



Higher-Order Supersymmetric Quantum Mechanics and Infinite Well Potential

J. Sadeghi ^{a,c,*}, A. R. Amani ^b and B. Pourhassan ^{c,†}

^a *Institute for Studies in Theoretical Physics and Mathematics (IPM),
P.O.Box 19395-5531, Tehran, Iran*

^b *Sciences Faculty Department of Physics, Ayatollah Amoli, Azad Univesity,
P. O. Box 678, Amol, Iran*

^c *Sciences Faculty, Department of Physics, Mazandaran University,
P.O.Box 47415-416, Babolsar, Iran*

Abstract

The higher - order supersymmetric quantum mechanics is very important because that involves to differential operators of order greater than one. We use the iterations of first - order supersymmetry transformation which are simple way for obtaining to the higher - order case. We use the higher order supersymmetry and obtain the corresponding partner hamiltonian for infinite well potential. In order to obtain the hamiltonian partner we introduce the supercharges which are corresponding to the raising and lowering operators for higher order supersymmetry.

PACS Nos: 21.60.Cs; 21.60.Fw

Keywords: Well Potential; Higher Supersymmetry; Operators; Supercharges

We should stress that supersymmetry in quantum mechanic by factorization method is based upon the framework of shape invariance . If one quantum mechanic problem obtains supersymmetry framework , we must then factorize hamiltonian equation quantum states in terms of a multiplication differential operator equation order one as the shape invariance equation. In this approach, hamiltonian is decomposed once in successive multiplication of raising and lowering operators and also again in successive multiplication of lowering and raising operators. In such a way that corresponding quantum states of successive levels, are eigen - states corresponding to them. These hamil-

tonian are called partner and supersymmetry of each other. In fact, the three sperate subject of factorization method, supersymmetry quantum mechanic and shape invariance nowadays converged at one point. Also we know that the most of exactly solvable problems can be dealt with the factorization method. Initially the factorization method have suggested by Darboux [1], and later the application of this method was provided by Schrödinger in quantum mechanic [2]. Infeld and Hull in their review article [3] showed a large variety of second - order differential equations with boundary conditions set in six different types of factorization. The idea of supersymmetry in context of quantum mechanics were first studied by Nicolai and witten and later Cooper and freedman et al[4]. The resulting supersymmetric quantum mechanics revived the study of exactly solvable hamiltonians [5]. An additional step was Mielnik's factorization through which the general supersym-

* pouriya@ipm.ir

† bpourhassan@yahoo.com

© a GNPHE publication 2005, *ajmp@fsr.ac.ma*

metry partner for the oscillator for a certain factorization energy was found [6], this method immediately applied to the hydrogen potential[7]. Meanwhile Nieto [8], Andianov et al [9] and Sukumar [10-14] put the method on its natural background discovering the link between supersymmetry, factorization and Darboux method [15-22] for recent review see [23]. These procedures can be recovered from a general setting in which a first - order differential operator intertwines two hamiltonians [24]. This so - called first - order intertwining method suggests further generalizations, the most obvious one involves a k-th quantum mechanics .It can be achieved by iterations of first order supersymmetry transformation . Now we are ready to apply those method in infinite well potential.

Let us start with two Schrödinger hamiltonians

$$H_i = -\frac{1}{2} \frac{d^2}{dx^2} + V_i(x) \quad i = 0, 1, \quad (1)$$

and suppose the existence of a first order differential operator A_1^+ intertwining them

$$H_1 A_1^+ = A_1^+ H_0, \quad (2)$$

where

$$A_1^+ = \frac{1}{\sqrt{2}} \left(-\frac{d}{dx} + W_1(x) \right). \quad (3)$$

From equation (1), (2) and (3) we obtain obviously

$$\begin{aligned} \sqrt{2} H_1 A_1^+ &= \frac{1}{2} \frac{d^3}{dx^3} - \frac{W_1}{2} \frac{d^2}{dx^2} \\ &\quad - (V_1 + W_1') \frac{d}{dx} + W_1 V_1 - \frac{W_1''}{2} \\ \sqrt{2} A_1^+ H_0 &= \frac{1}{2} \frac{d^3}{dx^3} - \frac{W_1}{2} \frac{d^2}{dx^2} \\ &\quad - V_0 \frac{d}{dx} + W_1 V_0 - V_0'. \end{aligned} \quad (4)$$

By using equation (4) and evaluating the coefficient $\frac{d}{dx}$ and also integrating , one obtain the following expression

$$W_1' + W_1^2 = 2(V_0 - \epsilon_1). \quad (5)$$

We express the explicitly form of superpotential in terms of the factorization energy ϵ in the way $W_1(x, \epsilon_1)$. If a function $\Psi^{(0)}(x)$ such that $W_1(x, \epsilon) = \frac{\Psi^{(0)'}}{\Psi^{(0)}}$ is used, we finally have the following,

$$-\frac{1}{2} \Psi_0^{(0)''} + V_0 \Psi_0^{(0)} = \epsilon_1 \Psi_0^{(0)}, \quad (6)$$

and

$$V_1 = V_0 - \left(\frac{\Psi_0^{(0)'}}{\Psi_0^{(0)}} \right)'. \quad (7)$$

As we know $\Psi^{(0)}$ is a solution of the initial stationary Schrödinger equation associated to ϵ . The equations (1) (2) ,(3) and(7) lead us to factorized the H_0 and H_1 ;

$$H_0 = A_1 A_1^+ + \epsilon_1 \quad H_1 = A_1^+ A_1 + \epsilon_1. \quad (8)$$

Now we are going to use the iteration method partner for the infinite well potential and also obtain the higher - order supersymmetry for the corresponding potential. In order to obtain the higher - order supersymmetry , we consider the following potential,

$$\begin{aligned} V_0(x) &= 0 \quad 0 \leq x \leq L \\ &= \infty \quad otherwise, \end{aligned} \quad (9)$$

and the wave function and eigen value for the above potential are;

$$\begin{aligned} \Psi_n^{(0)}(x) &= \sqrt{\frac{2}{L}} \sin \frac{(n+1)\pi x}{L} \\ E_n^{(0)} &= \frac{\pi^2}{L^2} (n+1)^2, \end{aligned} \quad (10)$$

where $n = 0, 1, \dots$, and for simplicity we take $\hbar^2 = 2m = 1$.

In here we want to obtain the higher order potential $V_1(x), V_2(x), \dots$, so we need to know the W and ϵ (which are superpotential and energy spectrum respectively). The superpotential is very important because it will be correspond to laddering operators A^+ and A .

From equations (6) and (7) we first obtain the $W_1(x, \epsilon_1)$, ϵ_1 and V_1 respectively

$$\begin{aligned} W_1(x, \epsilon_1) &= \frac{\pi}{L} \cot \frac{\pi x}{L} \\ \epsilon_1 &= \frac{1}{2} \left(\frac{\pi}{L} \right)^2 \leq E_0^{(0)} \\ V_1(x) &= \left(\frac{\pi}{L} \right)^2 \csc^2 \frac{\pi x}{L}, \end{aligned} \quad (11)$$

and also H_0, H_1, A_1^+ and A_1 , are;

$$\begin{aligned} H_0 &= -\frac{1}{2} \frac{d^2}{dx^2} \\ H_1 &= -\frac{1}{2} \frac{d^2}{dx^2} + \left(\frac{\pi}{L} \right)^2 \csc^2 \left(\frac{\pi}{L} \right) \\ A_1^+ &= \frac{1}{\sqrt{2}} \left(-\frac{d}{dx} + \frac{\pi}{L} \cot \frac{\pi x}{L} \right) \\ A_1 &= \frac{1}{\sqrt{2}} \left(\frac{d}{dx} + \frac{\pi}{L} \cot \frac{\pi x}{L} \right). \end{aligned} \quad (12)$$

Now we are going to higher - order wave function and we also use the following expression;

$$\begin{aligned} \Psi_n^{(1)}(x) &= \frac{A_1^+ \Psi_n^{(0)}(x)}{\sqrt{E_n - \epsilon_1}} \\ &= \frac{1}{\sqrt{E_n - \epsilon_1}} \frac{1}{\sqrt{2}} \left(-\frac{d}{dx} + \frac{\pi}{L} \cot \frac{\pi x}{L} \right) \\ &\quad * \sqrt{\frac{2}{L}} \sin \frac{(n+1)\pi x}{L}, \end{aligned} \tag{13}$$

so we have

$$\begin{aligned} \Psi_n^{(1)}(x) &= \frac{\pi}{L\sqrt{E_n - \epsilon_1}} \frac{1}{\sqrt{2}} \\ &\quad * \left[-(n+1) \cos \frac{(n+1)\pi x}{L} \right. \\ &\quad \left. + \cot \frac{\pi x}{L} \sin \frac{(n+1)\pi x}{L} \right]. \end{aligned} \tag{14}$$

In order to obtain the W_2 we need to know the $\Psi_1^{(1)}$ as following,

$$W_2(x, \epsilon_2) = \frac{\Psi_1^{(1)'}}{\Psi_1^{(1)}}, \tag{15}$$

we obtain

$$W_2(x, \epsilon_2) = 2 \frac{\pi}{L} \cot \left(\frac{\pi x}{L} \right). \tag{16}$$

In here we also obtain the corresponding potential $V_2(x)$

$$\begin{aligned} V_2(x) &= V_1(x) - W_2(x, \epsilon_2)' \\ &= 3 \left(\frac{\pi}{L} \right)^2 \csc^2 \left(\frac{\pi x}{L} \right). \end{aligned} \tag{17}$$

According to following relation one can obtain the ϵ_2

$$H_0 \Psi_1^{(0)}(x) = \epsilon_2 \Psi_1^{(0)}(x), \tag{18}$$

so we have

$$\epsilon_2 = 2 \left(\frac{\pi}{L} \right)^2, \tag{19}$$

We note that in this case the W_1 can be function of both ϵ_1 and ϵ_2 which obtain the following form,

$$\begin{aligned} \Psi_0^{(0)}(x) &= \exp \left(-\int_0^x W_1(x, \epsilon_1) dx \right) \\ \Psi_1^{(0)}(x) &= \exp \left(-\int_0^x W_1(x, \epsilon_2) dx \right) \end{aligned} \tag{20}$$

In fact $W_1(x, \epsilon_1)$ is known before as function of ϵ_1 but we also need to obtain the $W_1(x, \epsilon_2)$.

From equation (20) we obtain the $W_1(x, \epsilon_2)$ and the corresponding hamiltonian as the following,

$$W_1(x, \epsilon_2) = \frac{2\pi}{L} \cot \frac{2\pi x}{L}, \tag{21}$$

and

$$H_2 = -\frac{1}{2} \frac{d^2}{dx^2} + 3 \left(\frac{\pi}{L} \right)^2 \csc^2 \left(\frac{\pi x}{L} \right). \tag{22}$$

For the continue to do process we first to calculate the higher order wave function which is obtained by the raising operator as a following

$$\Psi_n^{(2)}(x) = \frac{1}{\sqrt{E_n - \epsilon_2}} A_2^+ \Psi_n^{(1)}(x), \tag{23}$$

where

$$A_2^+ = \frac{1}{\sqrt{2}} \left(-\frac{d}{dx} + \frac{2\pi}{L} \cot \frac{\pi x}{L} \right). \tag{24}$$

In here by using the $\Psi_n^{(2)}(x)$ we also obtain the W_3

$$W_3(x, \epsilon_3) = \left(\frac{\Psi_2^{(2)'}}{\Psi_2^{(2)}}(x) \right) = \frac{3\pi}{L} \cot \frac{\pi x}{L}, \tag{25}$$

by obtaining the W_3 we can write the clear form of $V_3(x)$ which is the following

$$V_3(x) = 6 \left(\frac{\pi}{L} \right)^2 \csc^2 \frac{\pi x}{L}. \tag{26}$$

If we continue this process we obtain the general form of potential which is the following,

$$V_n(x) = n \left(\frac{\pi}{L} \right)^2 \csc^2 \frac{\pi x}{L} + V_{n-1}(x). \tag{27}$$

And also one can obtain the operators A_n, A_n^+ and W_n which are helping us to obtain the higher order operators. So we are going to discuss the higher order of supersymmetry and supercharge which is obtained by

$$\begin{aligned} B_n^+ &= A_n^+ \cdots A_1^+ \\ B_n &= A_1 \cdots A_n, \end{aligned} \tag{28}$$

as we know the A_n and A_n^+ in case of infinite well potential

$$\begin{aligned} A_n &= \frac{1}{\sqrt{2}} \left(\frac{d}{dx} + \frac{n\pi}{L} \cot \frac{\pi x}{L} \right) \\ A_n^+ &= \frac{1}{\sqrt{2}} \left(-\frac{d}{dx} + \frac{n\pi}{L} \cot \frac{\pi x}{L} \right), \end{aligned} \tag{29}$$

and

$$W_n(x) = n \frac{\pi}{L} \cot \frac{\pi x}{L}, \tag{30}$$

by using equation (29) one can obtain the

$$B_n^+ = \left(\frac{1}{\sqrt{2}}\right)^n \left[-\frac{d}{dx} + \frac{n\pi}{L} \cot \frac{\pi x}{L}\right] \cdots \left[-\frac{d}{dx} + \frac{\pi}{L} \cot \frac{\pi x}{L}\right]$$

$$B_n = \left(\frac{1}{\sqrt{2}}\right)^n \left[\frac{d}{dx} + \frac{\pi}{L} \cot \frac{\pi x}{L}\right] \cdots \left[\frac{d}{dx} + \frac{n\pi}{L} \cot \frac{\pi x}{L}\right]$$

These operators lead to the higher - order supersymmetry quantum mechanics and also we have to introduce the supercharges Q and Q^+ which is the following,

$$Q = \begin{pmatrix} 0 & 0 \\ B_n & 0 \end{pmatrix}, \quad Q^+ = \begin{pmatrix} 0 & B_n^+ \\ 0 & 0 \end{pmatrix}. \quad (32)$$

In this treatment these supercharges give us the standard supersymmetry algebra with two generators,

$$[Q_i, H_{ss}] = 0, \quad \{Q_i, Q_j\} = \delta_{ij} H_{ss}, \quad i, j = 1, 2, \quad (33)$$

where

$$Q_1 = \frac{(Q^+ + Q)}{\sqrt{2}}, \quad Q_2 = i \frac{(Q - Q^+)}{\sqrt{2}}. \quad (34)$$

By using the equation (33) and operators B_n and B_n^+ one can obtain the supersymmetry quantum mechanics hamiltonian,

$$H_{ss} = \begin{pmatrix} B_n^+ B_n & 0 \\ 0 & B_n B_n^+ \end{pmatrix}, \quad (35)$$

and the higher order hamiltonian also obtain as the following

$$H_n B_n^+ = B_n^+ H_n, \quad (36)$$

where

$$H_n = A_n^+ A_n + \epsilon_n, \quad (37)$$

and the ϵ_n will be obtain by following formula,

$$H_0 \Psi_n^{(0)}(x) = \epsilon_n \Psi_n^{(0)}(x), \quad (38)$$

where

$$\epsilon_n = \frac{1}{2} n^2 \left(\frac{\pi}{L}\right)^2. \quad (39)$$

From equation (37) and (39) we also obtain the n-th hamiltonian,

$$H_n = \frac{1}{2} \left[-\frac{d^2}{dx^2} + (n+1)n \left(\frac{\pi}{L}\right)^2 \csc^2 \frac{\pi x}{L}\right], \quad (40)$$

by using the hamiltonian one can obtain the higher order hamiltonian for the infinite - well potential,

$$H_s^p = \begin{pmatrix} H_n & 0 \\ 0 & H_0 \end{pmatrix}. \quad (41)$$

also we conclude that the mentioning to this kind of higher supersymmetry method in non - linear equation in mathematical physics will be good progress in future and also very fruitful along the research. Also by using this method we can solve the other system as non-linear differential equation which is corresponding to non - linear potential.

References

- ¹ G. Darboux, "Théorie générale des Surfaces Vol. 2," (Paris, Gaut hier-Villars, 1896).
- ² E. Schrödinger, Proc. Roy. Irish Acad. A **46**, 9 (1940); *ibid.* **46**, 183 (1941); *ibid.* **47**, 53 (1941).
- ³ L. Infeld and T. E. Hull, Rev. Mod. Phys. **23**,
- ⁴ H. Nicolai, J. Phys. A, Math. Gen. **9**, 1497 (1976); E. Witten, Nucl. Phys. B **185**, 513 (1981); *ibid.* **202**, 253 (1982); F. Cooper and B. Freedman, Ann. Phys. (N.Y.) **146**, 262 (1983).
- ⁵ L. F. Urrutia, E. Hernández, Phys. Rev. Lett. **51**, 755 (1983).
- ⁶ B. Mielnik, J. Math. Phys. Rev. **25**, 3387 (1984).
- ⁷ D. J. Fernández, Lett. Math. Phys. **8**, 337 (1984).
- ⁸ M. M. Nieto, Phys. Lett. **145B**, 208(1984).
- ⁹ A. A. Andrianov, N. B. Borisov, M. V. Ioffe, Phys. Lett. **105A** 19 (1984).
- ¹⁰ C. V. Sukumar, J. Phys. **A 18** L57 (1985).
- ¹¹ C. V. Sukumar, J.Phys. **A 18** 2917 (1985).
- ¹² C. V. Sukumar, J. Phys. **A 18** 2937 (1985).
- ¹³ C. V. Sukumar, J. Phys. **A 19** 2297 (1986).
- ¹⁴ C. V. Sukumar, J. Phys. **A 20** 2461 (1987).
- ¹⁵ A. Frank, K. B. Wolf, J. Math. Phys. **26** 973 (1985).
- ¹⁶ Z. Dongpei, J. Phys. **A 20** 4331 (1987).
- ¹⁷ J. Beckers, D. Dehin, V. Hussin, J. Phys. **A 20** 1137 (1987).
- ¹⁸ J. Beckers, D. Dehin, V. Hussin, J. Phys. **A 21** 3215 (1988).
- ¹⁹ N. A. Alves, E. Drigo - Filho, J. Phys. **A 21** 3215(1988).
- ²⁰ E. Drigo - Filho, Eur. J. Phys. **A 21**, L1025 (1988).
- ²¹ L. J. Boya, Eur. J. Phys. **9** 139 (1988).
- ²² E. Drigo- Filho, R. M. Riccota, Mod. Phys. Lett. **A4** 2283 (1989).
- ²³ B. Mielnik, O. Rosas-Ortiz, J. Phys. **A. 37**, 10007 (2004).
- ²⁴ P. A. Deift, Duke, Math.**J. 45**, 276 (1978).