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Almost $k-$complex structures

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Abstract

We introduce and develop the notion of almost k-complex manifolds in analogy with the well known almost complex structure. In terms of G-structure, the integrability of this structure is equivalent to the vanishing of the Bernard-Chern tensor.

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I. INTRODUCTION

An almost complex structure on an even dimensional manifolds M is a tensor on M of type $(1,1)$ such that $J^2 = -id_{TM}$. We say that J is integrable if, around each point of M , we can find a local coordinates system $(x^i, y^i)_{1\leq i\leq n}$ such that

$$J\left(\frac{\partial}{\partial x^i}\right) = -\frac{\partial}{\partial y^i}, \qquad J\left(\frac{\partial}{\partial y^i}\right) = \frac{\partial}{\partial x^i}.$$

The theorem of Newlander-Nirenberg proves that J is integrable if and only if the Nijenhuis tensor of J $(N(Y,Y) = [JX,JY] - J[JX,Y] - J[X,JY] - [X,Y])$ is vanish.

On a symplectic manifold (M,θ) , there exists a Riemannian metric g and an almost complex structure J such that :

$$\theta(X,Y) = g(JX,Y).$$

For the canonical $k-$symplectic structure $(\theta^1,\ldots,\theta^k;E)$ defined on the model space $\mathbb{R}^{n(k+1)}$ by the two forms $\theta^p = dx^{p1}\wedge dx^1 + \ldots + dx^{pn}\wedge dx^n$ and the sub-bundle given by the equations $dx^1 = \ldots = dx^n = 0$, there exist k canonical tensors fields $J^1,\cdots,J^k \in T^1_1(\mathbb{R}^{n(k+1)})$ such that

$$\theta^p(X,Y) = \langle J^pX, Y\rangle,$$

where $\langle,\rangle$ is standard inner product. With respect to the coordinates (x^{pi}, x^i) , $1\leq p\leq k,\ 1\leq i\leq n$, we have

$$J^p = \sum_{i=1}^{n}\left(\frac{\partial}{\partial x^i}\otimes dx^{pi} - \frac{\partial}{\partial x^{pi}}\otimes dx^i\right). \qquad (1.1)$$

And also there exists an analog structure defined by k canonical tensors fields $J^1,\cdots,J^k$ of type $(1,1)$ on the manifold of all $1-$jets of mapping from an n-dimensional manifold M into $\mathbb{R}^k$ with target $0\in\mathbb{R}^k$ and take the same form (1.1) with respect the natural coordinates system.

The study of these examples have led us to introduce and study the notion of k-complex structure, which is equivalent to G-structure where G is the subgroup of $Gl(n(k+1),\mathbb{R})$ formed by matrices of the form :

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$$\begin{pmatrix} A & 0 & \cdots & 0 \\ 0 & \ddots & \cdots & \vdots \\ \vdots & \cdots & \ddots & 0 \\ 0 & \cdots & 0 & A \end{pmatrix}$$

where $A \in GL(n \times n, \mathbb{R})$.

For $k = 1$, we obtain an almost complex structure J on a $2n-$dimensional manifold with a subbundle E of TM of dimension n such that $TM = E \oplus J(E)$.

An almost k-complex structure on an $n(k+1)$-dimensional manifold is integrable if and only if about every point of M we can find a coordinate neighborhood U with a local coordinates system $(x^{pi}, x^i)_{1 \le p \le k, 1 \le i \le n}$ such that

$$\left(\frac{\partial}{\partial x^{pi}}, \frac{\partial}{\partial x^i}\right)_{1 \le p \le k, 1 \le i \le n}$$

is a cross section of the bundle of frames over U lie in the G-structure.

In this case, an almost $k-$complex structure is integrable when and only when the Bernard-Chern tensor is vanish, or equivalently, the manifold M possesses an adapted connection whitout torsion.

II. $K-$COMPLEX STRUCTURE OF VECTOR SPACES

A. $k-$complex structures

Let V be an $n(k+1)-$dimensional real vector space, let $J^1, \cdots, J^k \in T_1^1(V)$ be k tensors of type $(1,1)$ of rank $2n$, let E be an $n-$codimensional subspace of V and Let E^p be the subspaces of V given by

$$E^p = \bigcap_{q \neq p} \ker J^q.$$

By convention, for $k = 1$, $E^1 = E$ is an $n-$dimensional subspace of V such that $V = E \oplus J(E)$.

Definition II.1 *In the previous notations, we say that* $(J^1, \cdots, J^k; E)$ *is a* $k-$*complex structure on* V *if, for each* $p(p = 1, \ldots, k)$*, the following conditions are satisfied :*

1. *the system* $\{J^1, \cdots, J^k\}$ *is non degenerate, that is,* $\ker J^1 \cap \ldots \cap \ker J^k = (0)$,
2. $\ker J^p$ *is contained in* E,
3. $J^1(E^1) = \cdots = J^k(E^k) \equiv F$, *and* $J^p(F) = E^p$ *for each* p,
4. *for each* p, $Im\, J^p = E^p \oplus F$ *and the restriction* $\widehat{J}^p$ *of* J^p *to* $E^p \oplus F$ *verifies* $\left(\widehat{J}^p\right)^2 = -id_{E^p \oplus F}$.

Proposition II.1 *If* $(J^1, \cdots, J^k; E)$ *is a* $k-$*complex structure on* V*, then :*

1. $\dim F = \dim E^p = n$,
2. $E = E^1 \oplus \cdots \oplus E^k$

Definition II.2 *A* $1-$*complex structure on a* $2n-$*dimensional vector space* V *is a pair* (J, E^1) *such that* J *is a complex structure on* V *and* E^1 *is an* $n-$*dimensional subspace of* V *such that* $V = E^1 \oplus J(E^1)$.

remark II.1 *If* $(J^1, \cdots, J^k; E)$ *is a* $k-$*complex structure on* V *then* $\widehat{J}^p$ *gives a* $1-$*complex structure on* $E^p \oplus F$.

examples II.1

1. **The canonical $k-$complex structure on $\mathbb{R}^{n(k+1)}$.**

   We endow the space $\mathbb{R}^{n(k+1)}$ *with its canonical basis* $(e_{pi}, e_i)_{1 \le p \le k, 1 \le i \le n}$. *Let* E *be the subspace of* $\mathbb{R}^{n(k+1)}$ *spanned by the vectors* $(e_{pi})_{1 \le p \le k, 1 \le i \le n}$ *and let* $J^1, \cdots, J^k \in T_1^1(\mathbb{R}^{n(k+1)})$ *defined by :*

   $$J^p(e_{qi}) = \delta_{pq} e_i, \qquad J^p(e_i) = -e_{pi}.$$

   We verify that the $(k+1)-$*tuple* $(J^1, \cdots, J^k; E)$ *is a* $k-$*complex structure on* $\mathbb{R}^{n(k+1)}$.

2. **Whitney sum of several $1-$complex structure.**

   Let V_1 *be a* $2n-$*dimensional real vectors space,* (j, E^1) *an* $1-$*complex structure on* V_1*, and* $\pi_1 : V_1 \longrightarrow j(E^1) = B$ *the mapping defined by*

   $$\pi_1 = pr_{j(E^1)} \circ \zeta$$

   where ζ *is the canonical isomorphism* $\zeta : E^1 \oplus j(E^1) \longrightarrow E^1 \times j(E^1)$ *and* $pr_{j(E^1)}$ *the canonical projection* $pr_{j(E^1)} : E^1 \times j(E^1) \longrightarrow B = j(E^1)$. *Thus we have* $\ker \pi_1 = E^1$.

   Let V *be the vector subspace of* $V_1 \times \cdots \times V_1$ *(k times) defined by :*

   $$\begin{aligned} V &= \{(u_1, \ldots, u_k) \in V_1 \times \cdots \times V_1 \mid \pi_1(u_1) = \cdots = \pi_1(u_k)\}, \\ &= \{(x_1 + y, \ldots, x_k + y) \mid x_1, \ldots, x_k \in E^1 \text{ and } y \in B\}. \end{aligned}$$

   This space is called the Whitney sum of V_1 *(k-times) and will be denoted by :*

$$V = V_1 \otimes_B \ldots \otimes_B V_1.$$

The linear mapping $\pi : V \longrightarrow B = j(E^1)$ *such that*

$$\pi(u_{1,\cdots,}u_k) = \pi_1(u_1) = \cdots = \pi_1(u_k)$$

for each $(u_{1,\cdots,}u_k) \in V$, *is surjective and satisfies*

$$\ker \pi = E^1 \times \cdots \times E^1 (k - times).$$

It not difficult to see that the space V *is conically isomorphic to* $E^1 \times \cdots \times E^1 \times B$:

$$V \approx E^1 \times \cdots \times E^1 \times B.$$

Let $(e_{pi}, e_i)_{1\le p\le k,\, 1\le i\le n}$ *a basis of* V *with dual basis* $(\omega^{pi}, \omega^i)_{1\le p\le k,\ 1\le i\le n}$.
For

$$J^p = \sum_{i=1}^{n} \left(e_{pi} \otimes \omega^i - e_i \otimes \omega^{pi}\right),$$

then the $(k+1)-tuple$ $(J^1, \cdots, J^k; \ker \pi)$ *is a* $k-complex$ *structure on* V.

Proposition II.2
Let $(J^1, \cdots, J^k; E)$ *a* $k-complex$ *structure on* V, *then there exists a basis* $(e_{pi}, e_i)_{1\le p\le k, 1\le i\le n}$ *of* V *such that :*

1. *for each* $p = 1, \ldots, k$, *the subspace* E^p *is spanned by* $(e_{pi})_{1\le i\le n}$ *and the subspace* F *is spanned by* $(e_i)_{1\le i\le n}$,
2. *for each* $p, q = 1, \ldots, k$, $i = 1, \ldots, n$, *we have* $J^p(e_{qi}) = \delta_{pq} e_i$ *and* $J^p(e_i) = -e_{pi}$.

Proof. Let $(e_i)_{1\le i\le n}$ be a basis of F. For each $p, i (p = 1, \ldots, k,\ i = 1, \ldots, n)$, we take

$$e_{pi} = -J^p(e_i).$$

It is clear that we have $J^p(e_{qi}) = \delta_{pq} e_i$ and $J^p(e_i) = -e_{pi}$, the vectors $e_{p1}, \ldots, e_{pn}$ are independent and consequently constitute a basis of E^p for each p.

Now, let $\lambda^{qj}, \lambda^j \in \mathbb{R}$ $(q = 1, \ldots, k,\ j = 1, \ldots, n)$ verifying :

$$\lambda^{qj} e_{qj} + \lambda^j e_j = 0,$$

then, for each $p = 1, \ldots, k$, we have,

$$J^p\left(\lambda^{qj} e_{qj} + \lambda^j e_j\right) = \lambda^{pj} e_j - \lambda^j e_{pj} = 0$$

Consequently,

$\lambda^{pj} = \lambda^j = 0$ for all $p = 1, \ldots, k$ and $j = 1, \ldots, n$.

This shows the proposition.

corollary II.1 *In the previous hypothesis and notations we have a direct sum* $V = E^1 \oplus \cdots \oplus E^k \oplus F$.

B. The group of linear invariants of the canonical k-complex structure

Let $G_{k,n}(V)$ be the group of automorphisms of V leaving $(J^1, \cdots, J^k; E)$ invariant, that is :

$$G_{k,n}(V) = \left\{ \begin{array}{c} u \in Gl(V) \mid u(E) = E \\ \text{and } u \circ J^p = J^p \circ u \text{ for all } p \end{array} \right\}.$$

In terms of matrices, the group $G_{k,n}(\mathbb{R})$ is formed by matrices of the form :

$$\begin{pmatrix} A & 0 & \cdots & 0 \\ 0 & \ddots & \cdots & \vdots \\ \vdots & \cdots & \ddots & 0 \\ 0 & \cdots & 0 & A \end{pmatrix}$$

where $A \in GL(n \times n, \mathbb{R})$.

It is clear that $G_{k,n}(\mathbb{R})$ is a Lie group, the Lie algebra $\mathcal{G}_{k,n}(\mathbb{R})$ of this group is formed by matrices of the form :

$$\begin{pmatrix} A & 0 & \cdots & 0 \\ 0 & \ddots & \cdots & \vdots \\ \vdots & \cdots & \ddots & 0 \\ 0 & \cdots & 0 & A \end{pmatrix}$$

where $A \in \mathfrak{M}(n \times n, \mathbb{R})$.

III. ALMOST K-COMPLEX MANIFOLDS

A. Almost k-complex manifolds

Let M be a smooth manifold of dimension $n(k+1)$. We say that M is an almost k-complex manifold if for every $x \in M$ the tangent space $T_x M$ is equipped with a k-complex structure of vector spaces:

$$((J^1)_x, \ldots, (J^k)_x; E_x).$$

Of course, we assume that this structure is smooth, that is, for every $x_0 \in M$ there exists an open neighborhood U_0 of x_0 in M and a smooth cross-section $(e_{pi}, e_i)_{1\le p\le k, 1\le i\le n}$ of the bundle of frames of M such that we have on U_0 :

1. for each $p = 1, \ldots, k$, the subspace E_x^p is spanned by $(e_{pi}(x))_{1\le i\le n}$ and the subspace F_x is spanned by $(e_i(x))_{1\le i\le n}$,
2. for each $p, q = 1, \ldots, k$, $i = 1, \ldots, n$, we have $(J^p)_x(e_{qi}(x)) = \delta_{pq} e_i(x)$ and $(J^p)_x(e_i(x)) = -e_{pi}(x)$.

Such a cross-section of the bundle of frames of M is called adapted to the almost k-complex structure of M.

B. k-complex manifolds

An almost k-complex structure

$$(J^1, \ldots, J^k; E)$$

on M is called integrable, or it defines a k-complex structure on M, if, about every point of M we can find a coordinate neighborhood U with a local coordinate system $(x^{pi}, x^i)_{1\leq p\leq k, 1\leq i\leq n}$ such that

$$J^p\left(\frac{\partial}{\partial x^{qi}}\right) = \delta_{pq}\frac{\partial}{\partial x^i}$$

$$J^p\left(\frac{\partial}{\partial x^i}\right) = -\frac{\partial}{\partial x^{pi}}$$

at each point of U.

examples III.1 *1.* **Canonical almost $k-$complex structure on $\mathbb{R}^{n(k+1)}$.**

Consider the real space $\mathbb{R}^{n(k+1)}$ equipped with its Cartesian coordinates $(x^{pi}, x^i)_{1\leq p\leq k, 1\leq i\leq n}$. Let E be the sub-bundle of $T\mathbb{R}^{n(k+1)}$ defined by the equations

$$dx^1 = 0, \cdots, dx^n = 0$$

and let $J^p\,(p = 1, \cdots k)$ be the tensor field on M given by

$$J^p\left(\frac{\partial}{\partial x^{qi}}\right) = \delta_{pq}\frac{\partial}{\partial x^i}$$

$$J^p\left(\frac{\partial}{\partial x^i}\right) = -\frac{\partial}{\partial x^{pi}}$$

The $(k+1)-$tuple $(J^1, \cdots, J^k; E)$ defines an almost $k-$complex structure on $\mathbb{R}^{n(k+1)}$ called the canonical almost $k-$complex structure. This structure induces a natural almost $k-$complex structure on the torus $\mathbb{T}^{n(k+1)}$.

2. **Almost $k-$complex structure on the bundle of $k-(1-$covelocities)**

Let M be an $n-$dimensional manifold. We denote by $T_{k^1}M$ the tangent bundle of $k-(1-$covelocities) of M, that is, the manifold of all $1-$jets of mapping from M to $\mathbb{R}^k$ with target $0 \in \mathbb{R}^k$.

For each coordinate system $(x^j)_{1\leq j\leq n}$ on M we associate the local coordinates

$$(x^i, x_i^1, \cdots, x_i^k)_{1\leq i\leq n}$$

on $T_{k^1}M$ defined by

$$x^i\left(J_{x,0^1}f\right) = x^i(x),$$

$$x_i^p\left(J_{x,0^1}f\right) = \left(\frac{\partial f^p}{\partial x^i}\right)(x),$$

where $J_{x,0^1}f$ is the $1-$jet at $x \in M$ of the map $f = (f^1, \cdots, f^k) : M \longrightarrow \mathbb{R}^k$ such that $f(x) = 0$.

We have $T_{k^1}M$ is an $n(k+1)-$dimensional vector bundle with standard fibre type $\mathbb{R}^{nk}$; the canonical projection is the map $\pi : T_{k^1}M \longrightarrow M$ defined by

$$\pi\left(J_{x,0^1}f\right) = x.$$

For each $p\,(p = 1, \cdots, k)$ we take

$$J^p = \sum_{i=1}^{n}\left(\frac{\partial}{\partial x^{pi}} \otimes dx^i - \frac{\partial}{\partial x^i} \otimes dx^{pi}\right).$$

The $(k+1)-$tuple $(J^1, \cdots, J^k; \ker\pi_*)$ defines a almost $k-$complex structure on $T_{k^1}M$.

C. Integrability of an almost k-complex structure

Let M be a smooth manifold of dimension $n(k+1)$ equipped with an almost k-complex structure

$$(J^1, \ldots, J^k; E).$$

A linear connection π on an almost k-complex manifold is adapted to the almost k-complex structure if, with respect to an adapted cross-section of the bundle of coframes of M, the connection form (π_v^u) takes its values in the algebra $\mathcal{G}_{k,n}(\mathbb{R})$; that is, the components of the connection

$$(\pi_j^s, \pi_j^{ps}, \pi_{pj}^s, \pi_{qj}^{ps})$$

with respect to an adapted cross-section satisfy

$$\begin{aligned} \pi_{pj}^s &= \pi_j^{pi} = 0, \\ \pi_{qj}^{ps} &= 0 \text{ if } p \neq q, \\ \pi_{pj}^{ps} &= \pi_s^j. \end{aligned}$$

The components Ω^u of the torsion Ω of the linear connection π are related to those of the connection form (π_f^u) and the fundamental form ω^u of the frames bundle by the relation

$$\Omega^u = d\omega^u + \sum_f \pi_f^u \wedge \omega^f.$$

An almost $k-$complex structure on an $n(k+1)-$dimensional manifold M is equivalent to a given G-structure with $G = G_{k,n}(\mathbb{R})$. Such a $G_{k,n}(\mathbb{R})$-structure is integrable if this almost $k-$complex structure corresponds to a k-complex structure.

Consequently we can return to the calculation of the Bernard tensor in order to integrate this G-structure. But the vanishing of this tensor is equivalent to the existence of an adapted connection without torsion. Therefore we are going to study the problem of the existence of such a connection.

Note that the integrability of a G-structure implies the vanishing of the Bernard tensor; the reverse is false in the general case. In the case of an almost $k-$complex structure there is an equivalence between the integrability and the vanishing of this tensor (or the existence of an adapted connection without torsion).

The almost k-complex structure is integrable if and only if about every point of M we can find a coordinate neighborhood U with a local coordinate system $(x^{pi}, x^i)_{1\le p\le k, 1\le i\le n}$ such that :

$$J^p\left(\frac{\partial}{\partial x^{qi}}\right) = \delta_{pq}\frac{\partial}{\partial x^i}$$

$$J^p\left(\frac{\partial}{\partial x^{i}}\right) = -\frac{\partial}{\partial x^{pi}}$$

at each point of U.

Proposition III.1 *The almost $k-$complex structure $(J^1, \dots, J^k; E)$ is integrable when and only when the manifold M possesses an adapted connection whitout torsion.*

Proof. Let π be an adapted connection whitout torsion, then for any adapted cross-section $\left(\omega^{pi}, \omega^i\right)_{1\le p\le k, 1\le i\le n}$ we have,

$$d\omega^{pi} = -\pi_{pj}^{pi} \wedge \omega^{pj},$$
$$d\omega^{i} = -\pi_{j}^{i} \wedge \omega^{j}.$$

It results from the Frobenius theorem, that the sub-bundles E^p, E, and $F = J^1\left(E^1\right) = \dots = J^k\left(E^k\right)$ are integrable, thus, around each point of M, there exists a coordinates system $\left(x^{pi}, x^i\right)_{1\le p\le k, 1\le i\le n}$ defined on an open neighborhood U of M such that

$$E^p_{|U} = Vect\left\{\frac{\partial}{\partial x^{pi}},\quad 1\le i\le n\right\},\ \text{for each } p = 1, \dots, k$$
$$F_{|U} = Vect\left\{\frac{\partial}{\partial x^{i}},\quad 1\le i\le n\right\}.$$

For each $p = 1, \dots, k$ we can write

$$J^p\left(\frac{\partial}{\partial x^{pi}}\right) = A_{pi}^j\frac{\partial}{\partial x^j},\ J^p\left(\frac{\partial}{\partial x^{qi}}\right) = 0$$

and

$$J^p\left(\frac{\partial}{\partial x^{i}}\right) = A_i^{pj}\frac{\partial}{\partial x^{pj}}$$

We must determine an adapted coordinate system $\left(y^{pi}, y^i\right)_{1\le p\le k, 1\le i\le n}$, that is

$$y^{pi} = f^{pi}\left(x^{p1}, \dots, x^{pn}\right) \text{ and } y^i = g^i\left(x^1, \dots, x^n\right),$$

such that

$$J^p\left(\frac{\partial}{\partial y^{pi}}\right) = \frac{\partial}{\partial y^{i}},\ J^p\left(\frac{\partial}{\partial y^{qi}}\right) = 0$$

and

$$J^p\left(\frac{\partial}{\partial y^{i}}\right) = -\frac{\partial}{\partial y^{pi}}.$$

In these conditions we have,

$$J^p\left(\frac{\partial}{\partial y^{pi}}\right) = \frac{\partial x^{pj}}{\partial y^{pi}}J^p\left(\frac{\partial}{\partial x^{pj}}\right) = \frac{\partial x^{pj}}{\partial y^{pi}}A_{pj}^l\frac{\partial}{\partial x^l} = \frac{\partial x^l}{\partial y^i}\frac{\partial}{\partial x^l}, \tag{3.1}$$

$$J^p\left(\frac{\partial}{\partial y^{qi}}\right) = \frac{\partial x^{pj}}{\partial y^{qi}}J^p\left(\frac{\partial}{\partial x^{pj}}\right) = \frac{\partial x^{pj}}{\partial y^{qi}}A_{pj}^l\frac{\partial}{\partial x^l} = 0 \text{ for } p\neq q \tag{3.2}$$

and

$$J^p\left(\frac{\partial}{\partial y^{i}}\right) = \frac{\partial x^{l}}{\partial y^{i}}J^p\left(\frac{\partial}{\partial x^{l}}\right) = \frac{\partial x^{l}}{\partial y^{i}}A_l^{pj}\frac{\partial}{\partial x^{pj}} = -\frac{\partial x^{pj}}{\partial y^{pi}}\frac{\partial}{\partial x^{pj}}. \tag{3.3}$$

This proves that we have,

$$\frac{\partial x^{pj}}{\partial y^{pi}}A_{pj}^l = \frac{\partial x^l}{\partial y^i}$$

$$\frac{\partial x^{pj}}{\partial y^{qi}}A_{pj}^l = 0 \text{ for } p\neq q$$

$$\frac{\partial x^{qj}}{\partial y^{pi}} = 0 \text{ for } p \neq q$$

and

$$\frac{\partial x^l}{\partial y^i} A_l^{pj} = -\frac{\partial x^{pj}}{\partial y^{pi}}$$

Consequently the matrices (A_{pj}^l) and $\left(A_j^{pl}\right)$ are invertible and

$$(A_{pj}^l) = -\left(A_j^{pl}\right)^{-1}$$

and also

$$\left(\frac{\partial x^{pi}}{\partial y^{pj}}\right)(A_{pj}^i) = \left(\frac{\partial xi}{\partial y^j}\right) \text{ (product of matrices).}$$

We take $y^i = x^i$, then we have :

$$\left(\frac{\partial x^{pi}}{\partial y^{pj}}\right)(A_{pj}^i) = I_n$$

then,

$$\frac{\partial y^{pi}}{\partial x^{pj}} = A_{pj}^i$$

Finally,

$$y^{pi} = \int_0^{x^{pj}} A_{pj}^i \left(x^{p1}, \ldots, x^{p(i-1)}, t, x^{p(i+1)}, \ldots, x^{pn}\right) dt.$$

It is not difficult to see that (y^{pi}, y^i) is a coordinates system satisfying to :

$$J^p\left(\frac{\partial}{\partial y^{pi}}\right) = \frac{\partial}{\partial y^i},\ J^p\left(\frac{\partial}{\partial y^{qi}}\right) = 0,$$

$$J^p\left(\frac{\partial}{\partial y^i}\right) = -\frac{\partial}{\partial y^{pi}},$$

for each $p\,(p = 1, \ldots, k)$, E^p is spanned by the derivatives $\frac{\partial}{\partial y^{pi}}\,(i = 1, \ldots, n)$

and

F is spanned by the derivatives $\frac{\partial}{\partial y^i}\,(i = 1, \ldots, n)$

this proves the proposition.

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