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Polarized Poisson manifolds and its generalizations

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Abstract

We introduce and study the basic notions of polarized Poisson manifold and the vectorial polarized Poisson manifold which give a sense to the Poisson structure subordinate to a polarized manifold in the sense of P.Molino and to a k-symplectic manifold respectively, and which generalize the classical case of the Poisson manifold. And also, we show that for any vectorial polarized Hamiltonian mapping, the associated generalized Hamiltonian system of Nambu and the polarized Hamiltonian system are connected by relations characterizing the mechanical aspect of the $k-$*symplectic geometry.*

Keywords: *Polarized manifold. Polarized Poisson manifold. k-symplectic structure. Vector polarized Poisson manifold. Poisson manifold. Hamiltonian system. Generalized Hamiltonian system of Nambu.*

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I. INTRODUCTION

A polarized structure on an even dimensional smooth manifold M is a pair (θ, E) constituted by a closed differential $2-$form θ of maximum rank and by an $n-$codimensional integrable subbundle E of TM which is Lagrangian with respect to the $2-$form θ.

The notion of a polarized manifold plays an important role in theory of the geometric quantization of Kostant-Souriau, see for example P.Molino ([8]) and N.Woodhouse[13]). Interesting properties were putting in evidence by A. Weinstein, P. Dazord, J.M. Morvan, P. Molino, etc,...

The polarized Hamiltonian mappings of (θ, E) constitute a submodule $\mathcal{H}(M, \mathcal{F})$ of $\mathcal{C}^{\infty}(M)$ of smooth real functions on M over the ring of basic functions for $\mathcal{F}$. The Poisson tensor P associated to the symplectic structure θ is, in addition, zero on the annihilator E^o of E and verifies the following relation :

$$P(dH, dK) \in \mathcal{H}(M, \mathcal{F}) \text{ for all } H, K \in \mathcal{H}(M, \mathcal{F}),$$

which led us to introduce in this work the notion of polarized Poisson structure on a foliated manifold, allowing to study the properties of these new objects and to find a usual Poisson manifold in the case where $\mathcal{F}$ is the trivial foliation of dimension 0 in which the leaf $\mathcal{F}_x$ passing through x is reduced to $\{x\}$.

Let us call back that one of the main motivations which led to introduce the notion of $k-$symplectic structure, as extension of the geometry of polarization ([4]), is to propose a geometric support of the equations of Nambu ([10]), in analogy with the well known symplectic geometry as support of classic Hamiltonian mechanics. Some properties of the Poisson structure subordinate to a $k-$symplectic

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manifold have led us to put in evidence the notion of vectorial polarized Poisson structure; for a fixed finite dimensional real vector space V, this last structure is defined on a foliated manifold $(M,\mathcal{F})$ by a pair $(\mathcal{H}(M,\mathcal{F}),P)$, such that $\mathcal{H}(M,\mathcal{F})$ is a submodule of the space $\mathcal{C}^{\infty}(M,V)$ of V-valued smooth functions on M, over the ring of basic functions for the foliation $\mathcal{F}$, and P is $\mathcal{C}^{\infty}(M)$ $-$bilinear antisymmetric mapping

$$P:\bigwedge\nolimits_1(M,V)\times\bigwedge\nolimits_1(M,V)\longrightarrow\mathcal{C}^{\infty}(M,V)$$

allowing to find usual case for $V=\mathbb{R}$.

In the last part of this work, we put in evidence, some links between the Hamiltonian system and the generalized Hamiltonian system of Nambu associated to a same Hamiltonian mapping $H\in\mathcal{H}(M,\mathcal{F})$, where $(\mathcal{H}(M,\mathcal{F}),P)$ is the Poisson structure subordinate to the $k-$symplectic structure.

In all of the following, smoothness should be understood to mean C^{∞}. The manifolds considered are Hausdorff and second countable.

II. POLARIZED POISSON MANIFOLDS

A. Definitions

Let M be a smooth manifold of dimension n equipped with a foliation $\mathcal{F}$ of dimension m and let E be the $m-$dimensional integrable subbundle of TM defined by the tangent vectors to the leaves of $\mathcal{F}$. We denote by $\Gamma(E)$ the set of all cross-sections of the $M-$bundle $E\longrightarrow M$, $\bigwedge_r(M)$ the space of all differential $r-$forms on M and E^o the annihilator of E.

Recall that a smooth real function f on M is said to be basic for $\mathcal{F}$ if, for each $Y\in\Gamma(E)$, the derivative $Y(f)$ of f along Y is identically zero; this is equivalent to, f is constant on each leaf of $\mathcal{F}$.

The set of basic functions for $\mathcal{F}$ will be denoted by $\mathcal{B}(M,\mathcal{F})$. It is clear that $\mathcal{B}(M,\mathcal{F})$ is a subring of $\mathcal{C}^{\infty}(M)$ of all real smooth functions, and $\mathcal{C}^{\infty}(M)$ is a module over $\mathcal{B}(M,\mathcal{F})$.

Let us begin with the case where M is of dimension $2n$, equipped with a real polarization $(\theta,\mathcal{F})$, that is, θ is a symplectic form on M and $\mathcal{F}$ is a Lagrangian foliation. The Darboux theorem shows that every point of M has an open neighborhood U with local coordinates system $(x^1,\ldots,x^n,y^1,\ldots,y^n)$, such that $\theta=dx^1\wedge dy^1+\ldots+dx^n\wedge dy^n$ and the foliation $\mathcal{F}$ is defined on U by the equations $dy^1=\ldots=dy^n=0$.

A Hamiltonian system X of the symplectic structure θ is said to be polarized, if in addition, X is foliate for the subbundle E, that is, for all cross section $Y\in\Gamma(E)$, the Lie bracket $[X,Y]$ belongs to $\Gamma(E)$; locally, on an open neighborhood of each point of M, there exists a smooth real function H, such that $i(X)\theta=-dH$. With respect to an adapted coordinates system $(x^1,\ldots,x^n,y^1,\ldots,y^n)$, the mapping H takes the form :

$$H=\sum_{i=1}^{n}a_i(y^1,\ldots,y^n)x^i+b(y^1,\ldots,y^n)\qquad(2.1)$$

where $a_1,\ldots,a_n,b$ are basic functions for $\mathcal{F}$.

A smooth mapping $H:M\longrightarrow\mathbb{R}$ is said to be a polarized Hamiltonian mapping if there exists a polarized Hamiltonian system $X_H\in\mathcal{X}(M)$ such that $i(X_H)\theta=-dH$. We denote by $\mathcal{H}(M,\mathcal{F})$ the space of all polarized Hamiltonian mappings.

Recall that, by the symplectic duality $\zeta:X\longmapsto i(X)\theta$, between the tangent bundle TM and the cotangent bundle T^*M, we associate to θ a non degenerate bivector P (the Poisson tensor) defined by :

$$P(\alpha,\beta)=\theta\left(\zeta^{-1}(\alpha),\zeta^{-1}(\beta)\right)\text{ for all }\alpha,\beta\in\bigwedge\nolimits_1(M),$$

and, we have an antisymmetric linear mapping $\underline{P}:\bigwedge_1(M)\longrightarrow\mathcal{X}(M)$, given by

$$\langle\beta,\underline{P}(\alpha)\rangle=P(\alpha,\beta).$$

We observe here that we have the following properties :

1. $\mathcal{H}(M,\mathcal{F})$ is strictly contained in $\mathcal{C}^{\infty}(M)$,
2. $\mathcal{H}(M,\mathcal{F})$ is a submodule of $\mathcal{C}^{\infty}(M)$ over the ring $\mathcal{B}(M,\mathcal{F})$ of basic functions. On an open neighborhood U of M, endowed with an adapted coordinates system $(x^1,\ldots,x^n,y^1,\ldots,y^n)$, $\mathcal{H}(U,\mathcal{F}_U)$ is a free submodule of $\mathcal{C}^{\infty}(U)$, over $\mathcal{B}(U,\mathcal{F}_U)$, of finite type of rank $n+1$, spanned by the real functions $x^1,\ldots,x^n,1$. The associated Poisson tensor P is given by

$$P=\sum_{i=1}^{n}\frac{\partial}{\partial x^i}\wedge\frac{\partial}{\partial y^i},\qquad(2.2)$$

and we have $\underline{P}(dx^i)=\frac{\partial}{\partial y^i}$, $\underline{P}(dy^i)=-\frac{\partial}{\partial x^i}$.

Let $(M,\mathcal{F})$ be a foliated manifold, let $\mathcal{H}(M,\mathcal{F})$ be a submodule of $\mathcal{C}^{\infty}(M)$ over $\mathcal{B}(M,\mathcal{F})$ and $P:\bigwedge_1(M)\times\bigwedge_1(M)\longrightarrow\mathcal{C}^{\infty}(M)$ be an antisymmetric $\mathcal{C}^{\infty}(M)-$ bilinear mapping. We say that $(\mathcal{H}(M,\mathcal{F}),P)$ is a polarized Poisson structure on M, if the following properties are satisfied :

1. for all $\alpha, \beta \in E^o$, $P(\alpha, \beta) = 0$,
2. for all $H, K \in \mathcal{H}(M, \mathcal{F})$, $P(dH, dK) \in \mathcal{H}(M, \mathcal{F})$,
3. the correspondence $(H, K) \longmapsto \{H, K\} = P(dH, dK)$, from $\mathcal{H}(M, \mathcal{F}) \times \mathcal{H}(M, \mathcal{F})$ with values in $\mathcal{H}(M, \mathcal{F})$ confers to $\mathcal{H}(M, \mathcal{F})$ a law of Lie algebra,
4. each $H \in \mathcal{H}(M, \mathcal{F})$ corresponds to a vector field X_H such that :
$$\langle dK, X_H \rangle = X_H(K) = \{H, K\}$$
for all $K \in \mathcal{H}(M, \mathcal{F})$, this vector field is defined by $X_H = \underline{P}(dH)$.

We observe that we have $P(df, dg) = 0$ for all $f, g \in \mathcal{B}(M, \mathcal{F})$.

Let $(\mathcal{H}(M, \mathcal{F}), P)$ be a polarized Poisson structure on a smooth manifold M equipped with a $m-$dimensional foliation $\mathcal{F}$. Since the bivector P is zero on the annihilator E^o, then, with respect to an adapted coordinates system $(x^1, \ldots, x^m, y^1, \ldots, y^{n-m})$ defined on an open U, we have :

$$P = \sum_{i=1}^{m} \frac{\partial}{\partial x^i} \wedge \left(\sum_{j=1}^{m} A^{ij}(x, y) \frac{\partial}{\partial x^j} + \sum_{j=1}^{n-m} B^{ij}(x, y) \frac{\partial}{\partial y^j} \right) \tag{2.3}$$

where $A^{ij} = P(dx^i, dx^j) = \{x^i, x^j\}$, $B^{ij} = P(dx^i, dy^j) = \{x^i, y^j\}$ and $P(dy^i, dy^j) = \{y^i, y^j\} = 0$. The Jacobi identity is equivalent to :

$$\begin{array}{l} \frac{\partial A^{rs}}{\partial x^u} A^{ut} - \frac{\partial A^{rs}}{\partial y^v} B^{tv} + \frac{\partial A^{st}}{\partial x^u} A^{ur} \\ - \frac{\partial A^{st}}{\partial x^v} B^{rv} + \frac{\partial A^{tr}}{\partial x^u} A^{us} - \frac{\partial A^{tr}}{\partial y^v} B^{sv} = 0, \\ \frac{\partial A^{rs}}{\partial x^u} B^{ut} + \frac{\partial B^{rs}}{\partial y^v} A^{ur} - \frac{\partial B^{st}}{\partial y^v} B^{rv} \\ - \frac{\partial B^{rt}}{\partial x^u} A^{us} + \frac{\partial B^{rt}}{\partial y^v} B^{sv} = 0, \\ \frac{\partial B^{rs}}{\partial x^u} A^{ut} - \frac{\partial B^{rt}}{\partial x^u} B^{us} = 0. \end{array} \tag{2.4}$$

B. Model polarized Poisson manifolds

Consider the real space $\mathbb{R}^n = \mathbb{R}^m \times \mathbb{R}^{n-m}$ equipped with the $p-$dimensional foliation $\mathcal{F}$ defined by $dy^1 = \ldots dy^{n-m} = 0$, where $(x^1, \ldots, x^m, y^1, \ldots, y^{n-m})$ are the Cartesian here the coordinates system.

Let $(\mathcal{H}(M, \mathcal{F}), P)$ be a polarized Poisson structure on $\mathbb{R}^n$. The bivector P takes the form (2.3) and satisfies the equations (2.4), and $\mathcal{H}(M, \mathcal{F})$ is a submodule of $\mathcal{C}^\infty(\mathbb{R}^n)$ over the ring $\mathcal{B}(\mathbb{R}^n, \mathcal{F})$ of basic functions. We give now some examples of polarized Poisson manifolds.

1. Let $\mathcal{H}(\mathbb{R}^n, \mathcal{F})$ be the $\mathcal{B}(M, \mathcal{F})$ $-$submodule of $\mathcal{C}^\infty(\mathbb{R}^n)$ spanned by the constant function 1, thus $\mathcal{H}(\mathbb{R}^n, \mathcal{F}) = \mathcal{B}(\mathbb{R}^n, \mathcal{F})$. Then the pair $(\mathcal{H}(M, \mathcal{F}), \{,\})$ is an abelian Lie algebra.
2. Let $\mathcal{H}(\mathbb{R}^n, \mathcal{F})$ be the $\mathcal{B}(M, \mathcal{F})$ $-$submodule of $\mathcal{C}^\infty(\mathbb{R}^n)$ spanned by the real functions $1, x^1, \ldots, x^m$. Each element H of $\mathcal{H}(\mathbb{R}^n, \mathcal{F})$ has the form :
$$H = \sum_{i=1}^{m} a_i(y^1, \ldots, y^{n-m}) x^i + b(y^1, \ldots, y^{n-m})$$
where $a_1, \ldots, a_m, b$ are basic functions for $\mathcal{F}$. One finds here the case of polarized Hamiltonian mappings for $n = 2m$.
3. Let $\mathcal{H}(\mathbb{R}^n, \mathcal{F})$ be the $\mathcal{B}(M, \mathcal{F})$-submodule of $\mathcal{C}^\infty(\mathbb{R}^n)$ spanned by $1, x^i x^j$ $(1 \le i, j \le m)$. Each element H of $\mathcal{H}(\mathbb{R}^n, \mathcal{F})$ has the form :
$$H = \sum_{i=1}^{m} a_{ij}(y^1, \ldots, y^{n-m}) x^i x^j + b(y^1, \ldots, y^{n-m})$$
where a_{ij} $(1 \le i, j \le m)$, b are basic functions.

C. Remarks

1. If the foliation $\mathcal{F}$ has a dense leaf, then only basic functions are constant, consequently, $\mathcal{H}(M, \mathcal{F})$ is a real vector subspace of $\mathcal{C}^\infty(M)$.
2. The polarized Poisson structure subordinate to a real polarization is the polarized Poisson structure defined on a polarized manifold $(M; \theta, E)$ by the space $\mathcal{H}(M, \mathcal{F})$ is formed by the polarized Hamiltonian mappings, and P is the Poisson tensor associated to θ.
3. Let M be a smooth manifold equipped with the trivial foliation $\mathcal{F}$ of dimension 0 whose the leaf $\mathcal{F}_x$ passing through the point x is reduced to $\{x\}$. The ring $\mathcal{B}(M, \mathcal{F})$ coincides with $\mathcal{C}^\infty(M)$ of all smooth functions, consequently $\mathcal{H}(M, \mathcal{F})$ is a $\mathcal{C}^\infty(M)$-submodule of $\mathcal{C}^\infty(M)$. Let us give here some fundamental examples :
   (a) $\mathcal{H}(M, \mathcal{F}) = (0)$, this case corresponds to the trivial Poisson structure,
   (b) $\mathcal{H}(M, \mathcal{F}) = \mathcal{C}^\infty(M)$ this case corresponds to the classic Poisson manifold,
   (c) $\mathcal{H}(M, \mathcal{F})$ is an ideal of $\mathcal{C}^\infty(M)$.

4. Consider now the case where M is connected and equipped with the trivial $n-$dimensional foliation,where $n = \dim M$. In this case, the basic functions are constant on M, then $\mathcal{B}(M,\mathcal{F}) = \mathbb{R}$, consequently, $\mathcal{H}(M,\mathcal{F})$ is a vector subspace of the real vector space $\mathcal{C}^\infty(M)$.

III. VECTORIAL POLARIZED POISSON MANIFOLDS

A. Vectorial polarized Poisson tensor

Let M be an $n-$dimensional smooth manifold equipped with a foliation $\mathcal{F}$ of dimension m and let V be a real vector space of dimension k.

Let us fix a basis $(e_r)_{1\le r\le k}$ of V with dual basis $(\omega^r)_{1\le r\le k}$, and let $\bigwedge_1(M,V) = \bigwedge_1(M)\otimes V$ be the space of $V-$valued differential forms of degree 1, that is, the space of elements of the form $\alpha = \alpha^1\otimes e_1 + \ldots + \alpha^k\otimes e_k$ where $\alpha^1,\ldots,\alpha^k$ are Pfaffian forms on M. Locally, on an open neighborhood U endowed with local coordinates system $(x^1,\ldots,x^n)$, each element $\alpha\in\bigwedge_1(M,V)$ has the form :

$$\alpha_{|U} = \sum_{r=1}^{k}\sum_{i=1}^{n}\alpha_i^r dx^i\otimes e_r$$

where $\alpha_i^r : U\longrightarrow\mathbb{R}$ are smooth mappings.

We denote by E_V^o the annihilator of the subbundle E in $\bigwedge_1(M,V)$, it is the space of $V-$valued $1-$forms on M vanishing on the cross sections of E.

Let $\mathcal{H}(M,\mathcal{F})$ be a submodule of $\mathcal{C}^\infty(M,V)$ over the ring $\mathcal{B}(M,\mathcal{F})$, and let

$$P : \bigwedge\nolimits_1(M,V)\times\bigwedge\nolimits_1(M,V)\longrightarrow\mathcal{C}^\infty(M,V)$$

be an antisymmetric $\mathcal{C}^\infty(M)-$bilinear. We say that $(\mathcal{H}(M,\mathcal{F}),P)$ is a vectorial polarized Poisson structure on M, if the following properties are satisfied :

1. $P(\alpha,\beta) = 0$ for all $\alpha,\beta\in E_V^o$,
2. for all $H,K\in\mathcal{H}(M,\mathcal{F})$, $P(dH,dK)\in\mathcal{H}(M,\mathcal{F})$,
3. the correspondence $(H,K)\longmapsto\{H,K\} = P(dH,dK)$, from $\mathcal{H}(M,\mathcal{F})\times\mathcal{H}(M,\mathcal{F})$ with values in $\mathcal{H}(M,\mathcal{F})$, confers to $\mathcal{H}(M,\mathcal{F})$ a law of Lie algebra,
4. each $H\in\mathcal{H}(M,\mathcal{F})$ corresponds to a vector field X_H such that :

$$\langle dK, X_H\rangle = \{H,K\},$$

for all $K\in\mathcal{H}(M,\mathcal{F})$.

P will be called a vectorial polarized Poisson tensor.

Let us consider an open U of M endowed with an adapted local coordinates system $(x^1,\ldots,x^m,y^1,\ldots y^{n-m})$. Since P is zero on the annihilator E_V^o of the subbundle E in $\bigwedge_1(M,V)$, then the tensor P has the form :

$$P = A_{pq}^{ijr}\left((\frac{\partial}{\partial x^i}\otimes\omega^p)\wedge(\frac{\partial}{\partial x^j}\otimes\omega^q)\right)\otimes e_r \quad (3.1)$$

$$+B_{pq}^{ijr}\left((\frac{\partial}{\partial x^i}\otimes\omega^p)\wedge(\frac{\partial}{\partial y^j}\otimes\omega^q)\right)\otimes e_r \quad (3.2)$$

where $A_{pq}^{ijr} = P^r(dx^i\otimes e_p, dx^j\otimes e_q)$, $B_{pq}^{ijr} = P^r(dx^i\otimes e_p, dy^j\otimes e_q)$ and P^r are the components of P. The relation $P(\alpha,\beta) = 0$ for all $\alpha,\beta\in E_V^o$ implies that we have $P^r(dy^i\otimes e_p, dy^j\otimes e_q) = 0$. The Jacobi identity is equivalent to

$$\begin{gathered}
\frac{\partial A_{uv}^{abr}}{\partial x^l}A_{rw}^{lcv} - \frac{\partial A_{uv}^{abr}}{\partial y^m}B_{wr}^{cmv} + \frac{\partial A_{vw}^{bcr}}{\partial x^l}A_{ru}^{lav} \\
-\frac{\partial A_{vw}^{bcr}}{\partial y^m}B_{ur}^{amv} + \frac{\partial A_{wu}^{car}}{\partial x^l}A_{rv}^{lbv} - \frac{\partial A_{wu}^{car}}{\partial y^m}B_{vr}^{bmv} = 0, \\
\frac{\partial A_{uv}^{abr}}{\partial x^l}B_{rw}^{lcv} + \frac{\partial B_{vw}^{bcr}}{\partial x^l}A_{ru}^{lav} - \frac{\partial B_{vw}^{bcr}}{\partial y^m}B_{ur}^{amv} \\
-\frac{\partial B_{uw}^{acr}}{\partial x^l}A_{rv}^{lbv} + \frac{\partial B_{uw}^{acr}}{\partial y^m}B_{vr}^{bmv} = 0, \\
\frac{\partial B_{uv}^{abv}}{\partial x^l}A_{rv}^{lbv} - \frac{\partial B_{uw}^{acr}}{\partial x^l}B_{rv}^{lbv} = 0.
\end{gathered} \quad (3.3)$$

For each element $\alpha\in\bigwedge_1(M,V)$ we can associate a $\mathcal{C}^\infty(M)-$linear mapping

$$P(\alpha,.) : \bigwedge\nolimits_1(M,V)\longrightarrow\mathcal{C}^\infty(M,V)$$

such that $P(\alpha,.)(\beta) = P(\alpha,\beta)$ for each $\beta\in\bigwedge_1(M,V)$, the linear mapping $P(\alpha,.)$ coincides with the vector field $\underline{P}(\alpha)$ for $k = 1$.

In the general case, we have a canonical linear mapping

$$\Xi : \mathcal{X}(M)\longrightarrow\mathcal{L}_{\mathcal{C}^\infty(M)}\left(\bigwedge\nolimits_1(M,V),\mathcal{C}^\infty(M,V)\right)$$

defined by :

$$\Xi(X)(\beta) = \langle\beta, X\rangle = \sum_{p=1}^{k}\beta^p(X)e_p = \sum_{p=1}^{k}(\beta^p\otimes e_p)(X)$$

for all $X\in\mathcal{X}(M)$ and $\beta = \sum_{p=1}^{k}\beta^p\otimes e_p\in\bigwedge_1(M,V)$. The mapping Ξ is injective and it is an isomorphism if and only if $k = 1$. In particular, $\langle dK,\Xi(X_H)\rangle = P(dH,dK) = \{H,K\}$ for each $K\in\mathcal{H}(M,\mathcal{F})$.

B. Vectorial polarized Poisson structure associated to a $k-$symplectic structure

Let M be an $n(k+1)-$dimensional smooth manifold equipped with an $n-$codimensional foliation $\mathcal{F}$, let $\theta^1,\ldots,\theta^k$ be closed differential 2−forms on M and let E be the subbundle of TM associated to $\mathcal{F}$.

For each $x\in M$, we denote by $C_x(\theta^1),\ldots,C_x(\theta^k)$ the characteristic subspaces of $\theta^1,\ldots,\theta^k$ at x, recall that $C_x(\theta^p)$ is the set of $X_x\in T_xM$ such that $i(X_x)\theta^p=0$ and $i(X_x)d\theta^p=0$ where $i(X_x)\theta^p$ is the interior product of X_x by the 2−form θ^p.

And also, we recall that $(\theta^1,\ldots,\theta^k;E)$ is a $k-$symplectic structure on M if for each $x\in M$, the following properties are satisfied :

1. $C_x(\theta^1)\cap\cdots\cap C_x(\theta^k)=\{0\}$,
2. $\theta^p(X,Y)=0$ for all $X,Y\in\Gamma(E)$ and $p=1,\ldots,k$.

The Darboux theorem shows that, for any point of M, there is an open neighborhood U of M containing this point, with a local coordinates system $(x^{pi},x^i)_{1\le p\le k,1\le i\le n}$ such that the differential forms θ^p are represented on U by $\theta^p_{|U}=dx^{p1}\wedge dx^1+\ldots+dx^{pn}\wedge dx^n$ and the subbundle E is defined by the equations $dx^1=\ldots=dx^n=0$ (see for example[1],[2] and[4]).

A vector field X on M is called a Hamiltonian system if X is an infinitesimal automorphism for $\mathcal{F}$ and for the 2−forms θ^p; that is, if X is foliate for $\mathcal{F}$ and $L_X\theta^1=\ldots=L_X\theta^k=0$.

The Poincaré lemma shows that around each point of M, there exists a smooth mapping $H:U\longrightarrow\mathbb{R}^k$ whose components H^p verify $i(X)\theta^p=-dH^p$.

A Hamiltonian system X is called strongly if there exists a smooth mapping $H:M\longrightarrow\mathbb{R}^k$ whose components H^p verifie the relation $i(X)\theta^p=-dH^p$ on M. The mapping H is called a Hamiltonian mapping and the associated Hamiltonian system X, denoted by X_H, is called the Hamiltonian system associated to H.

We denote by $\mathcal{H}(M,\mathcal{F})$ the subspace of $\mathcal{C}^\infty\left(M,\mathbb{R}^k\right)$ formed by the Hamiltonian mappings.

With respect to an adapted coordinates system $(x^{pi},x^i)_{1\le p\le k,1\le i\le n}$, the components H^p of H takes the form :

$$H^p=\sum_{j=1}^n f_j(x^1,\ldots,x^n)x^{pj}+g^p(x^1,\ldots,x^n)$$

where f_j and g^p are basic functions for $\mathcal{F}_{|U}$ ([4]).

Let H, K be two Hamiltonian mappings, and X_H, X_K be the corresponding Hamiltonian systems. The Lie bracket $[X_H,X_K]$ is a strongly Hamiltonian system, and more precisely, the mapping $\{H,K\}:M\longrightarrow\mathbb{R}^k$ given by $\{H,K\}=(\theta^1(X_H,X_K),\ldots,\theta^k(X_H,X_K))$ belongs to $\mathcal{H}(M,\mathcal{F})$ and satisfies to $[X_H,X_K]=X_{\{H,K\}}$.

The space $\mathcal{H}(M,\mathcal{F})$ endowed with the bracket $\{,\}$ is an infinite real Lie algebra.

Let us consider the case where M is the real space $\mathbb{R}^{n(k+1)}$ endowed with the canonical $k-$symplectic structure $\left(\theta^1,\ldots,\theta^k,E\right)$ given by $\theta^p=dx^{p1}\wedge dx^1+\ldots+dx^{pn}\wedge dx^n$ and E is defined by $dx^1=\ldots=dx^n=0$. $(x^{pi},x^i)_{1\le p\le k,1\le i\le n}$ being the Cartesian coordinates system.

In this case, the space $\mathcal{H}(M,\mathcal{F})$ is the submodule of $\mathcal{C}^\infty\left(M,\mathbb{R}^k\right)$, over the ring of basic functions for $\mathcal{F}$, spanned by the mappings

$$X^i:(x^{pi},x^i)_{1\le p\le k,1\le i\le n}\longmapsto x^{1i}e_1+\ldots+x^{ki}e_k,\quad i=1,\ldots,n$$

and by the vectors $(e_r)_{1\le r\le k}$ considered as $\mathbb{R}^k-$valued constant functions $x\longmapsto e_r$ $(1\le r\le k)$, defined on M; $(e_r)_{1\le r\le k}$ being the canonical basis of $\mathbb{R}^k$, and let

$$P=\sum_{p=1}^k\sum_{i=1}^n\left((\frac{\partial}{\partial x^{pi}}\otimes\omega^p)\wedge(\frac{\partial}{\partial x^i}\otimes\omega^p)\right)\otimes e_p.$$

The pair $(\mathcal{H}(M,\mathcal{F}),P)$ defines a vector polarized Poisson structure called associated to the canonical $k-$symplectic structure of $\mathbb{R}^{n(k+1)}$. For all $H,K\in\mathcal{H}(M,\mathcal{F})$ we have :

$$\begin{aligned}P(dH,dK)&=P(dH^q\otimes e_q,dK^r\otimes e_r)\\&=\sum_{p=1}^k\sum_{i=1}^n\left(\frac{\partial H^p}{\partial x^{pi}}\frac{\partial K^p}{\partial x^i}-\frac{\partial H^p}{\partial x^i}\frac{\partial K^p}{\partial x^{pi}}\right)e_p\\&=\{H,K\}.\end{aligned}$$

IV. MODEL VECTORIAL POLARIZED POISSON MANIFOLDS

Let us consider the model space $\mathbb{R}^n=\mathbb{R}^m\times\mathbb{R}^{n-m}$ endowed with the $p-$dimensional model foliation $\mathcal{F}$ defined by the equations $dy^1=\ldots dy^{n-m}=0$, where $\left(x^i,y^j\right)$, with $i=1,\ldots,m$ and $j=1,\ldots,n-m$, are the Cartesian coordinates system and let $V=\mathbb{R}^k$ be the real space in which one fixes the canonical basis $(e_r)_{1\le r\le k}$ with dual basis $(\omega^r)_{1\le r\le k}$.

Let $(\mathcal{H}(M,\mathcal{F}),P)$ be a vectoriazl polarized Poisson structure on $\mathbb{R}^n$. The vector polarized Poisson bivector P takes the form (3.1) and satisfies the Jacobi identity (3.3), and $\mathcal{H}(M,\mathcal{F})$ is a submodule of $\mathcal{C}^\infty(\mathbb{R}^n,V)$ over the ring $\mathcal{B}(\mathbb{R}^n,\mathcal{F})$ of basic functions. We give now some examples of polarized Poisson structures widening the space of vector polarized Hamiltonian mappings subordinate to the $k-$symplectic manifolds.

1. Let $\mathcal{H}(\mathbb{R}^n,\mathcal{F})$ be the $\mathcal{B}(M,\mathcal{F})-$submodule of $\mathcal{C}^\infty(\mathbb{R}^n,V)$ spanned by $e_1\ldots,e_k$. thus $\mathcal{H}(\mathbb{R}^n,\mathcal{F})=\mathcal{B}(\mathbb{R}^n,\mathcal{F})\times\ldots\times\mathcal{B}(\mathbb{R}^n,\mathcal{F})$ (k times) and it results that the associated Lie algebra $(\mathcal{H}(M,\mathcal{F}),\{,\})$ is abelian.

2. For $V=\mathbb{R}^2$ we consider the $\mathcal{B}(M,\mathcal{F})-$submodule $\mathcal{H}(\mathbb{R}^n,\mathcal{F})$ of $\mathcal{C}^\infty(\mathbb{R}^n,V)$ spanned by the mappings :

$$X^i:(x,y)\longmapsto x^ie_1\ \ (i=1,\ldots,m).$$

$$X^{ij}:(x,y)\longmapsto x^ix^je_2\ \ (i,j=1,\ldots,m)$$

and by the vectors e_1,e_2. The components H^1 and H^2 of each element H of $\mathcal{B}(M,\mathcal{F})$ take the form :

$$\begin{aligned}H^1&=\sum_{i=1}^m f_i(y^1,\ldots,y^{n-m})x^i+g^1(y^1,\ldots,y^{n-m})\\&\quad(i=1,\ldots,m)\\H^2&=\sum_{i,j=1}^m f_{ij}(y^1,\ldots,y^{n-m})x^ix^j\\&\quad+g^2(y^1,\ldots,y^{n-m})\\&\quad(i,j=1,\ldots,m)\end{aligned}$$

where $f_i,f_{ij},g^1,g^2\in\mathcal{B}(M,\mathcal{F})$.

3. Suppose that $m=lk$. We denote by $(x,y)=(x^{ra},y^1,\ldots,y^{n-m})_{1\leq a\leq m,\,1\leq r\leq k}$ the Cartesian coordinates system of $\mathbb{R}^n$ and let $\mathcal{H}(\mathbb{R}^n,\mathcal{F})$ be the submodule of $\mathcal{C}^\infty(\mathbb{R}^n,V)$, over $\mathcal{B}(M,\mathcal{F})$, spanned by the mappings

$$X^a:(x,y)\longmapsto\sum_{r=1}^k x^{ra}e_r\ \ (a=1,\ldots,l)$$

and by the vectors $e_1,\ldots,e_k$. The component H^r of each element H of $\mathcal{H}(M,\mathcal{F})$ takes the form :

$$\begin{aligned}H^r&=\sum_{a=1}^l f_a(y^1,\ldots,y^{n-m})x^{ra}\\&\quad+g^r(y^1,\ldots,y^{n-m})\quad(r=1,\ldots,k).\end{aligned}$$

where $f_a,g^r\in\mathcal{B}(M,\mathcal{F})$.

4. In the previous notations, we consider the $\mathcal{B}(M,\mathcal{F})-$submodule $\mathcal{H}(\mathbb{R}^n,\mathcal{F})$ of $\mathcal{C}^\infty(\mathbb{R}^n,V)$ spanned by the mappings

$$X^{ab}:(x,y)\longmapsto\sum_{r=1}^k x^{ra}x^{rb}e_r\ \ (a,b=1,\ldots,l)$$

and by the vectors $e_1,\ldots,e_k$. The component H^r of each element H of $\mathcal{B}(M,\mathcal{F})$ has the form :

$$\begin{aligned}H^r&=\sum_{a,b=1}^l\{f_{ab}(y^1,\ldots,y^{n-m})x^{ra}x^{rb}\\&\quad+b^r(y^1,\ldots,y^{n-m})\}\\&\quad(r=1,\ldots,k).\end{aligned}$$

where $f_{ab},g^r\in\mathcal{B}(M,\mathcal{F})$.

5. In the previous notations, we consider the $\mathcal{B}(M,\mathcal{F})-$submodule $\mathcal{H}(\mathbb{R}^n,\mathcal{F})$ of $\mathcal{C}^\infty(\mathbb{R}^n,V)$ spanned by the mappings :

$$X^a:(x,y)\longmapsto\sum_{r=1}^k x^{ra}e_r\ \ (a=1,\ldots,l)$$

$$X^{ab}:(x,y)\longmapsto\sum_{r=1}^k x^{ra}x^{rb}e_r\ \ (a,b=1,\ldots,l)$$

and by the vectors $e_1,\ldots,e_k$. The components H^r of each element H of $\mathcal{B}(M,\mathcal{F})$ take the form :

$$\begin{aligned}H^r&=\sum_{a=1}^l f_a(y^1,\ldots,y^{n-m})x^{ra}\\&\quad+\sum_{a,b=1}^l\{f_{ab}(y^1,\ldots,y^{n-m})x^{ra}x^{rb}\\&\quad+b^r(y^1,\ldots,y^{n-m})\},\end{aligned}$$

where $(r=1,\ldots,k)$ and $f_a,f_{ab},g^r\in\mathcal{B}(M,\mathcal{F})$.

V. THE GENERALIZED HAMILTONIAN DYNAMICS OF NAMBU ASSOCIATED TO THE $K-$SYMPLECTIC HAMILTONIAN MAPPING

The equations of Nambu-Hamilton governing the movement of the generalized Hamiltonian dynamics of Nambu in the 3−dimensional case are given by :

$$\frac{dx}{dt} = \frac{D(H,G)}{D(y,z)}\ ,\ \frac{dy}{dt} = \frac{D(H,G)}{D(z,x)}\ ,\ \frac{dz}{dt} = \frac{D(H,G)}{D(x,y)} \tag{5.1}$$

where H and G are two real functions defined on the phase space M described by the coordinates system (x, y, z).

In 1973, Y. Nambu proposed a mechanics, in his paper on the generalized Hamiltonian dynamics ([10]), which has so far only partial geometric formuations. However, if we consider the vector polarized Poisson structure $(\mathcal{H}(M,\mathcal{F}), P)$ subordinate to the canonical $k-$symplectic structure of $\mathbb{R}^{n(k+1)}$, the submodule $\mathcal{B}(M,\mathcal{F})$ is strictly contained in $\mathcal{C}^\infty(M,\mathbb{R}^k)$, and it turns out that for every polarized Hamiltonian mapping $H \in \mathcal{H}(M,\mathcal{F})$, the associated Hamiltonian system X_H and the generalized Hamiltonian system of Nambu X_H^N are related by :

1. $X_H^N = (-1)^k (f(z))^{k-1} X_H$ where $f \in \mathcal{B}(M,\mathcal{F})$, for $M = \mathbb{R}^{k+1}$,
2. $X_H^N = \sum_{i=1}^n f_i(z^1,\ldots,z^n) X_H^i$ where $X_H^i = X_H(x^i)\frac{\partial}{\partial x^i} + X_H(y^i)\frac{\partial}{\partial y^i} + X_H(z^i)\frac{\partial}{\partial z^i}$, for each $i = 1,\ldots,n$, for $M = \mathbb{R}^{3n}$, where $f_1,\ldots,f_n$ are basic functions.

In the first stage, we consider the case where M is the real space $\mathbb{R}^3$ ($n = 1$ and $k = 2$) equipped with the canonical $2-$symplectic structure $(\theta^1,\theta^2;E)$ defined by $\theta^1 = dx\wedge dz$, $\theta^2 = dy\wedge dz$ and $E = \ker dz$. The components H^1 and H^2 of each polarized Hamiltonian mapping $H \in \mathcal{H}(M,\mathcal{F})$ are given by $H^1 = f(z)x + g^1(z)$ and $H^2 = f(z)y + g^2(z)$, where f, g^1 and g^2 are basic functions defined on this space M. The integral curves of the associated Hamiltonian system X_H are given by the following equations :

$$\frac{dx}{dt} = -\frac{\partial H^1}{\partial z},\ \frac{dy}{dt} = -\frac{\partial H^2}{\partial z}$$
$$\text{and } \frac{dz}{dt} = \frac{\partial H^1}{\partial x} = \frac{\partial H^2}{\partial y}$$

We deduct that the Hamiltonian system X_H and the generalized Hamiltonian dynamics of Nambu X_H^N are related by :

$$X_H^N = f(z)X_H.$$

In the second stage, we consider the case where M is the real space $\mathbb{R}^{k+1}$ ($n = 1$) equipped with the canonical $k-$symplectic structure $(\theta^1,\ldots,\theta^k;E)$ defined by $\theta^1 = dx^1\wedge dz,\ldots,\theta^k = dx^k\wedge dz$ and E is given by the equation $dz = 0$, $(x^1,\ldots,x^k, z = x^{k+1})$ being the Cartesian coordinates system of $\mathbb{R}^{k+1}$. The Hamiltonian mappings of this structure, that is, the elements of $\mathcal{H}(M,\mathcal{F})$, are the maps $H : M \longrightarrow \mathbb{R}^k$ whose components are given by $H^1 = f(z)x^1 + g^1(z),\ldots,H^k = f(z)x^k + g^k(z)$, where f, $g^1,\ldots,g^k$ are smooth basic functions defined on M.

The integral curves of the generalized Hamiltonian system of Nambu X_H^N associated to H are given by the following equations :

$$\frac{dx^j}{dt} = \sum_{i_1,i_2,\ldots,i_k=1}^{k+1} \varepsilon_{ji_1i_2\ldots i_k}\frac{\partial H^1}{\partial x^{i_1}}\frac{\partial H^2}{\partial x^{i_2}}\cdots\frac{\partial H^k}{\partial x^{i_k}},$$

where $\varepsilon_{i_1i_2\ldots i_{k+1}}$ is the Levi-Civita tensor. Then we have,

$$\frac{dz}{dt} = \varepsilon_{(k+1)123\ldots k}\frac{\partial H^1}{\partial x^1}\frac{\partial H^2}{\partial x^2}\cdots\frac{\partial H^k}{\partial x^k} = (-1)^k (f(z))^k .$$

and for each $j = 1,\ldots,k$, we have

$$\frac{dx^j}{dt} = \varepsilon_{12\ldots(j-1)(k+1)(j+1)\ldots k}\frac{\partial H^1}{\partial x^1}\frac{\partial H^2}{\partial x^2}\cdots\frac{\partial H^j}{\partial z}\cdots\frac{\partial H^k}{\partial x^k} = (-1)^{k-1}\left(\frac{\partial f(z)}{\partial z}x^1 + \frac{\partial g^1(z)}{\partial z}\right)(f(z))^{k-1}.$$

The integral curves of the Hamiltonian system X_H, are given by :

$$\frac{dx^j}{dt} = -\left(\frac{\partial f(z)}{\partial z}x^j + \frac{\partial g^1(z)}{\partial z}\right) \text{ for } j = 1,\ldots,k$$
$$\text{and } \frac{dz}{dt} = f(z).$$

We deduct that for each $H = (H^1,\ldots,H^k) \in \mathcal{H}(M,\mathcal{F})$, the Hamiltonian system X_H and the generalized Hamiltonian system of Nambu X_H^N are related by :

$$X_H^N = (-1)^k (f(z))^{k-1} X_H.$$

Finally, for $M = \mathbb{R}^{3n}$, we endow this space with the canonical $2-$symplectic structure given by $\theta^1 = \sum_{i=1}^n dx^i\wedge dz^i$, $\theta^2 = \sum_{i=1}^n dy^i\wedge dz^i$ and E is defined by the equations $dz^1 = \cdots = dz^n = 0$, $(x^i,y^i,z^i)_{1\le i\le n}$ being the Cartesian coordinates system of $\mathbb{R}^{3n}$. The components of each Hamiltonian mappings is defined by $H^1 = \sum_{i=1}^n f_i(z)x^i + g^1(z)$ and $H^2 = \sum_{i=1}^n f_i(z)y^i + g^2(z)$, where $z = (z^1,\ldots,z^n)$ and $f_1,\ldots,f_n$, g^1, g^2 are basic functions for $\mathcal{F}$.

The integral curves of the generalized Hamiltonian system of Nambu X_H^N associated to H are given by the following equations :

$$\frac{dx^i}{dt}=\frac{D\left(H^1,H^2\right)}{D\left(y^i,z^i\right)},\quad \frac{dy^i}{dt}=\frac{D\left(H^1,H^2\right)}{D\left(z^i,x^i\right)}$$

$$\text{and}\quad \frac{dz^i}{dt}=\frac{D\left(H^1,H^2\right)}{D\left(x^i,y^i\right)}$$

and we verify that for each smooth function $f: M\longrightarrow\mathbb{R}$, we have :

$$\frac{df}{dt}=\sum_{i=1}^{n}\frac{D(f,H^1,H^2)}{D\left(x^i,y^i,z^i\right)}=:(H^1,H^2,f),$$

then,

$$\frac{dx^i}{dt}=\sum_{j=1}^{n}\frac{D\left(x^i,H^1,H^2\right)}{D\left(x^j,y^j,z^j\right)}$$
$$=\frac{D\left(H^1,H^2\right)}{D\left(y^i,z^i\right)}=-\left(\sum_{j=1}^{n}\frac{\partial f_j}{\partial z^i}x^j+\frac{\partial g^1}{\partial z^i}\right)f_i$$

$$\frac{dy^i}{dt}=\sum_{j=1}^{n}\frac{D\left(y^i,H^1,H^2\right)}{D\left(x^j,y^j,z^j\right)}$$
$$=-\frac{D\left(H^1,H^2\right)}{D\left(x^i,z^i\right)}=-\left(\sum_{j=1}^{n}\frac{\partial f_j}{\partial z^i}y^j+\frac{\partial g^2}{\partial z^i}\right)f_i$$

$$\frac{dz^i}{dt}=\sum_{j=1}^{n}\frac{D\left(z^i,H^1,H^2\right)}{D\left(x^j,y^j,z^j\right)}$$
$$=\frac{D\left(H^1,H^2\right)}{D\left(x^i,y^i\right)}=(f_i)^2$$

Whereas the integral curves of the Hamiltonian system X_H are given by :

$$\frac{dx^i}{dt}=-\frac{\partial H^1}{\partial z^i}=-\left(\sum_{j=1}^{n}\frac{\partial f_j}{\partial z^i}x^j+\frac{\partial g^1}{\partial z^i}\right),$$
$$\frac{dy^i}{dt}=-\frac{\partial H^2}{\partial z^i}=-\left(\sum_{j=1}^{n}\frac{\partial f_j}{\partial z^i}y^j+\frac{\partial g^2}{\partial z^i}\right),$$
$$\frac{dz^i}{dt}=\frac{\partial H^1}{\partial x^i}=\frac{\partial H^2}{\partial y^i}=f_i.$$

We deduct the following relation

$$X_H^N=\sum_{i=1}^{n}f_{\ i}\left(z^1,\ldots,z^n\right)X_H^i$$

where $X_H^i = X_H\left(x^i\right)\frac{\partial}{\partial x^i}+X_H(y^i)\frac{\partial}{\partial y^i}+X_H\left(z^i\right)\frac{\partial}{\partial z^i}$, for each $i=1,\ldots,n$.

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