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The Non–commutative Geometry of the Matrix Algebra $M_3(C)$

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Abstract

The main aim of this work is to investigate the non–commutative differential geometry of matrix algebras $M_3(C)$. Firstly, we give a presentation of these algebras viewed as *cyclic* Manin planes $\mathcal{M}_3$. Then, the table of the Hodge-star operator on the invariants is computed. Finally, the quantum group G_3, its associated quantum algebra $\mathcal{G}_3$ and the Wess–Zumino complex $\Omega_{WZ}(\mathcal{M}_3; d)$ corresponding to $M_3(C)$, viewed as a cyclic Manin plane, are constructed.

I. INTRODUCTION

In non–commutative differential geometry, the role of the C^*-algebra $\mathcal{A}_0 = C^\infty(M)$ of smooth complex functions on a smooth manifold M is played by an abstract associative (not necessarily commutative) C^*–algebra $\mathcal{A}$ as analog of functions on non–commutative spaces,[1].

From this point of view, that originates from quantum mechanics[2], where the Hamiltonian vector fields are replaced by derivations[3], it is natural to consider that the non–commutative generalization of the notion of vector field is that of *derivation* and that the analog of the differentiable structure is encoded in the *Lie algebra of derivations*. In order to define *connections* on $\mathcal{A}$*–modules*, which generalize the notion of connections on vector bundles, and consequently to define a *non–commutative differential calculus*, one needs to define a non–commutative generalization of the graded differential algebra of differential forms. Following Connes' procedure[1], modules of sections of vector bundles generalize in *finite projective* $\mathcal{A}$*–modules*. The fact that in this generalization the Lie algebra $Der(\mathcal{A})$ of derivations of $\mathcal{A}$ is no more an $\mathcal{A}$–module, possibly justify the existence of several non–commutative generalization procedures in the literature ([1],[4],[5]).

Furthermore, one finds in literature variours approaches to define the notion of *quantum group*, ([6–10]), and *non–commutative differential calculus on quantum groups*, ([11–14]).

In[15], a detailed review on Dubois Violette's approch to non-commutative differential calculus is presented and some applications are discussed. The most simple non–trivial example wich was treated in this context is the case of the matrix algebra $M_N(C)$,[16]. Since all derivations of this algebra are inner, the complex Lie algebra $Der(M_N(C))$ reduces to the algebra $sl(N; C)$ and consequently, the real Lie algebra $Der_R(M_N(C))$ reduces to the algebra $su(N)$.

It is also shown that there is an operation of the Lie algebra $Der(M_N(C))$ in the graded differential algebra $\Omega_{Der}(M_N(C))$ of differential forms on $M_N(C)$ in the sense of H. Cartan,[17].

Choosing a basis $\{\mathbf{1}, V_k\}$ of $M_N(C)$ consisting of Hermitian matrices, where $\{V_k\}$ is a set of $N^2 - 1$ self–adjoint traceless $N \times N$ matrices, we can define a *presentation* of $\Omega_{Der}(M_N(C))$ associated with this

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basis. This presentation is determined as follows :
a)– Setting the multiplication table of the matrices V_k,
b)– Stating that $\Omega^1_{Der}(M_N(C))$ is a $M_N(C)$–module,
c)– Introducing a Grassmannian structure on $\Omega_{Der}(M_N(C))$ by defining an exterior product on differential 1-forms θ_k, and finally,
d)– Determining the relations between the differentials of the matrices V_k and the 1-forms θ_k.
In order to complete the study of the non–commutative differential geometry of the matrix algebra $M_N(C)$, it is necessary to define the notions of integration of p–forms and of canonical Riemmanian structure for $M_N(C)$. The simple case of $M_2(C)$ has been already treated in[18].
A non–commutative differential geometry of $M_N(C)$ and particulary $M_3(C)$ was determined,[19], by using the fact that the matrix algebra $M_N(C)$ can be viewed as the quotiented algebra $Fun_q(C^N)/I$ (of polynomials over the quantum hyperplane) generated by the elements x, y obeying the following relations:

$$xy = qyx$$

where the ideal I is defined by :

$$x^N = y^N = \mathbf{1}$$

which means that q is a generic root of unity, i.e. $q^N = 1$.
The resulting differential algebra is the smallest differential sub–algebra of the universal differential algebra of $M_N(C)$.
The main purpose of this work is to investigate some aspects of the non–commutative differential geometry of $M_N(C)$ following the Dubois–Violette's approach. The resulting differential algebra is the biggest differential sub–algebra of the universal differential algebra of $M_N(C)$. The cases $N = 2$ and $N = 3$ are treated in details.
This work is organized as follows. In section 2, we describe the space $\mathcal{M}_N \equiv M_N(C)$, and in particular the cases $N = 2$ and $N = 3$, as a cyclic Manin plane. In section 3, we give a presentation of $\Omega_{Der}(M_N(C))$ relatively to some basis for an arbitrary N. In section 4, we present respectively the notions of integration of p–forms and the canonical Riemannian stucture for $M_N(C)$ for an arbitrary N. In section 5, we determine the expressions of the fundamental invariants of $M_N(C)$ for the cases $N = 2, 3$. In section 6, we construct the quantum group $G_3 = Fun(GL_q(2))/\tilde{I}$ as the symmetry group on the *cyclic Manin plane* $\mathcal{M}_3$ and develop its Hopf algebra structure. Here, we give an explanation for an arbitrary choice fixed in[19] for the ideal $\tilde{I}$ generated by:

$$a^3 = d^3 = 1 \qquad , \qquad b^3 = c^3 = 0$$

where $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is a quantum matrix generating G_3. In section 7, we define the dual $\mathcal{G}_3$ of the quantum group G_3 as well as its Hopf algebra structure. In section 8, we briefly describe the Grassmannian Manin plane $\tilde{\mathcal{M}}_3$ associated with $\mathcal{M}_3$ and in section 9, we present the Wess–Zumino complex $\Omega_{WZ}(\mathcal{M}_3)$. Finally, the section 10 is devoted to some concluding remarks.

II. THE SPACE $\mathcal{M}_N \equiv M_N(C)$

It is well known,[20], that the algebra $\mathcal{M}_N$ of $N \times N$ complex matrices can be generated by two elements x and y obeying the following relations :

$$xy = qyx \tag{2.1}$$

$$x^N = y^N = \underline{1} \tag{2.2}$$

with q a generic (N–th) root of unity :

$$q^N = 1 \ , \ q = e^{\frac{i2\pi}{N}} \ , \ \sum_{n}^{N-1} q^n = 0 \ , \ q^n = q^{n-N} \ , \ n \in Z$$

and $\underline{1}$ is the $N \times N$ unit matrix.
The associative (and not necessarily commutative) algebra generated by abstract elements x, y satisfying the relations (2.1) is called the quantum Manin plane C_q, or more precisely the "algebra of polynomials over the quantum plane", and is often denoted by $Fun_q(C^2)$ or by $C_q[x,y]$,[9]. We shall just call it C_q. When $q = 1$, this algebra reduces to the commutative algebra of polynomials $C[x,y]$ over the usual plane, x and y being the two coordinate functions. The dimension of C_q is infinite since it is generated by the infinite set $\{x^r y^s, r, s \in Z\}$.
Now, $\mathcal{M}_N$ is generated by elements x, y satisfying the relations (2.2) in addition to (2.1). Its dimension is then equal to N^2 and it can be considered as the quotient of the associative algebra C_q, by the bilateral Ideal I generated by the relations $x^N - \underline{1} = 0 \quad , \; y^N - \underline{1} = 0$:

$$\mathcal{M}_N \equiv C_q[x,y]/I$$

and with

$$x = \begin{pmatrix} 0 & 1 & . & . & ... & . & . \\ 0 & 0 & 1 & . & ... & . & . \\ . & . & 0 & 1 & ... & . & . \\ . & . & . & . & ... & . & . \\ . & . & . & . & ... & . & . \\ 0 & . & . & . & ... & 0 & 1 \\ 1 & 0 & . & . & ... & . & 0 \end{pmatrix} \quad , \quad y = \begin{pmatrix} 1 & . & . & . & . & . & . \\ . & q & . & . & . & . & . \\ . & . & q^2 & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & q^{N-1} \end{pmatrix}.$$

Thus, we shall call $\mathcal{M}_N$ the *cyclic quantum Manin plane.*
It is well known that the algebra $\mathcal{M}_N$ of $N \times N$ complex matrices is generated by, say, the basis $\{E_{ij}, i, j = 1, 2, ..., N\}$ where E_{ij} is the elementary matrix whose single non–zero entry 1 is in position (i,j) and is filled with zeros elsewhere. One can also generate $\mathcal{M}_N$ by $1_{N\times N}$ plus the usual Hermitian traceless matrices $\{V_i\}, \;\; i = 1, 2, ..., N^2 - 1,$ that generate the Lie algebra $su(N)$ such that :

$$< V_i, V_j >= tr(V_i V_j) = 2\delta_{ij}$$

where $<,>$ means scalar product on $su(N)$ and the normalisation fixed here means that for $su(2)$ the V_i are the Pauli matrices σ_i and for $su(3)$ the V_i are the Gell–Mann matrices λ_i.
We set:

$$V_i V_j = G_{ij}\mathbf{1} + (S_{ij}^k + \frac{1}{2}C_{ij}^K)V_k \tag{2.3}$$

where the quantities $G_{ij}, S_{ij}^k, C_{ij}^k$ are to be determined.
Using the above relation, one may deduce the quantity G_{ij} as follows :

$$G_{ij} = \frac{1}{N} tr(V_i.V_j) = \frac{2}{N}\delta_{ij}.$$

From the relation (2.3), we derive the following defining commutation relation of the $su(N)$ Lie algebra:

$$[V_i, V_j] = C_{ij}^k V_k$$

where the structure constants C_{ij}^k are given by :

$$C_{ijk} = \frac{1}{2} tr([V_i, V_j]V_k) = C_{ij}{}^{l}\delta_{lk}$$

and satisfy the following relations :

$$\begin{cases} C_{iq}{}^{p} C_{jp}{}^{q} = 4N\delta_{ij} = 2N^2 G_{ij} \\ C_{ij}{}^{p} C_{mnp} = \frac{8}{N}(\delta_{in}\delta_{jm} - \delta_{im}\delta_{jn}) + 4(S_{in}{}^{p} S_{jmp} - S_{im}{}^{p} S_{jnp}) \\ C_{ij}{}^{p} C_{pkl} + C_{ki}{}^{p} C_{pjl} + C_{jk}{}^{p} C_{pil} = 0 \\ C_{il}{}^{m} C_{mj}{}^{k} C_{kp}{}^{i} = -2N C_{ljp} \end{cases}$$

Finally, the quantities $S_{ij}\; k$ can be determined as follows :

$$\begin{array}{l} \{V_i, V_j\} = \frac{4}{N}\delta_{ij}\mathbf{1} + 2S_{ij}\ ^{k}V_k \\ S_{ijk} = \frac{1}{4}tr(\{V_i, V_j\}V_k) = S_{ij}\ ^{l}\delta_{lk} \end{array} \tag{2.4}$$

They also obey the following relations :

$$\left\{ \begin{array}{l} S_i\ ^{km}S_{jkm} = \left(\frac{N^2-4}{N}\right)\delta_{ij} \\ C_{il}\ ^{m}S_{jkm} + C_{jl}\ ^{m}S_{kim} + C_{kl}\ ^{m}S_{ijm} = 0 \\ S_{pi}\ ^{q}C_{qj}\ ^{r}C_{rk}\ ^{p} = -2NS_{ijk} \\ S_{pi}\ ^{q}S_{qj}\ ^{r}C_{rk}\ ^{p} = \left(\frac{N^2-4}{2N}\right)C_{ijk} \\ S_{pi}\ ^{q}S_{qj}\ ^{r}S_{rk}\ ^{p} = \left(\frac{N^2-12}{2N}\right)S_{ijk} \\ S_{ip}\ ^{q}C_{jp}\ ^{q} = 0. \end{array} \right.$$

Furthermore, the bilinear Killing form K_{ij} of $su(N)$ is defined by :

$$K(V_i, V_j) = tr(AdV_iAdV_j) = C_i\ ^{lk}C_{jkl}$$

Since the Lie algebra $su(N)$ is semi–simple, then K_{ij} is non–degenerate and so is equal to δ_{ij} up to a factor. Then, it is easy to see that the quantities G_{ij}, $S_{im}\ ^{n}S_{jn}\ ^{m}$ and $C_{im}\ ^{n}C_{in}\ ^{m}$ are all proportional to the Killing form. Morever, from the relation :

$$(S_{ij}\ ^{p} + \frac{1}{2}C_{ij}\ ^{p})G_{pk} = \frac{1}{N}tr(V_iV_jV_k)$$

it is clear that the quantities $S_{ijk} = S_{ij}\ ^{m}G_{mk}$ and $C_{ijk} = C_{ij}\ ^{m}G_{mk}$ are respectively totally symmetric and totally antisymmetric.

For instance, let us recall briefly the case of the algebra $\mathcal{M}_2$. Any element of this algebra can be generated by the matrices E_{ij}, $i, j = 1, 2$ whose only non–vanishing entry 1 is located in position (i, j) and is filled with zeros elsewhere. One can also consider the algebra $\mathcal{M}_2$ to be generated simply by the 2×2 unit matrix $\mathbf{1}$ and the Hermitian traceless 2×2 Pauli matrices σ_i, $i = 1, 2, 3$, obeying the following relation :

$$\sigma_i.\sigma_j = \delta_{ij}\mathbf{1} + i\epsilon_{ij}\ ^{k}\sigma_k.$$

Then, one easily deduces that :

$$\begin{array}{l} G_{ij} = \delta_{ij} \\ C_{ij}\ ^{k} = 2i\epsilon ij\ ^{k} \\ S_{ij}\ ^{k} = 0. \end{array}$$

In other hand, we know that $\mathcal{M}_2$ is also generated by $\{x^ry^s, r, s = 0, 1\}$ where x, y satisfy the following relations :

$$\begin{array}{l} xy = qyx \\ x^2 = y^2 = \mathbf{1} \end{array}$$

with $q = -1$ and :

$$x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \ , \ y = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Furthermore, the above matrices E_{ij} and the Pauli matrices σ_i can be expressed in function of x and y as follows :

$$\begin{array}{l} \mathbf{1} = E_{11} + E_{22} = x^2 = y^2 \\ \sigma_1 = E_{12} + E_{21} = x \\ \sigma_2 = i(-E_{12} + E_{21}) = ixy \\ \sigma_3 = E_{11} - E_{22} = y \end{array}$$

Now, let us treats the case $N = 3$, the algebra $\mathcal{M}_3$ of 3×3 complex matrices is generated by, say, the basis $\{E_{ij}, i, j = 1, 2, 3\}$ where E_{ij} is the elementary matrix whose single non-zero entry 1 is in position

(i,j) and is filled with zeros elsewhere.
One can also generate $\mathcal{M}_3$ by $\mathbb{1}_{3\times 3}$ plus the usual Hermitian traceless Gell-Mann matrices λ_i, $i = 1,2,...,8$ that generate the Lie algebra $su(3)$:

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \ , \ \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \ , \ \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \ ,$$

$$\lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \ , \ \lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} \ , \ \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \ ,$$

$$\lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \ , \ \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} \ ,$$

such that :

$$tr(\lambda_i \lambda_j) = 2\delta_{ij}.$$

The relations between the above two basis are as follows :

$$\begin{array}{lll} \mathbf{1} = (E_{11} + E_{22} + E_{33}) & \lambda_1 = E_{12} + E_{21} & \lambda_2 = i(-E_{12} + E_{21}) \\ \lambda_3 = E_{11} - E_{22} & \lambda_4 = E_{13} + E_{31} & \lambda_5 = i(-E_{13} + E_{31}) \\ \lambda_6 = E_{23} + E_{32} & \lambda_7 = i(-E_{23} + E_{32}) & \lambda_8 = \frac{1}{\sqrt{3}}(E_{11} + E_{22} - 2E_{33}). \end{array}$$

Moreover,the algebra $\mathcal{M}_3$ as a vector space of dimension nine, is spanned by the basis $\{x^r y^s, r, s = 0,1,2\}$ where x and y obey the following relations :

$$\begin{array}{l} xy = qyx \\ x^3 = y^3 = \mathbf{1} \end{array}$$

with $q = j = e^{\frac{i2\pi}{3}}$ and $1 + j + j^2 = 0$ and :

$$x = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \ , \ y = \begin{pmatrix} 1 & 0 & 0 \\ 0 & q & 0 \\ 0 & 0 & q^2 \end{pmatrix},$$

It is easy to see that the elementary matrices E_{ij} and the Gell-Mann matrices themselves can be expressed in terms of x and y as follows:

$$\begin{aligned} E_{11} &= \tfrac{1}{3}(\mathbf{1} + y + y^2) \\ E_{12} &= \tfrac{1}{3}(x + q^2xy + y^2x) \\ E_{13} &= \tfrac{1}{3}(x^2 + qx^2y + q^2x^2y^2) \\ E_{21} &= \tfrac{1}{3}(x^2 + x^2y + x^2y^2) \\ E_{22} &= \tfrac{1}{3}(\mathbf{1} + q^2y + qy^2) \\ E_{23} &= \tfrac{1}{3}(x + qxy + qy^2x) \\ E_{31} &= \tfrac{1}{3}(x + xy + q^2y^2x) \\ E_{32} &= \tfrac{1}{3}(x^2 + q^2x^2y + qx^2y^2) \\ E_{33} &= \tfrac{1}{3}(\mathbf{1} + qy + q^2y^2), \end{aligned}$$

and

$$\begin{aligned} \mathbf{1} &= x^3 = y^3 \\ \lambda_1 &= \tfrac{1}{3}(x + q^2xy + y^2x + x^2 + x^2y + x^2y^2) \\ \lambda_2 &= \tfrac{i}{3}(-x - q^2xy - y^2x + x^2 + x^2y + x^2y^2) \\ \lambda_3 &= \tfrac{1}{3}((1 - q^2)y + (1 - q)y^2) \\ \lambda_4 &= \tfrac{1}{3}(x^2 + qx^2y + q^2x^2y^2 + x + xy + q^2y^2x) \\ \lambda_5 &= \tfrac{i}{3}(-x^2 - qx^2y - q^2x^2y^2 + x + xy + q^2y^2x) \\ \lambda_6 &= \tfrac{1}{3}(x + qxy + qy^2x + x^2 + q^2x^2y + qx^2y^2) \\ \lambda_7 &= \tfrac{i}{3}(-x - qxy - qy^2x + x^2 + q^2x^2y + qx^2y^2) \\ \lambda_8 &= -\tfrac{1}{\sqrt{3}}(qy + q^2y^2). \end{aligned} \tag{2.5}$$

Moreover, since the set of 3×3 matrices is endowed with a usual Hermitian conjugacy, it results that x and y are unitary elements satisfying :

$$x^{+}=x^{2}=x^{-1} \quad , \quad y^{+}=y^{2}=y^{-1}.$$

Let us return now to the Gell–Mann basis and we set :

$$\lambda_i\lambda_j=\frac{2}{3}\delta_{ij}\mathbf{1}+(S_{ij}{}^{k}+\frac{1}{2}C_{ij}{}^{k})\lambda_k$$

Then, the defining commutation relations of the Lie algebra $su(3)$ reads:

$$[\lambda_i,\lambda_j]=C_{ij}{}^{k}\lambda_k$$

where the non–vanishing structure constants C_{ij}^{k} are given by :

$$\begin{cases} C_{12}{}^{3}=2i \\ C_{14}{}^{7}=C_{24}{}^{6}=C_{25}{}^{7}=C_{34}{}^{5}=-C_{15}{}^{6}=-C_{36}{}^{7}=-i \\ C_{45}{}^{8}=C_{67}{}^{8}=i\sqrt{3} \end{cases}$$

and verify the following relations:

$$\begin{cases} C_{iq}{}^{p}C_{jp}{}^{q}=12\delta_{ij} \\ C_{ij}{}^{p}C_{mnp}=\frac{2}{3}(\delta_{in}\delta_{jm}-\delta_{im}\delta_{jn})+(S_{in}{}^{p}S_{jmp}-S_{im}{}^{p}S_{jnp}) \\ C_{ij}{}^{p}C_{pkl}+C_{ki}{}^{p}C_{pjl}+C_{jk}{}^{p}C_{pil}=0 \\ C_{il}{}^{m}C_{mj}{}^{k}C_{kp}{}^{i}=-6C_{ljp} \end{cases}$$

Finally, from (2.4) one can derive all the quantities $S_{ij}{}^{k}$ as follows:

$$S_{14}{}^{6}=S_{15}{}^{7}=S_{25}{}^{6}=-S_{24}{}^{7}=\frac{1}{2}.$$

$$S_{kk}{}^{3}=\begin{cases} \frac{1}{2} & if\ k=4,5 \\ -\frac{1}{2} & if\ k=6,7 \end{cases} \quad , \quad S_{kk}{}^{8}=\begin{cases} \frac{1}{\sqrt{3}} & if\ k=1,2,3 \\ -\frac{1}{2\sqrt{3}} & if\ k=4,5,6,7 \\ -\frac{1}{\sqrt{3}} & if\ k=8 \end{cases}$$

III. PRESENTATION OF $\Omega_{DER}(M_N(C))$

Consider the matrix algebra $M_N(C)$ generated by the basis $\{\mathbf{1},V_i\}$ defined by (2.3). The aim of this section is to define what we call a *presentation* of $\Omega_{Der}(M_N)(C)$ relatively to this basis. The procedure used hereunder is valid for any arbitrary N.
Let us define a basis $\{e_j\}$, $j=1.......N^2-1$, of $Der_R(M_N(C))\sim su(N)$, such as :

$$\begin{aligned} e_i &= ad(V_i) \\ e_m(V_n) &= [V_m,V_n]=C_{mn}{}^{k}V_k \\ [e_m,e_n] &= C_{mn}{}^{k}e_k. \end{aligned}$$

In order to construct the graded vector space $\Omega_{Der}(M_N(C))$ of exterior forms α_p defined as linear mappings :

$$\alpha_p:\Lambda^p(Der_R(M_N(C)))\longrightarrow M_N(C)$$

we must first choose a basis of 1–forms and then define an exterior product on this basis. The most convenient basis of 1–forms is the one formed by the 1–forms θ^k dual to the real basic derivations e_k, i.e. :

$$\theta^m(e_n) = \delta^m_n \mathbf{1}$$

which also verify the following properties :

$$V_k\theta^m = \theta^m V_k \tag{3.1}$$

$$\theta^m \wedge \theta^n = -\theta^n \wedge \theta^m \tag{3.2}$$

for any $V_k \in M_N(C)$.
Moreover, the differential "d" of $\Omega_{Der}(M_N(C))$ is defined as :

$$d^2 = 0 \tag{3.3}$$

$$d\mathbf{1} = 0 \tag{3.4}$$

$$dV_n(e_m) = e_m(V_n) = [V_m, V_n] = C_{mn}{}^k V_k \Longrightarrow dV_m = -C_{mn}{}^k V_k\theta^n \tag{3.5}$$

$$d\theta^m = -\frac{1}{2}C^m{}_{pq}\theta^p \wedge \theta^q \tag{3.6}$$

An element θ of $\Omega_{Der}(M_N(C))$ defined by :

$$\theta = \sum_k V_k\theta^k \tag{3.7}$$

is invariant, i.e :

$$L_\chi(\theta) = 0$$

for any $\chi \in Der_R(M_N(C))$, and any invariant element of $\Omega^1_{Der}(M_N(C))$ is a scalar multiple of θ. This latter is called *the canonical invariant element* of $\Omega_{Der}{}^1(M_N(C))$. The relations (2.3), (3.1), (3.2), (3.5), and (3.6) for the generators V_k, θ^m and the differential "d" give a *presentation* of $\Omega_{Der}(M_N(C))$.

IV. INTEGRATION AND CANONICAL RIEMANNIAN STRUCTURE

Firstly, let us notice that all the recipients presented in this section are valid for an arbitrary N.
As a left $M_N(C)$-module, $\Omega_{Der}{}^{N^2-1}(M_N(C))$ is spanned by the unique generator $\theta^1 \wedge \theta^2 \wedge ... \wedge \theta^{N^2-1}$.
Let $\alpha \in \Omega_{Der}{}^{N^2-1}(M_N(C))$ given by:

$$\alpha = V\sqrt{|g|}\theta^1 \wedge \theta^2 .... \wedge \theta^{N^2-1}$$

with $V \in M_N(C)$ and $g = det(G)$. Let us take g to be equal to 1 since G_{ij} is proportional to δ_{ij} and so the (normalization) factor is irrelevant. Then, one may define the *integral* of α by using the trace :

$$\int \alpha = \frac{1}{N}tr(V) \tag{4.1}$$

The linear mapping $\int : \Omega_{Der}{}^{N^2-1}(M_N(C)) \longrightarrow C$ is a closed trace, i.e. in addition to equ. (4.1), one has :

$$\int d\alpha = 0 \qquad \text{for any } \alpha \in \Omega_{Der}{}^{N^2-1}(M_N(C))$$

$$\int \alpha_p \wedge \beta_q = (-1)^{pq}\int \beta_q \wedge \alpha_p \tag{4.2}$$

for any $\alpha_p \in \Omega_{Der}{}^p(M_N(C))$, $\beta_q \in \Omega_{Der}{}^q(M_N(C))$ and $p + q = N^2 - 1$, and :

$$L_\chi(\sqrt{|g|}\ \theta^1 \wedge \theta^2 \wedge ......\theta^{N^2-1}) = 0$$

for any $\chi \in Der(M_N(C))$.
The form of the volume element $\sqrt{|g|}\theta^1 \wedge \theta^2 \wedge ......\theta^{N^2-1}$, which depends only on the choice of the orientation of the chosen basis $\{V_i\}$, looks like the volume element of a metric.It suggests the introduction of a flat metric defined by the symmetric 2–form :

$$G = G_{mn}\theta^m \otimes \theta^n$$

which is really the analog of an invariant Riemannian metric for $M_N(C)$ and we will call it *the canonical Riemannian structure* . The Hodge–star isomorphism can be introduced as usual, i.e. :

$$\star : \Omega^p_{Der}(M_N(C)) \longrightarrow \Omega^{N^2-1-p}_{Der}(M_N(C))$$

such that :

$$\star(\theta^{i_1} \wedge ..... \wedge \theta^{i_p}) = \frac{\sqrt{|g|}}{(N^2-1-p)!}\epsilon^{i_1.....i_p}{}_{j_{p+1}....j_{N^2-1}}\theta^{j_{p+1}} \wedge ... \wedge \theta^{j_{N^2-1}}$$

where $\epsilon_{j_1....j_{N^2-1}}$ is the totally antisymmetric Levi–civita symbol defined as usual, with $\epsilon_{12....(N^2-1)} = 1$.
One also has :

$$\star(V\theta^{i_1} \wedge .... \wedge \theta^{i_p}) = V \star (\theta^{i_1} \wedge .... \wedge \theta^{i_p}) \tag{4.3}$$

$$\star(\star(\alpha_p)) = (-1)^{N^2 p}\alpha_p \quad \text{for any } \alpha_p \in \Omega_{Der}{}^p(M_N(C)) \tag{4.4}$$

Moreover, one can also define on the graded differential algebra $\Omega_{Der}(M_N(C))$ a *scalar product* between any two real exterior forms as follows :

$$(.|.) : \Omega_{Der}(M_N(C)) \times \Omega_{Der}(M_N(C)) \longrightarrow R$$

such that :

$$(\alpha|\beta) = \begin{cases} \int \alpha \wedge \star(\beta) & \text{if } \alpha,\ \beta\ \in\ \Omega_{Der}{}^p(M_N(C)) \\ 0 & \text{otherwise (i.e. if not of the same degree).} \end{cases}$$

In view of the graded trace property (see (4.2)), has :

$$(\alpha|\beta) = (\beta|\alpha)$$

and this inner product is a real positive–definite bilinear form on the real sub–space of elements of $\Omega_{Der}(M_N(C))$. This definition can be extended to the complex p-forms as follows :

$$< \alpha|\beta >= (\alpha^*|\beta)$$

where $< .|. >$ is a positive–definite Hermitian bilinear form on $\Omega_{Der}(M_N(C))$ and "$*$" is the antilinear involution defined on it , such that:

$$\alpha_p{}^*(\chi_1, ..., \chi_p) = [\alpha_p(\chi_1{}^*, ..., \chi_p{}^*)]^*$$
$$d(\alpha_p^*) = (d\alpha_p)^*$$

and,

$$(\alpha_p \wedge \beta_q)^* = (-1)^{pq}\beta_q{}^* \wedge \alpha_p{}^*$$

for any $\alpha_p \in \Omega_{Der}{}^p(M_N(C))$, $\beta_q \in \Omega_{Der}{}^q(M_N(C))$ and $\chi_i \in Der(M_N(C))$.
Define now an antidifferentiation :

$$\delta : \Omega_{Der}{}^p(M_N(C)) \longrightarrow \Omega_{Der}{}^{p-1}(M_N(C))$$

such that :

$$< d\alpha|\beta >=< \alpha|\delta\beta > \tag{4.5}$$

One easily verifies that :

$$\delta V_k = 0$$
$$\delta \theta^k = 0$$

and

$$\delta \alpha_p = (-1)^{(N^2-1)p+N^2-p^2+1} \star d \star \alpha_p$$

Let us notice here, that the sign in the expression of $\delta\alpha_p$ is adjusted such that all the eigenvalues of the the Laplace–Beltrami operator Δ become definite positive (see below).
As usual, one can define the Laplace–Beltrami operator Δ on $\Omega_{Der}(M_N(C))$ as follows :

$$\Delta = d\delta + \delta d$$

Using equ. (4.5) and the bilinearity of$< .|. >$, one obtains :

$$< \alpha|\Delta\alpha >=< d\alpha|d\alpha > + < \delta\alpha|\delta\alpha >$$

Then, it follows that Δ is a definite–positive operator on the Hilbert space $(\Omega_{Der}(M_N(C)); -- < .|. >)$, and that any element α of $\Omega_{Der}(M_N(C))$ is called harmonic, i.e. satisfy $\Delta\alpha = 0$, if and only if it is closed ($d\alpha = 0$) and co–closed ($\delta\alpha = 0$). The set of the harmonic elements forms the kernel of Δ and it results an analog of the Hodge–De Rham decomposition :

$$\Omega_{Der}(M_N(C)) = d\Omega_{Der}(M_N(C)) \oplus Ker(\Delta)$$

V. INVARIANTS

In general, the fundamental invariants of $su(N)$ are given by :

$$I_{k-1} = tr[(\theta)^{2k-1}] = tr[V_{j_1}.V_{j_2}...V_{j_{2k-1}}]\theta^{j_1} \wedge \theta^{j_2} \wedge .... \wedge \theta^{j_{2k-1}}$$

for $k = 2, 3, ..., N$ and θ is the canonical invariant element defined by (3.7). In fact, $\Lambda sl(N)^* \subset \Omega_{Der}(M_N(C))$ is stable under $\star$ and one always has :

$$\prod_{k=2}^{N} I_{k-1} \sim \text{ volume element} \tag{5.1}$$

with $\sum_{k=2}^{N}(2k-1) = N^2 - 1$.
For instance, in the case $N = 2$, the algebra $su(2)$ has only one fundamental invariant :

$$I = tr[\theta^3] = 12i\theta^1 \wedge \theta^2 \wedge \theta^3 \sim \text{ volume element.}$$

Therefore, in the case $N = 3$, we obtain two fundamental invariants :

$$\begin{aligned} I_1 &= tr[\theta^3] = tr[V_k.V_m.V_n]\theta^k \wedge \theta^m \wedge \theta^n = C_{kmn}\theta^k \wedge \theta^m \wedge \theta^n \\ &= 6i\left\{2\theta^1 \wedge \theta^2 \wedge \theta^3 - \theta^1 \wedge \theta^4 \wedge \theta^7 + \theta^1 \wedge \theta^5 \wedge \theta^6 - \theta^2 \wedge \theta^4 \wedge \theta^6 - \theta^2 \wedge \theta^5 \wedge \theta^7 \right. \\ &\left. - \ \theta^3 \wedge \theta^4 \wedge \theta^5 + \theta^3 \wedge \theta^6 \wedge \theta^7 + \sqrt{3}\left(\theta^4 \wedge \theta^5 \wedge \theta^8 + \theta^6 \wedge \theta^7 \wedge \theta^8\right)\right\} \end{aligned}$$

$$\begin{aligned} I_2 &= tr[\theta^5] = tr[V_k.V_m.V_n.V_p.V_q]\theta^k \wedge \theta^m \wedge \theta^n \wedge \theta^p \wedge \theta^q \\ &= \frac{1}{2}C_{km}{}^{a}C_{np}{}^{b}S_{abq}\theta^k \wedge \theta^m \wedge \theta^n \wedge \theta^p \wedge \theta^q \\ &= \frac{8}{\sqrt{3}}\left\{2\theta^4 \wedge \theta^5 \wedge \theta^6 \wedge \theta^7 \wedge \theta^8 - \theta^2 \wedge \theta^3 \wedge \theta^5 \wedge \theta^6 \wedge \theta^8 + \theta^2 \wedge \theta^3 \wedge \theta^4 \wedge \theta^7 \wedge \theta^8 \right. \\ &- \ \theta^1 \wedge \theta^3 \wedge \theta^5 \wedge \theta^7 \wedge \theta^8 - \theta^1 \wedge \theta^3 \wedge \theta^4 \wedge \theta^6 \wedge \theta^8 - \theta^1 \wedge \theta^2 \wedge \theta^6 \wedge \theta^7 \wedge \theta^8 \\ &\left. + \ \theta^1 \wedge \theta^2 \wedge \theta^4 \wedge \theta^5 \wedge \theta^8 + \sqrt{3}\left(\theta^1 \wedge \theta^2 \wedge \theta^3 \wedge \theta^4 \wedge \theta^5 + \theta^1 \wedge \theta^2 \wedge \theta^3 \wedge \theta^6 \wedge \theta^7\right)\right\}. \end{aligned}$$

Using (4.3), we can easily verify that :

$$\star(I_1) = \frac{3i\sqrt{3}}{4} I_2 \quad \text{and} \quad \star(I_2) = \frac{4i}{3\sqrt{3}} I_1$$

such that the relations (4.4) and (5.1) are fulfilled, i.e. :

$$\star(\star(I_1)) = -I_1 \quad , \quad \star(\star(I_2)) = -I_2$$

$$I_3 = I_1 \wedge I_2 \sim \theta^1 \wedge \theta^2 \wedge \theta^3 \wedge \theta^4 \wedge \theta^5 \wedge \theta^6 \wedge \theta^7 \wedge \theta^8 \sim \text{volume element}$$

VI. THE QUANTUM GROUP G_3

Let us construct the quantum group G_3 as the symmetry group on the *cyclic Manin plane* $\mathcal{M}_3$ generated by x and y satisfying (2.1) and (2.2), (see[9]).
Let G_3 be a free associative algebra generated by the quantum matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ coacting on the coordinate function doublet by *left and right coactions* :

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \delta_L \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \delta_L(x) \\ \delta_L(y) \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \odot \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a \otimes x + b \otimes y \\ c \otimes x + d \otimes y \end{pmatrix}$$

$$(x'' \; y'') = \delta_R\,(x \; y) = (\delta_R(x) \; \delta_R(y)) = (x \; y) \odot \begin{pmatrix} a & b \\ c & d \end{pmatrix} = (x \otimes a + y \otimes c \;\; x \otimes b + y \otimes d)\,.$$

These coactions extend to the whole of $\mathcal{M}_3(C)$ using of the homomorphism property :

$$\delta_{L,R}(f.g) = \delta_{L,R}(f).\delta_{L,R}(g) \quad , \quad f, g \in \mathcal{M}_3. \tag{6.1}$$

Furthermore, imposing to the quantities x', y', (and x'', y'') to satisfy the same relations (2.1) and (2.2) as x and y, i.e. :

$$\delta_L(xy - qyx) = 0 \quad , \quad \delta_R(xy - qyx) = 0$$

one obtains the following non–trivial commutation relations of $Fun(GL_q(2))$:

$$\begin{aligned} ab &= qba \\ ac &= qca \\ ad - da &= (q - q^{-1})bc \\ bc &= cb \\ bd &= qdb \\ cd &= qdc \end{aligned}$$

with $q = j = e^{i2\pi/3}$. In fact, one has :

$$\delta_{L,R}(x^m y^n) = \delta_{L,R}(x^m).\delta_{L,R}(y^n)$$

with

$$\delta_L(x^m) = [\delta_L(x)]^m = \begin{cases} \underline{1} \otimes \underline{1} & \text{for m} = 0 \;\; \text{mod[3]} \\ a \otimes x + b \otimes y & \text{for m} = 1 \;\; \text{mod[3]} \\ a^2 \otimes x^2 - q^2 ab \otimes xy + b^2 \otimes y^2 & \text{for m} = 2 \;\; \text{mod[3]} \end{cases}$$

$$\delta_L(y^n) = [\delta_L(y)]^n = \begin{cases} \underline{1} \otimes \underline{1} & \text{for n} = 0 \;\; \text{mod[3]} \\ c \otimes x + d \otimes y & \text{for n} = 1 \;\; \text{mod[3]} \\ c^2 \otimes x^2 - q^2 cd \otimes xy + d^2 \otimes y^2 & \text{for n} = 2 \;\; \text{mod[3]} \end{cases}$$

$$\delta_R(x^m) = [\delta_R(x)]^m = \begin{cases} \underline{1} \otimes \underline{1} & \text{for m} = 0 \ \text{ mod[3]} \\ x \otimes a + y \otimes c & \text{for m} = 1 \ \text{ mod[3]} \\ x^2 \otimes a^2 - q^2 xy \otimes ac + y^2 \otimes c^2 & \text{for m} = 2 \ \text{ mod[3]} \end{cases}$$

$$\delta_R(y^n) = [\delta_R(y)]^n = \begin{cases} \underline{1} \otimes \underline{1} & \text{for n} = 0 \ \text{ mod[3]} \\ x \otimes b + y \otimes d & \text{for n} = 1 \ \text{ mod[3]} \\ x^2 \otimes b^2 - q^2 xy \otimes bd + y^2 \otimes d^2 & \text{for n} = 2 \ \text{ mod[3]} \end{cases}$$

and one obtains the following table :

	δ_L	δ_R
x^2	$a^2 \otimes x^2$ $-q^2 ab \otimes xy$ $+b^2 \otimes y^2$	$x^2 \otimes a^2$ $-q^2 xy \otimes ac$ $+y^2 \otimes c^2$
y^2	$c^2 \otimes x^2$ $-q^2 cd \otimes xy$ $+d^2 \otimes y^2$	$x^2 \otimes b^2$ $-q^2 xy \otimes bd$ $+y^2 \otimes d^2$
xy	$ac \otimes x^2$ $+(\underline{1} - bc) \otimes xy$ $+bd \otimes y^2$	$x^2 \otimes ab$ $+xy \otimes (\underline{1} - bc)$ $+y^2 \otimes cd$
$\underline{1}$	$\underline{1} \otimes \underline{1}$	$\underline{1} \otimes \underline{1}$
xy^2	$[ac^2 + bd^2] \otimes \underline{1}$ $-c \otimes x^2 y$ $+d \otimes xy^2$	$\underline{1} \otimes [ab^2 + cd^2]$ $-x^2 y \otimes b$ $+xy^2 \otimes d$
$x^2 y$	$[a^2 c + b^2 d] \otimes \underline{1}$ $+a \otimes x^2 y$ $-b \otimes xy^2$	$\underline{1} \otimes [a^2 b + c^2 d]$ $+x^2 y \otimes a$ $-xy^2 \otimes c$
x	$a \otimes x + b \otimes y$	$x \otimes a + y \otimes c$
y	$c \otimes x + d \otimes y$	$x \otimes b + y \otimes d$
$x^2 y^2$	$[a^2 c^2 - bd + q^2 b^2 cd] \otimes x$ $+[b^2 d^2 - ac + q^2 abc^2] \otimes y$ $+[a^2 d^2 + q^2 bc - qb^2 c^2] \otimes x^2 y^2$	$x \otimes [a^2 b^2 - cd + q^2 bc^2 d]$ $y \otimes [c^2 d^2 - ab + q^2 ab^2 c]$ $x^2 y^2 \otimes [a^2 d^2 + q^2 bc - qb^2 c^2]$

one can easily see that, under the coactions $\delta_{L,R}$, there are three independent subsets : $\{x^2, y^2, xy\}$, $\{\underline{1}, xy^2, x^2 y\}$ and $\{x, y, x^2 y^2\}$.
Using:

$$\delta_L(x^3) = x'^3 = \underline{1} \quad , \quad \delta_L(y^3) = y'^3 = \underline{1} \quad , \quad \delta_R(x^3) = y"^3 = \underline{1} \quad , \quad \delta_R(y^3) = y"^3 = \underline{1},$$

One obtains the following bilateral ideal $\tilde{I}$:

$$a^3 = d^3 \quad , \quad b^3 = c^3 \quad , \quad a^3 + b^3 = \underline{1}.$$

We impose the following choice for $\tilde{I}$:

$$a^3 = d^3 = \underline{1} \quad , \quad b^3 = c^3 = 0. \tag{6.2}$$

This choice will be argumented latter, when we will discuss the Hopf nature of this ideal. Our quantum group is then defined as :

$$G_3 = Fun(GL_q(2))/\tilde{I}.$$

The element :

$$\mathcal{D} = ad - qbc = da - q^{-1} bc$$

is central and is called the q-determinant. If one adds the condition :

$$\mathcal{D} = \underline{1}$$

to the above relations, one obtains the quantum group $G_3 = Fun(SL_q(2,C))/\tilde{I}$. From the above requirement, one finds :

$$d = a^2(\underline{1} + qbc) \quad , \quad a = (\underline{1} + qbc)d^2$$

so that d (or a) can be eliminated. The quantum group G_3 can therefore be linearly generated, as a vector space, by the elements :

$$\{a^r b^s c^t \ , \ \ r,s,t = 0,1,2\}.$$

Then, the algebra G_3 is a finite–dimensional quotiented Hopf algebra $Fun(SL_q(2,C))$ whose dimension is :

$$dim(G_3) = 3^3 = 27.$$

Let us stress the fact that this very particular algebra G_3 is effectively a Hopf algebra :
<u>Product</u> : It is the (associative) application :

$$m : G_3 \otimes G_3 \to G_3$$

$$m \circ (m \otimes Id) = m \circ (Id \otimes m) \ \ , \ \text{(associativity)}$$

defined by

$$m(f \otimes g) = fg;$$

<u>Unit</u> : It is the application :

$$\eta : C \to G_3$$

defined by

$$\eta(k) = k\underline{1},$$

$$m \circ (\eta \otimes Id) = m \circ (Id \otimes \eta) = Id_{G_3} \equiv Id_{C\otimes G_3} \equiv Id_{G_3 \otimes C};$$

<u>Co–product</u> : It is the (co–associative) homomorphism :

$$\Delta : G_3 \to G_3 \otimes G_3$$

$$\begin{array}{rcl} \Delta(x) &=& \sum_i x^i_{(1)} \otimes x^i_{(2)} \\ (Id \otimes \Delta) \circ \Delta &=& (\Delta \otimes Id) \circ \Delta \quad \text{(co-associativity)} \end{array}$$

defined by

$$\begin{array}{l} \Delta(a) = a \otimes a + b \otimes c \\ \Delta(b) = a \otimes b + b \otimes d \\ \Delta(c) = c \otimes a + d \otimes c \\ \Delta(d) = c \otimes b + d \otimes d \\ \Delta(\underline{1}) = \underline{1} \otimes \underline{1}; \end{array}$$

<u>Co–unit</u> : It is the homomorphism :

$$\epsilon : G_3 \to C$$

$$(\epsilon \otimes Id) \circ \Delta = (Id \otimes \epsilon) \circ \Delta = Id_{G_3}$$

defined by

$$\begin{aligned}
&\epsilon(a) = 1\\
&\epsilon(b) = 0\\
&\epsilon(c) = 0\\
&\epsilon(d) = 1\\
&\epsilon(\underline{1}) = 1;
\end{aligned}$$

Antipode : It is the anti–homomorphism :

$$S : G_3 \rightarrow G_3$$

$$m \circ (S \otimes Id) \circ \Delta = m \circ (Id \otimes S) \circ \Delta = \eta \circ \epsilon$$

defined by

$$\begin{aligned}
&S(a) = d\\
&S(b) = -q^{-1}b\\
&S(c) = -qc\\
&S(d) = a\\
&S(\underline{1}) = \underline{1}.
\end{aligned}$$

We can easily verify that the two–sided ideal $\tilde{I}$ is a Hopf ideal, i.e. :

$$\begin{aligned}
&\Delta(\tilde{I}) \subset \tilde{I} \otimes Fun(SL_q(2,C)) \oplus Fun(SL_q(2,C)) \otimes \tilde{I}\\
&\epsilon(\tilde{I}) = 0\\
&S(\tilde{I}) \subset \tilde{I}.
\end{aligned}$$

In fact, from the following relations :

$$\begin{array}{lll}
\Delta(a^3) = a^3 \otimes a^3 + b^3 \otimes c^3 & \epsilon(a^3) = 1 & S(a^3) = d^3 = a^3\\
\Delta(b^3) = a^3 \otimes b^3 + b^3 \otimes d^3 & \epsilon(b^3) = 0 & S(b^3) = -b^3 = -c^3\\
\Delta(c^3) = c^3 \otimes a^3 + d^3 \otimes c^3 & \epsilon(c^3) = 0 & S(c^3) = -c^3 = -b^3\\
\Delta(d^3) = c^3 \otimes b^3 + d^3 \otimes d^3 & \epsilon(d^3) = 1 & S(d^3) = a^3 = d^3,
\end{array}$$

one can easily see, for instance, that :

$$2\Delta(a^3 - \underline{1}) = (a^3 - \underline{1}) \otimes a^3 \oplus a^3 \otimes (a^3 - \underline{1}) \oplus (a^3 - \underline{1}) \otimes \underline{1} \oplus \underline{1} \otimes (a^3 - \underline{1}) \oplus 2b^3 \otimes c^3$$

belongs effectively to $\tilde{I} \otimes Fun(SL_q(2,C)) \oplus Fun(SL_q(2,C)) \otimes \tilde{I}$, and

$$\begin{aligned}
&\epsilon(a^3 - \underline{1}) = 0\\
&S(a^3 - \underline{1}) = d^3 - \mathbf{1} = a^3 - \underline{1} \in \tilde{I}.
\end{aligned}$$

At this level, one can explain the choice (6.2) for $\tilde{I}$ by the following :

$$a^3 + b^3 = \underline{1} \rightarrow S(a^3 + b^3) = a^3 - b^3 = S(\underline{1}) = \underline{1} \rightarrow a^3 = d^3 = \underline{1}\ ,\ b^3 = c^3 = 0.$$

Furthermore, $\mathcal{M}_3$ is a left and right comodule algebra over the quantum group G_3, i.e. a corepresentation space of G_3. This means that the right and left coactions :

$$\begin{aligned}
&\delta_L : \mathcal{M}_3 \rightarrow G_3 \otimes \mathcal{M}_3\\
&\delta_R : \mathcal{M}_3 \rightarrow \mathcal{M}_3 \otimes G_3
\end{aligned}$$

must satisfy, in addition to the homomorphism property (6.1), the following extra conditions :

$$\begin{array}{lcl}
(Id \otimes \delta_L)\delta_L(f) = (\Delta \otimes Id)\delta_L(f) & , & (\epsilon \otimes Id)\delta_L(f) = f\\
(\delta_R \otimes Id)\delta_R(f) = (Id \otimes \Delta)\delta_R(f) & , & (Id \otimes \epsilon)\delta_R(f) = f
\end{array}$$

and

$$\delta_{L,R}(\underline{1}) = \underline{1}_{\otimes}$$

Using one of these two coactions, one can build a set of generalized Gell-Mann matrices with entries in the quantum group G_3, by replacing x and y by $\delta_{L,R}(x)$ and $\delta_{L,R}(y)$ respectively in the expressions (2.5). For instance, one has :

$$\lambda'_3 = 1/3 \begin{pmatrix} (1-q^2)d + (1-q)d^2 & (1-q^2)c + (q^2-q)dc & (1-q)c^2 \\ (1-q)c^2 & (q-1)d + (q^2-1)d^2 & (1-q^2)c + (1-q^2)dc \\ (1-q^2)c + (q-1)dc & (1-q)c^2 & (q^2-q)d + (q-q^2)d^2 \end{pmatrix},$$

$$\lambda'_8 = -1/\sqrt{3} \begin{pmatrix} qd + q^2d^2 & qc - dc & q^2c^2 \\ q^2c^2 & q^2d + qd^2 & qc - qdc \\ qc - q^2dc & q^2c^2 & d + d^2 \end{pmatrix}.$$

To obtain the matrices $\lambda''_i = \delta_R(\lambda_i)$, it is sufficient to replace c by b and vice–versa in the corresponding matrices $\lambda'_i = \delta_L(\lambda_i)$. In our case, where $q^3 = 1$, there is only one *–Hopf structure on $Fun(SL_q(2,C))$ given by :

$$a^* = a \quad , \quad b^* = b \quad , \quad c^* = c \quad , \quad d^* = d. \tag{6.3}$$

VII. THE QUANTUM ALGEBRA $\mathcal{G}_3$

Let us now describe the quantum group $\mathcal{G}_3$ for which $\mathcal{M}_3$ is a module, i.e. a representation space. Being the dual of the quantum group G_3, it is clear that $\mathcal{G}_3$ is a vector space of dimension 27. So, $\mathcal{G}_3$ can be defined as a quotiented quantum universal enveloping algebra $U_q(sl(2))$, the dual of the quotiented quantum group $Fun(SL_q(2,C))$. By duality, one understands the interchange of product and co–product and vice–versa. Let us recall that $U_q(sl(2))$ is a complex associative algebra generated by $\{H^\alpha {X_+}^\beta {X_-}^\gamma \ , \alpha, \beta, \gamma \in Z\}$, where X_+, X_- and H are defined by duality by means of the following pairing between generators :

$$\begin{array}{llll} <H,a>=q \ , & <H,b>=0 \ , & <H,c>=0 \ , & <H,d>=q^2 \\ <H^{-1},a>=q^2 \ , & <H^{-1},b>=0 \ , & <H^{-1},c>=0 \ , & <H^{-1},d>=q \\ <X_+,a>=0 \ , & <X_+,b>=\alpha=1 \ , & <X_+,c>=0 \ , & <X_+,d>=0 \\ <X_-,a>=0 \ , & <X_-,b>=0 \ , & <X_-,c>=\alpha^{-1}=1 \ , & <X_-,d>=0 \end{array}$$

where $<Y,u>\in C$ means the evaluation of $Y \in U_q(sl(2))$ on $u \in Fun(SL_q(2))$. The second line of this table can be obtained from the first one and the fourth column can be obtained from the others. The c–number α is arbitrary, the only condition one finds is $<X_+,b><X_-,c>=1$. So, one can choose $<X_+,b>=\alpha=1$.

Using this duality, let us now define the bilateral ideal J, dual to $\tilde{I}$, by :

$$H^3 = \mathbf{1} \quad , \quad {X_\pm}^3 = 0.$$

Then, the Hopf algebra structure of $\mathcal{G}_3 = U_q(sl(2))/J$ is defined as follows :

Product: It is the (associative) application :

$$\begin{array}{rcl} \tilde{m} : \mathcal{G}_3 \otimes \mathcal{G}_3 & \rightarrow & \mathcal{G}_3 \\ X \otimes Y & \rightarrow & \tilde{m}(X \otimes Y) = XY \end{array}$$

$$\tilde{m} \circ (\tilde{m} \otimes Id) = \tilde{m} \circ (Id \otimes \tilde{m}) \quad \text{(associativity)}$$

which, using the co–product Δ in G_3, is defined by :

$$< \tilde{m}(X_1 \otimes X_2), u > \;=\; < X_1 X_2, u > \;=\; < X_1 \otimes X_2, \Delta(u) > \;=\; \sum_i < X_1, u^i_{(1)} >< X_2, u^i_{(2} > .$$

This leads to the following defining relations :

$$\begin{array}{rcl} HH^{-1} & = & H^{-1}H \;=\; \underline{1} \\ HX_{\pm} & = & q^{\pm 2} X_{\pm} H \\ X_{\pm}H^{-1} & = & q^{\pm 2} H^{-1} X_{\pm} \\ X_+X_- - X_-X_+ & = & \frac{H-H^{-1}}{q-q^{-1}}. \end{array}$$

Unit: It is the application :

$$\begin{array}{rcl} \tilde{\eta} : C & \rightarrow & \mathcal{G}_3 \\ k & \rightarrow & \tilde{\eta}(k) = k\underline{1} \end{array}$$

$$\tilde{m} \circ (\tilde{\eta} \otimes Id) = \tilde{m} \circ (Id \otimes \tilde{\eta}) = Id_{\mathcal{G}_3} = Id_{C \otimes \mathcal{G}_3} = Id_{\mathcal{G}_3 \otimes C}$$

which, using the co–unit ϵ in G_3, is defined by :

$$< \tilde{\eta}(1), u > \;=\; < \underline{1}, u > \;=\; \epsilon(u).$$

Co–product: It is the (co–associative) homomorphism :

$$\begin{array}{rcl} \tilde{\Delta} : \mathcal{G}_3 & \rightarrow & \mathcal{G}_3 \otimes \mathcal{G}_3 \\ X & \rightarrow & \tilde{\Delta}(X) = \sum_i X^i_{(1)} \otimes X^i_{(2)} \end{array}$$

$$(Id \otimes \tilde{\Delta}) \circ \tilde{\Delta} = (\tilde{\Delta} \otimes Id) \circ \tilde{\Delta} \quad \text{(co-associativity)}$$

which, using the product m in G_3, is defined by :

$$< \tilde{\Delta}(X), u_1 \otimes u_2 > \;=\; < X, u_1 u_2 > \;=\; < X, m(u_1 \otimes u_2) > \;=\; \sum_i < X^i_{(1)}, u_1 >< X^i_{(2)}, u_2 > .$$

This leads to the following relations :

$$\begin{array}{rcl} \tilde{\Delta}(H) & = & H \otimes H \\ \tilde{\Delta}(H^{-1}) & = & H^{-1} \otimes H^{-1} \\ \tilde{\Delta}(X_+) & = & X_+ \otimes \underline{1} + H \otimes X_+ \\ \tilde{\Delta}(X_-) & = & \underline{1} \otimes X_- + X^- \otimes H_{-1} \\ \tilde{\Delta}(\underline{1}) & = & \underline{1} \otimes \underline{1}. \end{array}$$

Co–unit: It is the homomorphism :

$$\tilde{\epsilon} : \mathcal{G}_3 \rightarrow C$$

$$(\tilde{\epsilon} \otimes Id) \circ \tilde{\Delta} = (Id \otimes \tilde{\epsilon}) \circ \tilde{\Delta} = Id_{\mathcal{G}_3}$$

which, using the unit η in G_3, is defined by :

$$< X, \underline{1} > \;=\; < X, \eta(1) > \;=\; \tilde{\epsilon}(X).$$

This leads to the following relations :

$$\begin{array}{rcl} \tilde{\epsilon}(H) & = & 1 \\ \tilde{\epsilon}(H^{-1}) & = & 1 \\ \tilde{\epsilon}(X_+) & = & 0 \\ \tilde{\epsilon}(X_-) & = & 0 \\ \tilde{\epsilon}(\underline{1}) & = & 1. \end{array}$$

Antipode: It is the anti–homomorphism :

$$\tilde{S} : \mathcal{G}_3 \rightarrow \mathcal{G}_3$$

$$\tilde{m} \circ (\tilde{S} \otimes Id) \circ \tilde{\Delta} = \tilde{m} \circ (Id \otimes \tilde{S}) \circ \tilde{\Delta} = \tilde{\eta} \circ \tilde{\epsilon}$$

which, using the antipode S in G_3, is defined by :

$$< \tilde{S}(X), u > \quad = \quad < X, S(u) > .$$

This leads to the following equations:

$$\begin{aligned} \tilde{S}(H) &= H^{-1} \\ \tilde{S}(H^{-1}) &= H \\ \tilde{S}(X_+) &= -H^{-1}X_+ \\ \tilde{S}(X_-) &= -X_-H \\ \tilde{S}(\underline{1}) &= \underline{1}. \end{aligned}$$

In addition, one has :

$$S^2(X) \; = \; HXH^{-1} \quad , X \in \mathcal{G}_3$$

which means that :

$$\begin{aligned} S^2(H) &= H \\ S^2(H^{-1}) &= H^{-1} \\ S^2(X_+) &= q^2X_+ \\ S^2(X_-) &= q^{-2}X_-. \end{aligned}$$

Finally, $\mathcal{G}_3$, as a vector space over C, is spanned by the 27 elements $\{H^{\alpha}X_+^{\beta}X_-^{\gamma} \quad , \quad \alpha, \beta, \gamma \; = 0, 1, 2\}$. Note also that the element $\tilde{D} = X_+X_- - \frac{1}{3}(q^{-1}H + qH^{-1})$ is central and therefore plays the role of the usual Casimir operator.
Now, let us discuss the action of $\mathcal{G}_3$ on $\mathcal{M}_3$. In fact, since the quantum group G_3 coacts on $\mathcal{M}_3$, there is a natural definition of an action of the quantum algebra $\mathcal{G}_3$ on $\mathcal{M}_3$. For instance, using the pairing and the right coaction δ_R of G_3 on $\mathcal{M}_3$, the *left action* X_L of $\mathcal{G}_3$ on $\mathcal{M}_3$ is defined by :

$$\begin{aligned} X_L(z) &= (Id \otimes < X_L, . >) \circ \delta_R(z) \\ &= (Id \otimes < X_L, . >)(\textstyle\sum_i z_i \otimes u_i) \\ &= \textstyle\sum_i < X_L, u_i > z_i, \end{aligned}$$

for $z, z_i \in \mathcal{M}_3, X_L \in \mathcal{G}_3, u_i \in G_3$.
This gives the following results :

$$\begin{aligned} X_L(\underline{1}) &= \tilde{\epsilon}(X_L)\underline{1} \\ X_L(x) &= < X_L, a > x + < X_L, c > y \\ X_L(y) &= < X_L, b > x + < X_L, d > y \end{aligned}$$

and then, one obtains the following table :

Left	H	H^{-1}	X_+	X_-
$\underline{1}$	$\underline{1}$	$\underline{1}$	0	0
x	qx	q^2x	0	y
y	q^2y	qy	x	0

Using the pairing and the left coaction δ_L of G_3 on $\mathcal{M}_3$, one can also define the *right action* of $\mathcal{G}_3$ on $\mathcal{M}_3$ as follows :

$$\begin{aligned} X_R(z) &= (< X_R, . > \otimes Id) \circ \delta_L(z) \\ &= (< X_R, . > \otimes Id)(\textstyle\sum_i u_i \otimes z_i) \\ &= \textstyle\sum_i < X_R, u_i > z_i, \end{aligned}$$

with $z, z_i \in \mathcal{M}_3, X_R \in \mathcal{G}_3, u_i \in G_3$.
This leads to the following equations :

$$\begin{aligned} X_R(\underline{1}) &= \tilde{\epsilon}(X_R)\underline{1} \\ X_R(x) &= < X_R, a > x + < X_R, b > y \\ X_R(y) &= < X_R, c > x + < X_R, d > y \end{aligned}$$

and then, to the following table :

Right	H	H^{-1}	X_+	X_-
$\underline{1}$	$\underline{1}$	$\underline{1}$	0	0
x	qx	q^2x	y	0
y	q^2y	qy	0	x

To obtain the (left or right) action of $\mathcal{G}_3$ on any generator $x^m y^n$ of $\mathcal{M}_3$, one may use the above definition to deduce the following property :

$$\begin{aligned} X_L(f.g) &= (Id \otimes < X_L, . >)\delta_R(f).\delta_R(g) = \textstyle\sum_{i,j} f_i^R.g_j^R < \tilde{\Delta}(X_L), u_i^R \otimes v_j^R > \\ X_R(f.g) &= (< X_R, . > \otimes Id)\delta_L(f).\delta_L(g) = \textstyle\sum_{i,j} < \tilde{\Delta}(X_R), u_i^L \otimes v_j^L > f_i^L.g_j^L \end{aligned}$$

where $f, g, f_i^{(L,R)}, g_j^{(L,R)} \in \mathcal{M}_3$, $u_i^{(L,R)}, v_j^{(L,R)} \in G_3$, $X_{L,R} \in \mathcal{G}_3$ and

$$\begin{cases} \delta_L(f) = \sum_i u_i^L \otimes f_i^L \quad , \quad \delta_L(g) = \sum_j v_j^L \otimes g_j^L \\ \delta_R(f) = \sum_i f_i^R \otimes u_i^R \quad , \quad \delta_R(g) = \sum_j g_j^R \otimes v_j^R \end{cases}$$

Then, one obtains :

$$X_L(x^m) = \begin{cases} \tilde{\epsilon}(X_L)\underline{1} & \text{for m} = 0 \ \text{mod}[3] \\ < X_L, a > x + < X_L, c > y & \text{for m} = 1 \ \text{mod}[3] \\ < \tilde{\Delta}(X_L), a \otimes a > x^2 - & \\ q^2 < \tilde{\Delta}(X_L), a \otimes c > xy + & \\ < \tilde{\Delta}(X_L), c \otimes c > y^2 & \text{for m} = 2 \ \text{mod}[3] \end{cases}$$

$$X_L(y^n) = \begin{cases} \tilde{\epsilon}(X_L)\underline{1} & \text{for n} = 0 \ \text{mod}[3] \\ < X_L, b > x + < X_L, d > y & \text{for n} = 1 \ \text{mod}[3] \\ < \tilde{\Delta}(X_L), b \otimes b > x^2 - & \\ q^2 < \tilde{\Delta}(X_L), b \otimes d > xy + & \\ < \tilde{\Delta}(X_L), d \otimes d > y^2 & \text{for n} = 2 \ \text{mod}[3] \end{cases}$$

$$X_R(x^m) = \begin{cases} \tilde{\epsilon}(X_R)\underline{1} & \text{for m} = 0 \ \text{mod}[3] \\ < X_R, a > x + < X_R, b > y & \text{for m} = 1 \ \text{mod}[3] \\ < \tilde{\Delta}(X_R), a \otimes a > x^2 - & \\ q^2 < \tilde{\Delta}(X_R), a \otimes b > xy + & \\ < \tilde{\Delta}(X_R), b \otimes b > y^2 & \text{for m} = 2 \ \text{mod}[3] \end{cases}$$

$$X_R(y^n) = \begin{cases} \tilde{\epsilon}(X_R)\underline{1} & \text{for n} = 0 \ \text{mod}[3] \\ < X_R, c > x + < X_R, d > y & \text{for n} = 1 \ \text{mod}[3] \\ < \tilde{\Delta}(X_R), c \otimes c > x^2 - & \\ q^2 < \tilde{\Delta}(X_R), c \otimes d > xy + & \\ < \tilde{\Delta}(X_R), d \otimes d > y^2 & \text{for n} = 2 \ \text{mod}[3] \end{cases}$$

and finally, one obtains the following tables :

$Left$	$H^{(L)}$	$(H^{-1})^{(L)}$	$X_+^{(L)}$	$X_-^{(L)}$
x^2	q^2x^2	qx^2	0	$-q^2xy$
y^2	qy^2	q^2y^2	$-q^2xy$	0
xy	xy	xy	qx^2	qy^2
$\underline{1}$	$\underline{1}$	$\underline{1}$	0	0
xy^2	q^2xy^2	qxy^2	$-x^2y$	$q^2\underline{1}$
x^2y	qx^2y	q^2x^2y	$q^2\underline{1}$	$-xy^2$
x	qx	q^2x	0	y
y	q^2y	qy	x	0
x^2y^2	x^2y^2	x^2y^2	$-qy$	$-qx$

$Right$	$H^{(R)}$	$(H^{-1})^{(R)}$	$X_+^{(R)}$	$X_-^{(R)}$
x^2	q^2x^2	qx^2	$-xy$	0
y^2	qy^2	q^2y^2	0	$-xy$
xy	xy	xy	y^2	x^2
$\underline{1}$	$\underline{1}$	$\underline{1}$	0	0
xy^2	q^2xy^2	qxy^2	$\underline{1}$	$-x^2y$
x^2y	qx^2y	q^2x^2y	$-xy^2$	$\underline{1}$
x	qx	q^2x	y	0
y	q^2y	qy	0	x
x^2y^2	x^2y^2	x^2y^2	$-x$	$-y$

As $q^3 = 1$, we stick to the $G_3 = Fun(SL_q(2, R))/\tilde{I}$ case since it is the only one compatible with a complex q (see (6.3)). We therefore define the star involution on $\mathcal{M}_3$ by :

$$x^* = x \quad , \quad y^* = y.$$

Finally, using the pairing between G_3 and $\mathcal{G}_3$, one can obtain the corresponding involution in $\mathcal{G}_3$. In order to get a $*$–Hopf structure in the dual $\mathcal{G}_3$ of G_3, one should have,[21] :

$$< X^*, u >=< X, S(u)^* >^{cc} \quad , \quad X \in \mathcal{G}_3, u \in G_3$$

where "cc" means complex conjugate. This leads to the following star in $\mathcal{G}_3$:

$$H^* = H \ , \ (H^{-1})^* = H^{-1} \ , \ X_+^* = -q^2 X_+ \ , \ X_-^* = -qX_-.$$

VIII. THE GRASSMANNIAN MANIN PLANE $\tilde{\mathcal{M}}_3$

We define the Grassmannian Manin plane $\tilde{\mathcal{M}}_3$ associated with $\mathcal{M}_3$ as the algebra generated over C by the element ξ_1 and ξ_2 satisfying the relations:

$$\begin{gathered} \xi_1\xi_2 + q^{-1}\xi_2\xi_1 = \xi_1\xi_2 + q^2\xi_2\xi_1 = 0 \\ \xi_1^2 = \xi_2^2 = 0 \end{gathered}$$

It is clear that there is no need to introduce additional relations for ξ_1 and ξ_2 corresponding to the ideal I in $\mathcal{M}_3$ ([21]).
The quantum group G_3 and its dual $\mathcal{G}_3$ coact and act respectively on the left and on the right on $\tilde{\mathcal{M}}_3$ exactly in the same way as they do on $\mathcal{M}_3$. The only difference is that $\tilde{\mathcal{M}}_3$ is now, as a vector space, of dimension 3 since it is only generated by $\{\xi_1, \xi_2$ and $\xi_1\xi_2\}$.
For instance, one has :

$$\begin{aligned} \delta_L \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} &= \begin{pmatrix} a \otimes \xi_1 + b \otimes \xi_2 \\ c \otimes \xi_1 + d \otimes \xi_2 \end{pmatrix} \\ X_-^L(\xi_1) &= \xi_2 \end{aligned}$$

IX. THE WESS–ZUMINO COMPLEX $\Omega_{WZ}(\mathcal{M}_3)$

One constructs the universal differential algebra $\Omega(\mathcal{M}_3)$ as follows. Let μ be the multiplication map :

$$\mu(z_1 \otimes z_2) = z_1 z_2 \quad , \quad \text{for } z_1 \text{ and } z_2 \in \mathcal{M}_3$$

and let $\ker(\mu)$ be the kernel of this map, i.e:

$$z_1 \otimes z_2 \in \ker(\mu) \Longrightarrow z_1 z_2 = 0.$$

Then, the algebra $\Omega(\mathcal{M}_3)$ of universal forms is the graded vector space defined by:

$$\Omega(\mathcal{M}_3) = \oplus_{p=0}^{\infty}\Omega^p(\mathcal{M}_3)$$

with:

$$\begin{aligned}
&\Omega^0(\mathcal{M}_3) \equiv \mathcal{M}_3\\
&\Omega^1(\mathcal{M}_3) = \ker(\mu) \subset \mathcal{M}_3 \otimes \mathcal{M}_3\\
&\Omega^p(\mathcal{M}_3) = \Omega^1(\mathcal{M}_3) \otimes_{\mathcal{M}_3} \Omega^1(\mathcal{M}_3) \otimes_{\mathcal{M}_3} \ldots\ldots\ldots \otimes_{\mathcal{M}_3} \Omega^1(\mathcal{M}_3) \qquad \text{(p times)}
\end{aligned}$$

We shall write:

$$\xi_1 = dx \quad , \quad \xi_2 = dy$$

so $\tilde{\mathcal{M}_3}$ is defined by these two generators and the relations:

$$(dx)^2 = 0 \quad , \quad (dy)^2 = 0 \quad , \quad dxdy + q^2 dydx = 0$$

Since $\Omega^1(\mathcal{M}_3) = ker(\mu)$, one finds that:

$$\dim(\Omega^1(\mathcal{M}_3)) = \dim(\mathcal{M}_3 \otimes \mathcal{M}_3) - \dim(\mathcal{M}_3) = 9^2 - 9 = 72$$

Hereunder, we present the Wess–Zumino complex $\Omega_{WZ}(\mathcal{M}_3)$,[19]. It is the only one differential algebra on the quantum plane which is both quadratic and compatible with the coaction of the quantum group Fun$(SL_q(2, C))$.
It is constructed as follows :

- $\Omega_{WZ} = \Omega^0_{WZ} \oplus \Omega^1_{WZ} \oplus \Omega^2_{WZ}$ is a graded vector space
- $\Omega^0_{WZ} =$ {functions on the quantum plane, i.e. the polynomials in x and y}
- $\Omega^1_{WZ} = \{a_{rs}x^r y^s dx + b_{rs}x^r y^s dy\}$
- $\Omega^2_{WZ} = \{c_{rs}x^r y^s dxdy\}$

$\tilde{\mathcal{M}_3}$ is the sub–algebra $\{\lambda_{00} + \lambda_{10}dx + \lambda_{01}dy + \lambda_{22}dxdy\}$ of Ω_{WZ} where λ_{ij} belongs to the field of scalars. Ω_{WZ} is an algebra, and moreover a differential algebra. Indeed, setting :

$$\begin{cases} d(x) = dx \quad , \quad d(y) = dy \\ d \text{ satisfy Leibniz rule} \\ d\mathbf{1} = 0 \\ d^2 = 0 \end{cases}$$

one obtains the following relations between x, y, dx and dy are :

$$\begin{cases} xy = qyx \quad , \quad ydx = qdxy \\ xdx = q^2dxx \quad , \quad ydy = q^2dyy \\ xdy = qdyx + (q^2-1)dxy \\ (dx)^2 = 0 \quad , \quad (dy)^2 = 0 \\ dxdy + q^2dydx = 0 \end{cases}$$

$$\begin{cases} \dim\Omega^0_{WZ}(\mathcal{M}_3) = 9 \\ \dim\Omega^1_{WZ}(\mathcal{M}_3) = 18 \\ \dim\Omega^2_{WZ}(\mathcal{M}_3) = 9. \end{cases}$$

X. CONCLUSION

In this work, one has investigated the non–commutative differential geometry of $M_N(C)$ and in particular $M_3(C)$.
In the last section, the dimension of $\Omega(\mathcal{M}_3)$ itself is infinite ($\Omega^p(\mathcal{M}_3)$ never vanishes)!
Moreover, it does not remember anything of the coaction of G_3 on $\mathcal{M}_3$ (the 0–forms).
So, it is more convenient to build another differential algebra as the quotient of the universal one by some differentiable ideal. In literature, one finds several types of these differential algebras, see[22],[23],[24] and[19]. To complete this study, it remains to give the general expression of the Laplacian Δ acting on an arbitrary p–form belonging to $\Omega^p_{Der}(\mathcal{M}_N)$ and then to compute the spectrum of Δ for an arbitrary N and in particular for $N = 2, 3$. we plane to treat this aspect in a future paper.

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