alpha + \beta + \gamma = 0$, then (16) has the following 2-soliton solution (5) with $\eta_i = k_i x + k_i t + \eta_i^0$ and arbitrary A_{12}. By the transformation $w = 2\ln f$, (16) becomes

$$\begin{aligned}(w_{tt} - w_{xx})_t = &-(w_{tt} - w_{xx})w_t + \alpha w_{xx}\\ &+ \beta w_{xt} + \gamma w_{tt}\end{aligned} \tag{2.13}$$

A direct calculation shows the equation (17) fails to pass Painleve test if $|\alpha|^2+|\beta|^2+|\gamma|^2 \neq 0$. When $\alpha = \beta = \gamma = 0$, (17) can be transformed into

$$w_{tt} - w_{xx} = c(x)e^{-w} \tag{2.14}$$

where $c(x)$ is an arbitrary function of x. Furthermore (18) passes Painleve test iff $c(x)$ =constant.

To sum up, (17) is thought to be integrable iff $\alpha = \beta = \gamma = 0$ and $c(x)$ =constant. In this case, (18) is just the Liouville equation.

So far, we have only considered solutions of the form (5) for equation (4). Naturally, we can also consider other types of solutions.

For example consider the equation

$$\begin{aligned}&[D_t^2 + A_1D_x^2\partial_t + A_2D_xD_t\partial_x\\ &+ A_3D_x^4 + A_4D_x^2\partial_x^2]f \bullet f = 0\end{aligned} \tag{2.15}$$

If we seek a solution of this in the form

$$f = e^{\eta_1} + e^{\eta_2} + A_{12}e^{\eta_1+\eta_2}$$

with $\eta_i = k_i x + k_i^2 t + \eta_i^0$, A_{12} is arbitrary, then a direct calculation shows that $A_1 = A_2 = -1, A_3 = A_4 = \frac{1}{2}$ In this case, (19) becomes

$$[D_t^2 - D_x^2\partial_t - D_xD_t\partial_x + \frac{1}{2}D_x^4 + \frac{1}{2}D_x^2\partial_x^2]f \bullet f = 0 \tag{2.16}$$

Through the transformation $u = (\ln\ f)_x$, $v = -\frac{1}{2}(\partial_t - \partial_x^2)f/f$ (20) becomes

$$\begin{aligned}u_t &= u_{xx} + 2uu_x - 2v_x\\ v_t &= v_{xx} + 2u_xv\end{aligned}$$

which is just first equation of the so-called two-truncated KP hierarchy[15]. Now we consider solution of the form

$$f = e^{\eta_1} + e^{\eta_2} + A_{12}e^{\frac{1}{2}(\eta_1+\eta_2)}$$

with $\eta_i = k_i x + k_i^2 t + \eta_i^0$, A_{12} is arbitrary. A direct calculation shows that $A_1 = -1, A_2 = -2, A_3 = \frac{1}{2}, A_4 = \frac{3}{2}$. In this case, (19) becomes,

$$[D_t^2 - D_x^2\partial_t - 2D_xD_t\partial_x + \frac{1}{2}D_x^4 + \frac{3}{2}D_x^2\partial_x^2]f \bullet f = 0 \tag{2.17}$$

which can be transformed into

$$\begin{aligned}&u_{tt} - 3u_{xxt} - u_{xx}u_t - 2u_{xt}u_x\\ &+ 2u_{4x} + 3u_xu_{xxx} + 3u_{xx}^2 + \frac{3}{2}u_{xx}u_x^2 = 0\end{aligned} \tag{2.18}$$

by the transformation $u = 2\ln f$.

III. GENERALIZED MKDV-TYPE BILINEAR EQUATIONS

In this section, we will consider the following generalized mKdV-type bilinear equations

$$P_1(D_x, D_t, \partial_x, \partial_t)F \bullet G = 0 \quad (3.1)$$
$$P_2(D_x, D_t, \partial_x, \partial_t)F \bullet G = 0 \quad (3.2)$$

where P_1 and P_2 are polynomials of $D_x, D_t, \partial_x, \partial_t$, and

$$P_i(-D_x, -D_t, \partial_x, \partial_t) = (-1)^i P_i(D_x, D_t, \partial_x, \partial_t) \quad (3.3)$$
$$P_i(0, 0, \partial_x, \partial_t) = 0, \ i = 1, 2 \quad (3.4)$$

Equations (23,24) with (25,26) have one-soliton solution $F = 1 + ae^\eta$, $G = 1 - ae^\eta$ with $\eta = kx + \Omega t + \eta^0$, $P_1(k, \Omega, k, \Omega) = 0$. Futhermore, a direct calculation shows that (23,24) has 2-soliton solution

$$\begin{aligned} F &= 1 + e^{\eta_1} + e^{\eta_2} + Ae^{\eta_1+\eta_2} \\ G &= 1 - e^{\eta_1} - e^{\eta_2} + Be^{\eta_1+\eta_2} \end{aligned} \quad (3.5)$$

with $\eta_i = k_i x + \Omega_i t + \eta_i^0$, $P_1(k_i, \Omega_i, k_i, \Omega_i) = 0$ and,

$$A = B = \frac{P_2(k_1 - k_2, \Omega_1 - \Omega_2, k_1 + k_2, \Omega_1 + \Omega_2)}{P_2(k_1 + k_2, \Omega_1 + \Omega_2, k_1 + k_2, \Omega_1 + \Omega_2)}$$

provided that

$$P_1(k_1, \Omega_1, k_1 + 2k_2, \Omega_1 + 2\Omega_2) = 0$$

$$P_1(k_2, \Omega_2, k_2 + 2k_1, \Omega_2 + 2\Omega_1) = 0$$

In the following, we give some examples.

Example 4.

$$\begin{cases} (D_t\partial_x - \frac{3}{2}D_x^3 - \frac{1}{2}D_x\partial_x^2 + D_x\partial_t)F \bullet G = 0 \\ D_x^2 F \bullet G = 0 \end{cases} \quad (3.6)$$

has 2-soliton solution (27) with $\eta_i = k_i x + k_i^2 t + \eta_i^0$ and

$$A = B = \left(\frac{k_1 - k_2}{k_1 + k_2}\right)^2$$

By the transformation $u = \ln\,(f/g)$, (28) becomes

$$u_{xt} - \frac{1}{2}u_t\partial_x^{-1}(u_x^2) - u_x\partial_x^{-2}(u_x u_{xt}) - u_{xxx} + \frac{1}{2}u_{xx}\partial_x^{-1}(u_x^2) + \frac{7}{4}u_x^3 - \frac{1}{4}u_x(\partial_x^{-1}u_x^2)^2 = 0 \quad (3.7)$$

Example 5.

$$\begin{cases} (D_t\partial_x - \frac{3}{2}D_x^3 - \frac{1}{2}D_x\partial_x^2 + D_x\partial_t)F \bullet G = 0 \\ (D_x D_t - D_x^2\partial_x)F \bullet G = 0 \end{cases} \quad (3.8)$$

has 2-soliton solution (27) with $\eta_i = k_i x + k_i^2 t + \eta_i^0$ and $A = B = 0$. By the transformation $u = \ln(F/G)$, $v = \ln(FG)$, (30) becomes

$$u_{xt} + \frac{1}{2}u_t v_x + \frac{1}{2}u_x v_t - u_{xxx} - \frac{5}{2}u_x v_{xx} - \frac{3}{4}u_x^3 - \frac{1}{4}u_x v_x^2 - \frac{1}{2}u_{xx}v_x = 0 \quad (3.9)$$

$$v_{xt} - v_{xxx} + u_x u_t - v_{xx}v_x - u_x^2 v_x - 2u_x u_{xx} = 0 \quad (3.10)$$

In the following, we consider $P_1(D_x, D_t, \partial_x, \partial_t) = P_1(D_x, D_t)$. In this case, equation (23,24) becomes

$$\begin{cases} P_1(D_x, D_t)F \bullet G = 0 \\ P_2(D_x, D_t, \partial_x, \partial_t)F \bullet G = 0 \end{cases}$$

which have 2-soliton solution

$$\begin{aligned} F &= 1 + e^{\eta_1} + e^{\eta_2} + Ae^{\eta_1+\eta_2} \\ G &= 1 - e^{\eta_1} - e^{\eta_2} + Be^{\eta_1+\eta_2} \end{aligned}$$

with

$$\eta_i = k_i x + \Omega_i t + \eta_i^0, \ P_1(k_i, \Omega_i) = 0, \ (i = 1, 2)$$

$$A = B = \frac{P_2(k_1 - k_2, \Omega_1 - \Omega_2, k_1 + k_2, \Omega_1 + \Omega_2)}{P_2(k_1 + k_2, \Omega_1 + \Omega_2, k_1 + k_2, \Omega_1 + \Omega_2)}$$

Furthermore, for special P_1 and P_2, we have the following results.

Proposition 1: For equations

$$\begin{aligned} &(D_t - D_x^3)F \bullet G = 0 \\ &P_2(D_x, D_t, D_y)F \bullet G = 0, \ P_2 \ \textit{is even}, \end{aligned} \quad (3.11)$$

if there exists a positive integer $n \in Z_+$ such that

$$\frac{P_2(k_i - k_j, k_i^3 - k_j^3, k_i^2 - k_j^2)}{P_2(k_i + k_j, k_i^3 + k_j^3, k_i^2 + k_j^2)} = \pm\frac{(k_i - k_j)(k_i^n - k_j^n)}{(k_i + k_j)(k_i^n + k_j^n)}, \ i \leq i < j \leq 3 \quad (3.12)$$

then (33) has the following so-called 3-soliton-like solutions

$$F = 1 + e^{\eta_1} + e^{\eta_2} + e^{\eta_3} + A_{12}e^{\eta_1+\eta_2} + A_{13}e^{\eta_1+\eta_3} + A_{23}e^{\eta_2+\eta_3} + A_{123}e^{\eta_1+\eta_2+\eta_3} \quad (3.13)$$

$$G = 1 - e^{\eta_1} - e^{\eta_2} - e^{\eta_3} + A_{12}e^{\eta_1+\eta_2} + A_{13}e^{\eta_1+\eta_3} + A_{23}e^{\eta_2+\eta_3} - A_{123}e^{\eta_1+\eta_2+\eta_3} \quad (3.14)$$

with $\eta_i = k_i x + k_i^3 t + k_i^2 y + \eta_i^0$ and

$$A_{ij} = \frac{P_2(k_i - k_j, k_i^3 - k_j^3, k_i^2 - k_j^2)}{P_2(k_i + k_j, k_i^3 + k_j^3, k_i^2 + k_j^2)}, \quad A_{123} = A_{12}A_{13}A_{23} \quad (3.15)$$

Proof: direct calculation.

Example 6.

$$\begin{cases} (D_t - D_x^3)F \bullet G = 0 \\ (D_x D_t - D_x^4)F \bullet G = 0 \end{cases} \tag{3.16}$$

In this case,

$$A_{ij} = -\left(\frac{k_i - k_j}{k_i + k_j}\right)^2$$

therefore (35) has 3-soliton solution (34) with $\eta_i = k_i x + k_i^3 t + \eta_i^0$ By the transformation $u = \ln(F/G)$, $v = \ln(FG)$, (35) becomes

$$u_t = u_{xxx} + 3u_x v_{xx} + u_x^3 \tag{3.17}$$

$$v_{xt} - v_{xxxx} - 3v_{xx}^2 - 3u_x u_{xxx} - 3u_x^2 v_{xx} = 0 \tag{3.18}$$

Example 7.

$$\begin{cases} (D_t - D_x^3)F \bullet G = 0 \\ (D_t^2 - D_t D_x^3)F \bullet G = 0 \end{cases} \tag{3.19}$$

In this case,

$$A_{ij} = -\frac{(k_i - k_j)(k_i^3 - k_j^3)}{(k_i + k_j)(k_i^3 + k_j^3)}$$

therefore (38) has 3-soliton solution (34) with $\eta_i = k_i x + k_i^3 t + \eta_i^0$ By the transformation $u = \ln(F/G)$, $v = \ln(FG)$, (38) becomes

$$u_t = u_{xxx} + 3u_x v_{xx} + u_x^3 \tag{3.20}$$

$$v_{tt} - v_{xxxt} - 3v_{xx} v_{xt} - 3u_x u_{xxt} - 3u_x^2 v_{xt} = 0 \tag{3.21}$$

Remark: Equations (35) and (38) are included in the list of equations passing 3-soliton solutions[5].

Example 8.

$$\begin{cases} (D_t - D_x^3)F \bullet G = 0 \\ (D_x D_t + \frac{1}{2}D_x^4)F \bullet G = 0 \end{cases} \tag{3.22}$$

In this case,

$$A_{ij} = \left(\frac{k_i - k_j}{k_i + k_j}\right)^2$$

therefore (41) has 3-soliton-like solution (34) with $\eta_i = k_i x + k_i^3 t + \eta_i^0$. By the transformation $u = \ln(F/G)$, $v = \ln(FG)$, (41) becomes

$$u_t = u_{xxx} + 3u_x v_{xx} + u_x^3 \tag{3.23}$$

$$v_{xt} + \frac{1}{2}v_{xxxx} + \frac{3}{2}v_{xx}^2 + 3u_x u_{xxx} + 6u_x^2 v_{xx} + \frac{3}{2}u_x^4 = 0 \tag{3.24}$$

Equation (41) was first considered in [5]. Here we obtain the following result,

Proposition 2: If F is a solution of the KdV equation

$$(D_x D_t - D_x^4)F \bullet F = 0 \tag{3.25}$$

and G is another solution of (44) which is related to F by the following Backlund transformation

$$(D_t - D_x^3)F \bullet G = 0 \tag{3.26}$$

$$D_x^2 F \bullet G = 0 \tag{3.27}$$

Then (F, G) is a solution of (41).

Proof: Using bilinear operator identities

$$(D_x D_t a \bullet a)b^2 + a^2 D_x D_t b \bullet b = 2ab D_x D_t a \bullet b - 2(D_x a \bullet b)(D_t a \bullet b)$$

$$b^2 D_x^4 a \bullet a + a^2 D_x^4 b \bullet b = 3D_x^2(D_x^2 a \bullet b) \bullet ab - ab D_x^4 a \bullet b + 3(D_x^2 a \bullet b)^2 - 2(D_x^3 a \bullet b)(D_x a \bullet b) \tag{3.28}$$

We can easily obtain from $G^2(D_x D_t - D_x^4)F \bullet F + F^2(D_x D_t - D_x^4)G \bullet G = 0$ and by use of (45,46), that

$$(D_x D_t + \frac{1}{2}D_x^4)F \bullet G = 0$$

Therefore we have completed the proof of Proposition 2.

Using this result, we can constitute some other solutions of (41) just from solutions of the KdV equation (44). For example, we know that the following solution pairs of the KdV equation (44)

$$(x, x^3 - 12t),\ (x^3 - 12t, x^6 - 60x^3 t - 720t^2),$$
$$(x^6 - 60x^3 t - 720t^2, x^{10} - 180x^7 t + 302400xt^3) \tag{3.29}$$

and

$$(e^\eta + e^{-\eta}, (1 - px)e^\eta + (1 + px)e^{-\eta}),$$

where $\eta = px + 4p^3 t + \eta^0$ satisfy (45,46). Therefore they are all the solutions of (41).

Example 9.

$$\begin{cases} (D_t - D_x^3)F \bullet G = 0 \\ (D_t^2 + \frac{1}{2}D_t D_x^3)F \bullet G = 0 \end{cases} \tag{3.30}$$

In this case,

$$A_{ij} = \frac{(k_i - k_j)(k_i^3 - k_j^3)}{(k_i + k_j)(k_i^3 + k_j^3)}$$

therefore (47) has 3-soliton-like solution (34) with $\eta_i = k_i x + k_i^3 t + \eta_i^0$. By the transformation $u = \ln(F/G)$, $v = \ln(FG)$, (47) becomes

$$u_t = u_{xxx} + 3u_x v_{xx} + u_x^3 \tag{3.31}$$

$$v_{tt} + \frac{1}{2}v_{xxxt} + \frac{3}{2}v_{xx}v_{xt} + \frac{3}{2}u_x u_{xxt} + \frac{3}{2}u_t u_{xxx} + \frac{3}{2}u_x^2 v_{xt} + \frac{9}{2}u_x u_t v_{xx} + \frac{3}{2}u_x^3 u_t = 0 \tag{3.32}$$

It is noted that (47) was first given by Hietarinta [5]. Concerning (47), we obtain the following result,

Proposition 3: If F is a solution of the Ito equation

$$(D_t^2 - D_x^3 D_t) F \bullet F = 0 \tag{3.33}$$

and G is another solution of (50) which is related to F by the following Backlund transformation

$$(D_t - D_x^3) F \bullet G = 0 \tag{3.34}$$
$$D_x D_t F \bullet G = 0 \tag{3.35}$$

Then (F, G) is a solution of (47).

Proof: Using the following Hirota bilinear operator identities

$$\begin{aligned} &(D_t^2 a \bullet a) b^2 + a^2 D_t^2 b \bullet b = 2ab D_t^2 a \bullet b - 2(D_t a \bullet b)^2 \\ &b^2 D_x^3 D_t a \bullet a + a^2 D_x^3 D_t b \bullet b = 3 D_x^2 (D_x D_t a \bullet b) \bullet ab \\ &-ab D_x^3 D_t a \bullet b + 3(D_x^2 a \bullet b)(D_x D_t a \bullet b) \\ &- 2(D_x^3 a \bullet b)(D_t a \bullet b) \end{aligned} \tag{3.36}$$

We can easily obtain from $G^2(D_t^2 - D_x^3 D_t) F \bullet F + F^2 (D_t^2 - D_x^3 D_t) G \bullet G = 0$ and by use of (51,52), that

$$(D_t^2 + \frac{1}{2} D_x^3 D_t) F \bullet G = 0$$

Thus we have completed the proof of Proposition3.

Using this result, we can also obtain some other solutions from the Ito equation (50).

Example 10.

$$\begin{cases} (D_t - D_x^3) F \bullet G = 0 \\ (D_t^2 + D_x^6 - 2 D_x^3 D_t) F \bullet G = 0 \end{cases} \tag{3.37}$$

In this case,

$$A_{ij} = \left(\frac{k_i - k_j}{k_i + k_j} \right)^2$$

therefore (53) has 3-soliton solution (34) with $\eta_i = k_i x + k_i^3 t + \eta_i^0$

Next, we consider the following type of equations

$$\begin{cases} (D_t - D_x^3) \tilde{P}_1 (D_x, D_t) F \bullet G = 0, \ \tilde{P}_1 \ is \ odd \\ P_2 (D_x, D_t) F \bullet G = 0, \ P_2 \ is \ even \end{cases} \tag{3.38}$$

Concerning (54), we obtain,

Proposition 4: If $\tilde{P}_1$ and P_2 satisfy the following conditions

$$\begin{aligned} &\tilde{P}_1(k_1 + k_2 + k_3, k_1^3 + k_2^3 + k_3^3)(k_1 - k_2)(k_2 - k_3)(k_3 - k_1) \\ &+ \tilde{P}_1(k_1 + k_2 - k_3, k_1^3 + k_2^3 - k_3^3) \bullet \\ &(k_1 - k_2)(k_2 + k_3)(k_3 + k_1) \\ &+ \tilde{P}_1(k_1 - k_2 + k_3, k_1^3 - k_2^3 + k_3^3)(k_1 + k_2)(k_2 + k_3)(k_3 - k_1) \\ &+ \tilde{P}_1(-k_1 + k_2 + k_3, -k_1^3 + k_2^3 + k_3^3) \\ &\times (k_1 + k_2)(k_2 - k_3)(k_3 + k_1) = 0 \end{aligned} \tag{3.39}$$

and

$$\frac{P_2(k_i - k_j, k_i^3 - k_j^3, k_i^2 - k_j^2)}{P_2(k_i + k_j, k_i^3 + k_j^3, k_i^2 + k_j^2)} = \pm \left(\frac{k_i - k_j}{k_i + k_j} \right)^2, \ 1 \le i < j \le 3 \tag{3.40}$$

then (54) has the following 3-soliton-like solutions

$$\begin{aligned} F &= 1 + e^{\eta_1} + e^{\eta_2} + e^{\eta_3} + A_{12} e^{\eta_1 + \eta_2} \\ &+ A_{13} e^{\eta_1 + \eta_3} + A_{23} e^{\eta_2 + \eta_3} + A_{123} e^{\eta_1 + \eta_2 + \eta_3} \end{aligned} \tag{3.41}$$
$$\begin{aligned} G &= 1 - e^{\eta_1} - e^{\eta_2} - e^{\eta_3} + A_{12} e^{\eta_1 + \eta_2} \\ &+ A_{13} e^{\eta_1 + \eta_3} + A_{23} e^{\eta_2 + \eta_3} - A_{123} e^{\eta_1 + \eta_2 + \eta_3} \end{aligned} \tag{3.42}$$

with $\eta_i = k_i x + k_i^3 t + \eta_i^0$ and

$$\begin{aligned} A_{ij} &= \frac{P_2(k_i - k_j, k_i^3 - k_j^3, k_i^2 - k_j^2)}{P_2(k_i + k_j, k_i^3 + k_j^3, k_i^2 + k_j^2)}, \\ A_{123} &= A_{12} A_{13} A_{23} \end{aligned} \tag{3.43}$$

Proof:direct calculation.

It is easily verified that $\tilde{P}_1(D_x, D_t) = D_x^2, D_x^4, D_x D_t$ satisfy (55) and $P_2(D_x, D_t) = D_x^2, D_x D_t - D_x^4, D_x D_t + \frac{1}{2} D_x^4$ satisfy (56).Thus, using Proposition 4, we can constitute some equations which have 3-soliton-like solution (57).

Finally we consider following-type equation

$$\begin{cases} (D_t - D_x^5) F \bullet G = 0 \\ P_2 (D_x, D_t, D_{t_3}) F \bullet G = 0, P_2 \ is \ even \end{cases} \tag{3.44}$$

We have the following result,

Proposition 5: For equation (58), if P_2 satisfies

$$\begin{aligned} &\frac{P_2(k_i - k_j, k_i^5 - k_j^5, k_i^3 - k_j^3)}{P_2(k_i + k_j, k_i^5 + k_j^5, k_i^3 + k_j^3)} \\ &= \pm \left(\frac{k_i - k_j}{k_i + k_j} \right)^2, \ 1 \le i < j \le 3 \end{aligned} \tag{3.45}$$

then (58) has the following so-called 3-soliton-like solutions

$$\begin{aligned} F &= 1 + e^{\eta_1} + e^{\eta_2} + e^{\eta_3} + A_{12} e^{\eta_1 + \eta_2} + A_{13} e^{\eta_1 + \eta_3} \\ &+ A_{23} e^{\eta_2 + \eta_3} + A_{123} e^{\eta_1 + \eta_2 + \eta_3} \\ G &= 1 - e^{\eta_1} - e^{\eta_2} - e^{\eta_3} + A_{12} e^{\eta_1 + \eta_2} + A_{13} e^{\eta_1 + \eta_3} \\ &+ A_{23} e^{\eta_2 + \eta_3} - A_{123} e^{\eta_1 + \eta_2 + \eta_3} \end{aligned} \tag{3.46}$$

with $\eta_i = k_i x + k_i^5 t + k_i^3 t_3 + \eta_i^0$ and

$$A_{ij} = \frac{P_2(k_i - k_j, k_i^5 - k_j^5, k_i^3 - k_j^3)}{P_2(k_i + k_j, k_i^5 + k_j^5, k_i^3 + k_j^3)},$$
$$A_{123} = A_{12}A_{13}A_{23} \qquad (3.47)$$

Proof: direct calculation.

Example 11.

$$\begin{cases} (D_t - D_x^5)F \bullet G = 0 \\ (D_x D_t + \frac{1}{4}D_x^6)F \bullet G = 0 \end{cases} \qquad (3.48)$$

In this case,

$$A_{ij} = \left(\frac{k_i - k_j}{k_i + k_j}\right)^2$$

therefore (60) has 3-soliton-like solution (59) with $\eta_i = k_i x + k_i^5 t + \eta_i^0$

Example 12.

$$\begin{cases} (D_t - D_x^5)F \bullet G = 0 \\ (D_t^2 + \frac{1}{2}D_t D_x^5 + \frac{1}{16}D_x^{10})F \bullet G = 0 \end{cases} \qquad (3.49)$$

In this case,

$$A_{ij} = \left(\frac{k_i - k_j}{k_i + k_j}\right)^2$$

therefore (61) has 3-soliton-like solution (59) with $\eta_i = k_i x + k_i^5 t + \eta_i^0$

IV. CONCLUSION

In this paper, generalized KdV-type and mKdV-type bilinear equations are considered. We show that the equations have one-soliton solution. The conditions under which 2-soliton solutions exist are given. Furthermore, some special forms of the mKdV-type bilinear equations are considered. Under certain conditions, we show that the equations have 3-soliton-like solutions. Some examples are illustrated. It is known there are some other types of bilinear equations to be available. It would be interesting to consider generalized forms of these types of bilinear equations.

ACKNOWLEDGEMENT

This work of XBH was supported by Royal Society, Chinese Academy of Sciences and National Natural Science Foundation of China.

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