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# Characterization of the Poisson transform of an $L^2$-function on the Shilov boundary $S$ of the Lie ball in $\mathbf{C}^2$.

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Abstract

Let $\mathcal{D}_2$ be the Lie ball in $\mathbf{C}^2$ and let $\mathcal{H}$ be the associated Hua operator.
The purpose of the present work is to establish for every number $\lambda$ such that $\mathcal{R}](\rangle\lambda) > \prime$ that every $\mathbf{C}$-valued function $F$ on $\mathcal{D}_2$ satisfying the following $\mathcal{H}F(z) = -(\lambda^2+1)F(z).e$, is the Poisson transform of an $L^2$-function on the Shilov boundary $S$ of $\mathcal{D}_2$ if and only if it satisfies a growth condition of Hardy type.

## I. INTRODUCTION

Let $\mathcal{D}_2 = \{z = (z_1, z_2) \in \mathbf{C}^2; \quad \mathbf{1} - \mathbf{2}\overline{\mathbf{z}}\mathbf{z}^\mathbf{t} + |\mathbf{z}\mathbf{z}^\mathbf{t}|^\mathbf{2} > \mathbf{0}$ **and** $|\mathbf{z}\mathbf{z}^\mathbf{t}| < \mathbf{1}\}$ be the Lie ball in $\mathbf{C}^2$ and let $S$ be its Shilov boundary with $du$ its normalized invariant measure. . Then, for fixed $\lambda \in \mathbf{C}$ the Poisson transform $P_\lambda$ is defined by

$$[P_\lambda f](z) = \int_S (P_\lambda(z,u))f(u)du, \qquad f \in \mathcal{A}'(S),$$

where $A'(S)$ is the space of all hyperfunctions over the Shilov boundary $S$ of $\mathcal{D}_2$ and $P_\lambda(z,u) = \Big[P(z,u)\Big]^{\frac{i\lambda+1}{2}}$. Here $P(z,u)$ is the Poisson kernel of the Lie ball $\mathcal{D}_2$ with respect to its Shilov boundary $S$, given by (see [4])

$$P(z,u) = \frac{1 - 2\overline{z}z^t + |zz^t|^2}{|(z-u)(z-u)^t|^2}.$$

Let $E_\lambda(\mathcal{D}_2, \mathcal{H})$ we denote the space of all real analytic functions $F$ on $\mathcal{D}_2$ that satisfy the following Hua system $\mathcal{H}F(z) = -(\lambda^2+1)F(z).e$.
Then, Shimeno result reads in the case of the Lie ball $\mathcal{D}_2$ as follows

**Theorem I.1** *Let $\lambda \in \mathbf{C}$ such that $-i\lambda \notin \{1, 2, ...\}$. Then the Poisson transform $P_\lambda$ is a $G$-isomorphism from $\mathcal{A}'(S)$ onto $E_\lambda(\mathcal{D}_2, \mathcal{H})$.*

Now, since $P_\lambda(L^2(S))$ is a proper closed subspace of $E_\lambda(\mathcal{D}_2, \mathcal{H})$, it's natural to look for those $F$ in $E_\lambda(\mathcal{D}_2, \mathcal{H})$ such that $F = P_\lambda f$ for some $f \in L^2(S)$. To do this, we introduce the subspace $E^*_\lambda(\mathcal{D}_2, \mathcal{H})$ of $E_\lambda(\mathcal{D}_2, \mathcal{H})$ whose elements are the functions $F$ in $E_\lambda(\mathcal{D}_2, \mathcal{H})$ satisfying

$$||F||_{*,\lambda} = \sup_{0 \le r < 1} (1-r^2)^{(Rei\lambda - 1)} \times [\int_S |F(ru)|^2 du]^{\frac{1}{2}} < \infty$$

Now, with the help of the above notations the main result of this work can be state as follows.

**Theorem I.2** *Let $\lambda \in \mathbf{C}$ such that $Rei\lambda > 0$ and let $F$ be a $\mathbf{C}$-valued function satisfying the Hua system $\mathcal{H}F(z) = -(\lambda^2+1)F(z).e$. Then, we have $F$ is the Poisson transform by $P_\lambda$ of some $f \in L^2(S)$ (ie $F = P_\lambda f$) if and only if $||F||_{*,\lambda} < \infty$. Moreover, there exists a positive constant $\gamma(\lambda)$ such that for every function $f \in L^2(S)$ we have*

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$$|c(\lambda)| \|f\|_{L^2(S)} \leq \|P_\lambda f\|_{*,\lambda} \leq \gamma(\lambda)\|f\|_{L^2(S)}, \quad (1.1)$$

*where* $c(\lambda) = 4\pi^2 \frac{\Gamma^2(i\lambda)}{\Gamma^4(\frac{i\lambda+1}{2})}$. *Here* $\Gamma$ *is the usual Euler Gamma function on* $\mathbf{C}$.

Let note here that the hard part in the proof of Theorem 1.2 is to establish that every function $F \in E^*_\lambda(\mathcal{D}_2, \mathcal{H})$ is the Poisson transform by $P_\lambda$ of some function $f \in L^2(S)$. To do this, we use Shimeno's result [7] to write $F = P_\lambda f$, where $f$ is in the space $\mathcal{A}'(S)$ of all hyperfunctions over the Shilov boundary $S$. Then expanding $f$ with respect to an orthonormal basis of spherical harmonics of $L^2(S)$; we are led:
First to evaluate the following Hua type integral

$$\phi_{\lambda,m}(z) = \int_S P_\lambda(z,u)\phi_m(u)du,$$

where $\phi_m$ is the zonal spherical function and $m \in \wedge = \{m = (m_1, m_2) \in \mathbf{Z^2} \ ; \mathbf{m_1} \geq \mathbf{m_2}\}$, see Lemma 3.1 in section 3.
Second to establish the asymptotic behaviour of the generalized spherical function $\phi_{\lambda,m}(z)$ as $z = re \longrightarrow 1^-$, uniformly in $m \in \wedge$.

## II. PRELIMINARY RESULTS.

In this section, we review some well known results of harmonic analysis on the Lie ball $\mathcal{D}_2$ in $\mathbf{C^2}$ (referring to [4] for more details on this subject).
For any matrix $a$ we denote respectively by ${}^t a$ and $\bar{a}$ the transpose and conjugate of $a$.
The Lie ball in $\mathbf{C^2}$ is defined by

$$\mathcal{D}_2 = \{z = (z_1, z_2) \in \mathbf{C^2} \ ;$$

$$1 - 2\bar{z}z^t + |zz^t|^2 > 0 \ \text{ and } \ |zz^t| < 1\},$$

here $z$ is viewing as $1 \times 2$ matrix and $|w|^2 = w\bar{w}$ for any $w \in \mathbf{C}$.
The Shilov boundary of $\mathcal{D}_2$ is given by

$$S = \{u = e^{i\theta}x \in \mathbf{C^2}; \quad \mathbf{0} \leq \theta < \mathbf{2}\pi, \quad \mathbf{x} \in \mathbf{S^1}\},$$

with

$$S^1 = \{(x_1, x_2) \in \mathbf{R^2}; \ \ \mathbf{x_1^2} + \mathbf{x_2^2} = \mathbf{1}\}.$$

Let $G = SO(2,2)$ be the group of all matrices $g$ in $SL(4, \mathbf{R})$ such that ${}^t gJg = J$, where $J = \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix}$. then The group $G$ acts transitively on $\mathcal{D}_2$ by

$$g : z \mapsto g.z$$
$$= \left\{ \left[ \left( \frac{zz^t + 1}{2}, i\frac{zz^t - 1}{2} \right) A^t + zB^t \right] \begin{pmatrix} 1 \\ i \end{pmatrix} \right\}^{-1}$$
$$\left\{ \left( \left( \frac{zz^t + 1}{2} \right), i\left( \frac{zz^t - 1}{2} \right) \right) C^t + zD^t \right\}$$

for $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in SO(2,2)$.

Thus as homogeneous space, we have the identification $\mathcal{D}_2 = G/K$, where $K$ is the stabilizer in $G$ of 0 given by $K = S(O(2) \times O(2))$.
The action (2,1) of $G$ extends naturally to $\overline{\mathcal{D}_2}$ and under this action the group $K$ acts transitively on the Shilov boundary $S$ and we have $S = K/\{\pm I_2\}$.
Let $L^2(S)$ be the space of all square integrable $\mathbf{C}$-valued functions on $S$ with respect to the measure $du$. Then the group $K$ acts on $L^2(S)$ by

$$f \longmapsto \pi(k)f = f \circ k^{-1}, \quad k \in K,$$

and under this action the space $L^2(S)$ has the following Peter-Weyl decomposition (see [2,3])

$$L^2(S) = \bigoplus_{m \in \wedge} V_m,$$

where $\wedge$ is the set of all two-tuple, $m = (m_1, m_2) \in \mathbf{Z^2}$ with $m_1 \geq m_2$. The K-irreducible component $V_m$ is the finite linear span $\{\phi_m \circ k, k \in K\}$. Here the function $\phi_m \in V_m$ is the zonal spherical function given by

$$\phi_m(u_1, u_2) = \Big(u_1 - iu_2\Big)^{m_1 - m_2} \Big(u_1^2 + u_2^2\Big)^{m_2},$$

$$m = (m_1, m_2).$$

## III. THE PRECISE ACTION OF THE POISSON TRANSFORM $P_\lambda$ ON $L^2(S)$.

In this section, we give the explicit form of the Poisson transform $P_\lambda$ on each K-type of $L^2(S)$. For $\lambda \in \mathbf{C}$ and $k \in \mathbf{Z}^+$, let $\varphi_{\lambda,k}$ denote the following $\mathbf{C}$-valued function on $[0, 1[$

$$\varphi_{\lambda,k}(r) = (1 - r^2)^{(\frac{i\lambda+1}{2})} r^k \frac{(\frac{i\lambda+1}{2})_k}{(1)_k} \times$$
$$F\Big(\frac{i\lambda + 1}{2}, \frac{i\lambda + 1}{2} + k, 1 + k; r^2\Big),$$

where $(a)_k = a(a+1)(a+2)....(a+k-1)$ is the Pochammer symbol and $F(a,b,c;x)$ is the classical Gauss hypergeometric function.
Notice that $\varphi_{\lambda,k}$ is nothing but the generalized spherical function associated to the hyperbolic disk.

**Proposition III.1** *Let $m = (m_1, m_2) \in \wedge$ and let $f \in V_m$. Then we have*

$$(P_\lambda f)(ru) = \phi_{\lambda,m}(r) f(u),$$

*where the generalized spherical function $\phi_{\lambda,m}(r)$ is given by*

$$\phi_{\lambda,m}(r) = 4\pi^2 \Big[\varphi_{\lambda,|m_1|}(r)\varphi_{\lambda,|m_2|}(r)\Big].$$

*Furthermore, for $f(u) = \sum_{m\in\wedge} f_m(u)$ in $L^2(S)$ we have*

$$(P_\lambda f)(ru) = \sum_{m\in\wedge} \phi_{\lambda,m}(r) f_m(u) \quad \text{in } C^\infty([0,1[\times S).$$

We will now evaluate the generalized spherical function $\phi_{\lambda,m}(z)$ and establish it's asymptotic behaviour.

**Lemma III.1** *Let $m = (m_1, m_2) \in \wedge$. Then, the explicit expression of the generalized spherical function $\phi_{\lambda,m}(z)$ is given by*

$$\phi_{\lambda,m}(z) = 4\pi^2\Big(1 - 2\overline{z}z^t + |zz^t)\Big)^{\frac{i\lambda+1}{2}} (z_1 - iz_2)^{|m_1|}$$

$$\times (z_1 + iz_2)^{|m_2|} \frac{(\frac{i\lambda+1}{2})_{|m_1|}}{(1)_{|m_1|}} \frac{(\frac{i\lambda+1}{2})_{|m_2|}}{(1)_{|m_2|}}$$

$$\times F(\frac{i\lambda+1}{2}, \frac{i\lambda+1}{2} + |m_1|, |m_1| + 1; |z_1 - iz_2|^2)$$

$$\times F(\frac{i\lambda+1}{2}, \frac{i\lambda+1}{2} + |m_2|, |m_2| + 1; |z_1 + iz_2|^2)$$

**Lemma III.2** *For each $\lambda \in \mathbf{C}$ such that $\mathcal{R}]\rangle\lambda > \prime$, we have*

$$\lim_{r\to 1^-} (1 - r^2)^{-1+i\lambda} \phi_{\lambda,m}(r) = 4\pi^2 \frac{\Gamma^2(i\lambda)}{\Gamma^4(\dfrac{i\lambda+1}{2})}$$

*for all $m \in \wedge$.*

**Corollary III.1** *Let $\lambda \in \mathbf{C}$ such that $-i\lambda \notin \{1, 2, ...\}$ and let $F$ be a $\mathbf{C}$-valued function on $\mathcal{D}_2$ solution of the Hua system*

$$(\mathcal{H}F)(z) = -(\lambda^2 + 1)F(z).e.$$

*Then, there exists a sequence of spherical harmonic functions $(f_m)_{m\in\wedge}$ such that $F$ has the following expansion*

$$F(ru) = \sum_{m\in\wedge} \phi_{\lambda,m}(r) f_m(u),$$

*in $C^\infty([0,1[\times S)$.*

We end this work by giving the explicit Fourier series of solution of Hua system for any $z \in \mathcal{D}_2$, for this we need the Cartan decomposition of $G$ which is given by $G = KAK$.

**Theorem III.1** *Let $\lambda \in \mathbf{C}$ such that $-i\lambda \notin \{1, 2, ...\}$ and $F : D_2 \longmapsto \mathbf{C}$ be a function satisfying the Hua system*

$$(\mathcal{H}F)(z) = -(\lambda^2 + 1)F(z).e.$$

*Then, there exists a sequence of spherical harmonics $(f_m)_{m\in\wedge}$ such that for every $z = ka.0 \in \mathcal{D}_2, k \in K, a \in A$, $F$ may be written in the form as follows*

$$F(z) = \sum_{m\in\wedge} \phi_{\lambda,m}(a.0) f_m(k.e), \quad f_m \in V_m.$$

**Lemma III.3** *Let $m = (m_1, m_2) \in \wedge$. Then for every, $z = ka.0 \in \mathcal{D}_2,\ \ k \in K, a \in A$, we have*

$$(P_\lambda f)(ka.0) = \phi_{\lambda,m}(a.0) f(ke).$$

## IV. REFERENCES


[1] A. Boussejra, *$L^2$-Poisson integral representations of Solutions of the Hua system on the bounded symmetric domain $SU(n,n)/S(U(n)\times U(n))$,* J. Funct. Anal. 202(2003)25-43

[2] J. Faraut, A. Kouranyi, *Functions spaces and reproducing kernels on bounded symmetric domains,* J. Funct. Anal. 88(1990)64-89

[3] J. Faraut, A. Kouranyi, *Analysis on Symmetric Cones,* Clarendon Press, Oxford, 1994.

[4] L. K. Hua, *Harmonic analysis of functions of several variables in the classical domains,* American Mathematical Society, Providence, RI, 1963.

[5] M.Lassalle, *Les équations de Hua d'un domaine borné symétrique de type tube,* Invent. Math. 77 (1984) 129-161

[6] A. Nikiforov et V.Ouvarov. *Fonctions spécials de la physique Mathématique.* Edition Mir.Moscou.

[7] N. Shimeno, *Boundary value problems for the Shilov boundary of a bounded symmetric domain of tube type,* J. Funct. Anal. 140(1996)124-141

[8] S. Helgason, Geometric Analysis on Symmetric Spaces, Mathematical Surveys and Monographs,vol 39, Amer. Math. Soc., Providence, 1994.