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Study of some domain decomposition strategies.

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Abstract

Two interface techniques to solve the elliptic differential equations are considered. The first is the Robin method. The second one is the penalization technique which is theoritically attractive. our objective in this paper is to compute the performance of these two methods for classical problem like the Poisson equation. Numerical comparison of these two interface techniques are given.

Keywords: *decomposition, subdomain, relaxation, penalty.*

I. INTRODUCTION

Domain decomposition has proven an effective means of partitioning the task of solving Differential Equation (DE) problem numerically. It is mainly an algebraic approach and works by splitting the discrete DE domain into subdomains which can be coupled in many ways. The well established additive and multiplicative Schwartz methods are examples of typical domain decomposition approaches that have been analyzed extensively. Interface relxation (IR) is a step beyond domain decomposition. IR methods are characterized by the fact that they can be easily formulated as numerical procedure for solving differential equation problems while all their actions involve continuous data. They assume a splitting of the domain into a set of non-overlapping subdomains and consider the associated DE problem defined on them. These subproblems are coupled through relaxation mechanisms on the interfaces. IR methods naturally apply to multi-physics problem when the DE may change from one subdomain to another; we do not consider this applicatoin here. For a general introduction to the IR methodology the reader is refereed to[1,2].

The convegence of these schemes depends, as expected, on the differential operator, the geometry of the original domain, an in addition on the geometry of the subdomains chosen. This makes selection of "optimum" values for the relaxation parameters a hard and challenging problem.

In this paper, we expose an averaging scheme based on the domain decomposition method with overlapping introduced by Tomàs Chacòn Rebollo and Eliseo Chacòn Vera in[6], see also[8] (denoted by PENAL). We present a Robin-type IR scheme[3] (denoted by ROB). We restrict ourselves to the Poisson problem.

The rest of this paper is organized as following. In the next section we formulate the two IR schemes. Section 3 presents results from an experimental study. Section 4 contains our conclusion and perspective.

II. TWO DECOMPOSITION METHODS

Let $\Omega \subset IR^n (n = 2, 3)$ be a simply connected and bounded domain with Lipschitz boundary $\partial\Omega$. We restrict ourselves to a simple decomposition of Ω into two subdomains Ω_1 and Ω_2 that may not overlap and Γ denote the common interface. We assume that all of these boundaries are Lipschitz (n-1)-dimensional manifolds. We will denote by n_{ij} the outward normal vector on Γ pointing from Ω_i into Ω_j, $\partial_{n_{ij}}$ the partial derivative with respect to n_{ij}. From any $D \subset IR^n (n \geq 1)$ and $u, v \in L^2(D)$ we set $(u, v)_D = \int_D u(x)v(x)dx$ and $\|v\|^2_{0,D} = (v, v)_D$. We also consider the Sobolev spaces:

$$\begin{aligned} X_i &= H^1(\Omega_i; \Gamma_i) \\ &= \{v \in H^1(\Omega_i) s.t. v|\Gamma_i = 0\}, \quad i = 1, 2 \end{aligned} \tag{1}$$

normed by $|u|^2_{1,\Omega_i} = (\nabla u, \nabla u)_{\Omega_i}$, define $(u, v)_{1,\Omega_i} = (\nabla u, \nabla v)_{\Omega_i}$ and set $X = X_1 \times X_2$.

We introduce the domain decomposition method in the particular case of the Poisson problem in Ω with homogeneous boundary conditions:

$$L(u) \equiv -\Delta u(x, y) = f, \qquad (x, y) \in \Omega \tag{2}$$

subject to homogeous boundary conditions.

In the variational formulation, we look for $u \in H^1_0(\Omega)$ such that:

$$(\nabla u, \nabla v) = (f, v)_\Omega \tag{3}$$

for all $v \in H^1_0(\Omega)$.

A. The ROB method

The ROB scheme is defined, for the model problem under consideration, by the following algorithm:

$$\begin{cases} -\Delta u_1^{n+1} = f \quad in \quad \Omega_1 \\ u_1^{n+1} = 0 \quad on \quad \partial\Omega_1 \cap \partial\Omega \\ \frac{\partial u_1^{n+1}}{\partial n} + \gamma_1 u_1^{n+1} = \frac{\partial u_2^n}{\partial n} + \gamma_1 u_2^n \quad on \quad \Gamma \end{cases} \tag{4}$$

and

$$\begin{cases} -\Delta u_2^{n+1} = f \quad in \quad \Omega_2 \\ u_2^{n+1} = 0 \quad on \quad \partial\Omega_2 \cap \partial\Omega \\ \frac{\partial u_2^{n+1}}{\partial n} + \gamma_2 u_2^{n+1} = \frac{\partial u_1^{n+1}}{\partial n} + \gamma_2 u_1^{n+1} \quad on \quad \Gamma \end{cases} \tag{5}$$

where u_2^0 is given, and γ_1 and γ_2 are non-negative acceleration parameters satisfying $\gamma_1 + \gamma_2 > 0$.
For the sake of parallelization, in (3) we could also consider u_1^n instead of u_1^{n+1}, assigning in that case also u_1^0.
Along tha same lines, we mention the following iteration-by-subdomain algorithm that has been proposed by Aghoskov and Lebedev[4]:
Given u_1^0 and u_2^0, for each $k \geq 0$ we have to solve

$$\begin{cases} -\Delta u_1^{n+1/2} = f \quad in \quad \Omega_1 \\ u_1^{n+1/2} = 0 \quad on \quad \partial\Omega_1 \cap \partial\Omega \\ \frac{\partial u_1^{n+1}}{\partial n} + p_n u_1^{n+1/2} = \frac{\partial u_2^n}{\partial n} + p_n u_2^n \quad on \quad \Gamma \end{cases} \tag{6}$$

and

$$u_1^{n+1} = u_1^n + \alpha_{n+1}(u_1^{n+1/2} - u_1^n) \quad on \quad \Omega_1 \tag{7}$$

$$\begin{cases} -\Delta u_2^{n+1/2} = f \quad in \quad \Omega_2 \\ u_1^{n+1/2} = 0 \quad on \quad \partial\Omega_2 \cap \partial\Omega \\ -q_n \frac{\partial u_2^{n+1/2}}{\partial n} + u_2^{n+1/2} = -q_n \frac{\partial u_1^{n+1}}{\partial n} + u_1^{n+1} \quad on \quad \Gamma \end{cases} \tag{8}$$

and

$$u_2^{n+1} = u_2^n + \beta_{n+1}(u_1^{n+1/2} - u_2^n) \quad on \quad \Omega_2 \tag{9}$$

in the above procedure, $p_n \geq 0$, $q_n \geq 0$, α_{n+1} and β_{n+1} are free parameters.

B. Decomposition method with a penalty term

The basic idea of the method introduced in[6,8] is the following:
For any $\epsilon > 0$, we pose the problem (P_ϵ) of finding $(u_1, u_2) \in X$ such that for all $(v_1, v_2) \in X$

$$(P_\epsilon) \begin{cases} (\nabla u_1, \nabla v_1)_{\Omega_1} + \frac{1}{\epsilon} \int_\Gamma (u_1 - u_2) v_1 d\sigma = (f, v_1)_{\Omega_1} \\ (\nabla u_2, \nabla v_2)_{\Omega_2} + \frac{1}{\epsilon} \int_\Gamma (u_2 - u_1) v_2 d\sigma = (f, v_2)_{\Omega_2} \end{cases} \tag{10}$$

This problem is the variational formulation of the following coupled partial differential equations:

$$\begin{cases} -\Delta u_1 = f \quad in \quad \Omega_1 \\ u_1 = 0 \quad on \quad \partial\Omega_1 \cap \partial\Omega \\ \partial_{n1,2} u_1 = -\frac{1}{\epsilon}(u_1 - u_2) \quad on \quad \Gamma \end{cases} \tag{11}$$

$$\begin{cases} -\Delta u_2 = f \quad in \quad \Omega_2 \\ u_2 = 0 \quad on \quad \partial\Omega_2 \cap \partial\Omega \\ \partial_{n2,1} u_1 = -\frac{1}{\epsilon}(u_2 - u_1) \quad on \quad \Gamma \end{cases} \tag{12}$$

where $n_{i,j}$ is the outward normal vector on Γ pointing from Ω_i and $\partial_{ni,j}$ denotes the partial derivative with respect to $n_{i,j}$. It is easly seen that when we set $u = u_i$ in Ω_i we recover the solution of the original Poisson problem provided that the transmission conditions:

$$u_1 = u_2 \quad on \quad \Gamma \tag{13}$$

$$\partial_{n1,2}u_1 = -\partial_{n2,1}u_2 \quad on \quad \Gamma \tag{14}$$

are satisfied. As in our method (14) is always true, we observe that the role of ϵ is to enforce (13) as $\epsilon \to 0^+$.
To see this, one define for $u = (u_1, u_2)$, $v = (v_1, v_2) \in X$ the scalar product and norme on X

$$((u, v))_\epsilon = \sum_{i=1}^{2}(\nabla u_i, \nabla v_i)_{\Omega_i} + \frac{1}{\epsilon}(u_1 - u_2, v_1 - v_2)_\Gamma,$$
$$\|u\|_\epsilon^2 = ((u, u))_\epsilon, \tag{15}$$

where $(u_i, u_j)_\Gamma = \int_\Gamma u_i u_j d\sigma$. Then, the problem (P_ϵ) is:

$$\begin{cases} \text{Find } u^\epsilon \in X & \text{such that} \\ ((u^\epsilon, v))_\epsilon = F(v) & \text{for all } v \in X \end{cases} \tag{16}$$

where F is the $\|.\|_\epsilon$-continuous linear form on X given by $F(v) = \sum_{i=1}^{2}(f, v_i)_{\Omega_i}$. Therefore, via Lax-Milgram Lemma, problem (P_ϵ) has a solution $u^\epsilon = (u_1^\epsilon, u_2^\epsilon) \in X$ unique for each $\epsilon > 0$. Next, for the true solution of problem (3), $u \in H_0^1(\Omega)$, we write $u = (u|\Omega_1, u|\Omega_2)$ and define the consistency error by

$$E(v) = ((u, v))_\epsilon - F(v) \tag{17}$$

for all $v = (v_1, v_2) \in X$.
As the solution u^ϵ of problem (P_ϵ) satisfies $((u^\epsilon, v)) - F(v) = 0$, we have $((u - u^\epsilon))_\epsilon = E(v)$ for all $v \in X$. One can easily shown using the integration by parts on(17) the following result:

$$E(v) = \int_\Gamma \partial_n u(v_1 - v_2)d\sigma \tag{18}$$

where n is the normal vector on Γ pointing Ω. Therefore, the error is bounded by:

$$|E(v)| \leq \|\partial_n u\|_{0,\Gamma}\|v_1 - v_2\|_{0,\Gamma} \tag{19}$$

Moreover, if $\partial_n u = 0$ on Γ we have that $u_i^\epsilon = u|\Omega_i$, $i = 1, 2$ for all $\epsilon > 0$.
Then we have the following theorem:
Theorem 1. Let $C = \|\partial_n u\|_{0,\Gamma}$, then for all $\epsilon > 0$ the following estimates hold

$$\|u_1^\epsilon - u_2^\epsilon\| \leq C\epsilon,$$
$$\sum_{i=1}^{2} |u - u_i^\epsilon|_{1,\Omega_i} \leq \sqrt{2}C\sqrt{\epsilon} \tag{20}$$

Proof. The result follows from the last result about the error E and the fact that

$$E(u - u^\epsilon) = \|u - u^\epsilon\|_\epsilon^2$$
$$= \sum_{i=1}^{2} |u - u_i^\epsilon|_{1,\Omega_i}^2 + \frac{1}{\epsilon}\|u_1^\epsilon - u_2^\epsilon\|_{0,\Gamma}^2 \tag{21}$$

The iterative procedure that we propose (it may be seen as an iterative substructuring method, see[5]) is the following:
Starting from an initial guess u_1^0 and u_2^0, generates two sequences of functions $\{u_1^n\}$ and $\{u_2^0\}$, for $n \geq 1$, such that:

$$\begin{cases} -\Delta u_1^{n+1} = f \quad in \quad \Omega_1 \\ u_1^{n+1} = 0 \quad on \quad \Gamma_1 \\ \partial_{n1,2}u_1^{n+1} = -\frac{1}{\epsilon}(u_1^{n+1} - u_2^n) \quad on \quad \Gamma \end{cases} \tag{22}$$

and

$$\begin{cases} -\Delta u_2^{n+1} = f \quad in \quad \Omega_2 \\ u_2^{n+1} = 0 \quad on \quad \Gamma_2 \\ \partial_{n2,1}u_2^{n+1} = -\frac{1}{\epsilon}(u_2^{n+1} - u_1^n) \quad on \quad \Gamma \end{cases} \tag{23}$$

For n large enough and $\epsilon \to 0^+$, we show in[6] that the solution $u_1^{\epsilon,n}$ and $u_2^{\epsilon,n}$ will converge to the global solution of the Poisson problem (3).
In the framework of the finite element approximation, we assume that the domain Ω is polygonal and takes for $h > 0$ an admissible and regular triangulation T_h of $\overline{\Omega}$ formed by polygons ($d = 2$) or polyhedra ($d = 3$) elements such that Γ is formed by faces or sides of elements K in T_h. One consider a family of finite element subspaces of $H_0^1(\Omega)$, denoted by V_h, and the approximed solution $u_h \in V_h$ to the discrete version of (3) posed on V_h. For $k \geq 1$ we assume that when $u \in H^{k+1}(\Omega) \cap H_0^1(\Omega)$ the error commited with $u_h \in V_h$ is $O(h^k)$, i.e.,

$$|u_h - u|_{1,\Omega} \leq C_0 h^k, \tag{24}$$

for some constant $C_0 = C_0(u)$. We say that u_h is the finite element approximation of degree k of u. Let $T_h^i = T_h \cap \overline{\Omega_i}$, be triangulation of $\overline{\Omega_i}$ and use finite element subspaces of X_i, denoted by $X_{i,h}$ ($i = 1, 2$). These triangulations of $\overline{\Omega_i}$ are compatible on Γ, i.e. they share the same edges on Γ. One could use the restriction of the spaces V_h to each of the Ω_i. It is standard to show that each $h, \epsilon > 0$ the discrete version $(P_{\epsilon,h})$ of (P_ϵ on the space $X_h = X_{1,h} \times X_{2,h}$ has a unique solution $u_h^\epsilon = (u_{1,h}^\epsilon, u_{2,h}^\epsilon) \in X_h$.
The following result gives error estimates with respect to ϵ and h.
Theorem 2.
Let $u \in H^{k+1}(\Omega) \cap H_0^1(\Omega)$, ($k \leq 1$) be the solution to problem refpois) and $u_h \in V_h$ the approximated

solution satisfying (24). Now for each $\epsilon, h > 0$ let $(u^\epsilon_{1,h}, u^\epsilon_{2,h}) \in X_h$ solve $(P_{\epsilon,h})$. Then, the following bounds hold:

$$\begin{array}{l} \sum_{i=1}^2 |u_h - u^\epsilon_{i,h}|_{1,\Omega_i} \le C(h^k + \sqrt{\epsilon}), \\ \sum_{i=1}^2 |u - u^\epsilon_{i,h}|_{1,\Omega_i} \le C(h^k + \sqrt{\epsilon}) \\ \|u^\epsilon_{1,h} - u^\epsilon_{2,h}\|_{0,\Gamma} \le C(\sqrt{\epsilon} h^k + \epsilon) \end{array} \tag{25}$$

where $C = C(u)$ is a positive constant that just depends on u. In terms of the relation of ϵ and h the choice $\epsilon = O(h^{2k})$ gives

$$\begin{array}{l} \sum_{i=1}^2 |u_h - u^\epsilon_{i,h}|_{1,\Omega_i} \le Ch^k, \\ \sum_{i=1}^2 |u - u^\epsilon_{i,h}|_{1,\Omega_i} \le Ch^k \\ \|u^\epsilon_{1,h} - u^\epsilon_{2,h}\|_{0,\Gamma} \le Ch^{2k}. \end{array} \tag{26}$$

Proof. The idea of the proof of this theorem is the use of the consistency error of the discrete problem and to do some classical bounding using the Young's inequality. See[7] for more details.

For $n = 0, 1, 2..$, given u_1^n and u_2^n we compute $u_1^{n+1} = u^{\epsilon,n+1}_{1,h}$ and $u_2^{n+1} = u^{\epsilon,n+1}_{2,h}$ such that, we drop ϵ and h for simplicity,

$$\left\{ \begin{array}{l} (\nabla u_1^{n+1}, \nabla v_1)_{\Omega_1} + \frac{1}{\epsilon}(u_1^{n+1} - u_2^n, v_1)_{0,\Gamma} = (f, v_1)_{\Omega_1} \\ (\nabla u_2^{n+1}, \nabla v_2)_{\Omega_2} + \frac{1}{\epsilon}(u_2^{n+1} - u_1^n, v_2)_{0,\Gamma} = (f, v_2)_{\Omega_2} \end{array} \right. \tag{27}$$

for all $(v_1, v_2) \in X_h$. In[8], the following problem is proved:

Theorem 3.
When the true solution u is smooth enough, $u \in H^{k+1}(\Omega) \cap H^1_0(\Omega)$ for $k \ge 1$, then given $\epsilon, h > 0$ there exists a constant $C = C(u, f)$ only depending on u, f such that for all $n \ge 1$:

$$\sum_{i=1}^2 |u - u^{\epsilon,n+1}_{i,h}|_{1,\Omega_i} \le C(h^k + \sqrt{\epsilon} + \frac{1}{\epsilon(1 + 2C_0\epsilon)^{n/2}}) \tag{28}$$

where $u^{n,\epsilon}_{i,h}$ is the P_k finite element approximation for the solution $u^\epsilon_{i,h}$ obtained in the iteration process starting with $u^{0,\epsilon}_{i,h} = 0$. Therefore, for $\epsilon = O(h^{2k}$ and n large enough one can obtain:

$$\sum_{i=1}^2 |u - u^{n,\epsilon}_{i,h}| \le Ch^k \tag{29}$$

III. NUMERICAL EXPERIMENTS

The purpose of the numerical experiments performed in this study is twofold. First to verify and elucidate the theoritically determined penalty parameter values and compare the two methods.
In this test $\Omega =]0, 1[\times]0, 1[$ and the boundary condition is $u = 0$ on the boundary $\partial\Omega$ of Ω. The second member f is identically equal to 1.
We consider the interface Γ as the line $y = 0.5$ and then $\Omega_1 =]0, 1[\times]0, = .5[$ and $\Omega_2 =]0, 1[\times]0.5, 1[$. A uniform triangular mesh of mesh size h is considered and $\epsilon = h^2$. Finally the error is $eu(h) = (\sum_{i=1}^2 \int_\Omega |\nabla(u_h - u_{i,h}|^2 dx)^1/2$ and must be lower than 10^{-5}. The u_h is computed the IP_1 finite element approximation. The initial guess is identically null.
Our iteration results shows that $n \times h^2/(-log(h))$ is constant where n is the number of iterations as theoritical analysis had forecasted. The Robin method converges with the same precision but lower number of iterations (factor of ten). The factor γ is taken equal to 0.5.
The algorithm (6-9) is also numerically tested. This algorithm gives a good solution in comparison with the analytical one but the ratio of the number of tests and the good choice of the free parameters p_n, q_n, α_{n+1} and β_{n+1} lets the use of this algorithm in a general case more less attractive as the Robin method.

IV. CONCLUSION AND PERSPECTIVE

We have presented a numerical comparison of three subdomain methods: Robin method, Aghoskov and Lebedev method and the method based penalization technique (PENAL). The Robin method is proved to converge weakly[3] but the PENAL is proved to converge strongly in H^1[8].
Concerning the rate of convergence, the theoritical result is well respected numerically by the PENAL method but this one is very slow in comparison with the Robin one.
The extension of the PENAL method to solve the Stokes problem is proposed in[8]. The theoritical analysis gives a L^2 strong convergence of the pressure consequently this technique must give a good quality solution (velocity and pressure).
Our present work is the good management of the ratio quality and slownes to obtain the solution of the Stokes problem and we want to extend the same technique to solve the Navier-Stokes equation.

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