



## The Moyal Momentum Algebra

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### Abstract

We introduce in this short note some aspects of the Moyal momentum algebra introduced first by Das and Popowicz and that is denoted by Mm algebra. Our interest on this algebra is motivated by the central role that it can play in the formulation of integrable models and in higher conformal spin theories.

### I. INTRODUCTION

In the past few years, there has been a growth in the interest in non-commutative geometry (NCG), which appears in string theory in several ways [1]. Much attention has been paid also to field theories on NC spaces and more specifically Moyal deformed space-time, because of the appearance of such theories as certain limits of string, D-brane and M-theory [2]. One of the strong points of the NCG framework is its richness and also the fact that we can recover all the well known standard results just by requiring the vanishing of the deformed parameter which means also the vanishing of noncommutativity. Note that the passage from commutative to noncommutative space time is achieved by replacing the ordinary commutative product, in the space of smooth functions on  $R^2$  with coordinates  $x, t$ , by the noncommutative associative  $\star$ -product [3]. Moyal deformation is applied also to Lax equations and supersymmetric KdV hierarchies [4,5]. The aim of this letter is to introduce some aspects of the Moyal momentum algebra that we call the Moyal Momentum (Mm)algebra [4] shown to play, in the

NCG framework, an important role in formulating integrable models and higher conformal spin theories in a systematic way [6].

### II. THE MOYAL MOMENTUM ALGEBRA

Using our convention notations [7], we denote this algebra by  $\widehat{\Sigma}(\theta)$ . This is a non-commutative space based on arbitrary momentum Lax operators and which decomposes as:

$$\widehat{\Sigma}(\theta) = \oplus_{r \leq s} \oplus_{m \in \mathbb{Z}} \widehat{\Sigma}_m^{(r,s)}, \quad (1)$$

where  $\widehat{\Sigma}_m^{(r,s)}(\theta)$  is the space of  $\theta$ -Lax operators of fixed conformal spin  $m$  and degrees  $(r,s)$  given by

$$\mathcal{L}_m^{(r,s)}(u) = \sum_{i=r}^s u_{m-i}(x) \star p^i. \quad (2)$$

These are  $\theta$ -differentials whose operator character is inherited from the star product law defined as follows:

$$f(x, p) \star g(x, p) = \sum_{s=0}^{\infty} \sum_{i=0}^s \frac{\theta^s}{s!} (-)^i c_s^i (\partial_x^i \partial_p^{s-i} f) (\partial_x^{s-i} \partial_p^i g), \quad (3)$$

with  $c_s^i = \frac{s!}{i!(s-i)!}$  and  $f(x, p)$  are arbitrary functions on the two-dimensional phase space.

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Now, it is important to precise how the momentum operators act on arbitrary functions  $f(x, p)$  via the star product. Performing computations based on the relation (3), we find the following  $\theta$ - Leibniz rules:

$$p^n \star f(x, p) = \sum_{s=0}^n \theta^s c_n^s f^{(s)}(x, p) p^{n-s}, \tag{4}$$

and

$$p^{-n} \star f(x, p) = \sum_{s=0}^{\infty} (-)^s \theta^s c_{n+s-1}^s f^{(s)}(x, p) p^{-n-s}, \tag{5}$$

with  $n \in \mathbb{N}$  and  $f^{(s)} = (\partial_x^s f)$  is the prime derivative.

We find also that the Moyal bracket defined as [8]:

$$\{f(x, p), g(x, p)\}_\theta = \frac{f \star g - g \star f}{2\theta} \tag{6}$$

is subject to the following expressions

$$\begin{aligned} \{p^n, f\}_\theta &= \sum_{s=0}^n \theta^{s-1} c_n^s \left\{ \frac{1-(-)^s}{2} \right\} f^s p^{n-s}, \\ \{p^{-n}, f\}_\theta &= \sum_{s=0}^{\infty} \theta^{s-1} c_{n+s-1}^s \left\{ \frac{(-)^s-1}{2} \right\} f^s p^{-n-s}. \end{aligned} \tag{7}$$

Having defined the Moyal Momentum algebra, we present here bellow, some remarkable properties.

**P1:** We give here bellow the conformal dimensions of objects used in this study

$$[x] = -1, [p] = [\partial_x] = 1, [\theta] = 0, [u_m] = m, [\widehat{Res}] = [Res] = 1$$

**P2:** The momentum operators  $p^i$  satisfy

$$p^n \star p^m = p^{n+m}. \tag{8}$$

ensuring the suspected rule

$$p^n \star (p^{-n} \star f(x, p)) = f(x, p). \tag{9}$$

**P3:** We should note that formulas (4-5,7) are computed for integers values  $n = 0, 1, 2, 3, \dots$ . Now, what happens if half integers  $n = \pm \frac{1}{2}, \pm \frac{3}{2}, \pm \frac{5}{2}, \dots$  or arbitrary fractional powers were allowed?

Using the general formula (3) and performing algebraic computations we find the following formulas:

$$p^{\frac{a}{2}} \star f(x, p) = \sum_{s=0}^{\infty} \prod_{j=0}^{s-1} \left( \frac{a}{2} - j \right) \frac{\theta^s}{s!} f^{(s)}(x, p) p^{\frac{a}{2}-s}, \tag{10}$$

describing the  $\star$ -product action of half integer powers of the momentum operators on the phase space with

$$\prod_{j=0}^{s-1} \left( \frac{a}{2} - j \right) = \left( \frac{a}{2} \right) \left( \frac{a}{2} - 1 \right) \left( \frac{a}{2} - 2 \right) \dots \left( \frac{a}{2} - s + 1 \right). \tag{11}$$

Moreover, from eq.(10) one can recover eqs.(4-5) once even values  $a = 2n$ , (for n positive or negative) are considered. We can then conclude that these equations as

well as eq.(8) are special cases of (10). The more general situation consist in considering fractional objects type  $p^{\frac{a}{b}}$ , we have

$$p^{\frac{a}{b}} \star f(x, p) = \sum_{s=0}^{\infty} \prod_{j=0}^{s-1} \left( \frac{a}{b} - j \right) \frac{\theta^s}{s!} f^{(s)}(x, p) p^{\frac{a}{b}-s}, \tag{12}$$

Few remarks are in order. If eqs.(4-5) are associated to 'bosonic' behavior of the Moyal Momentum algebra  $\widehat{\Sigma}$ , the half conformal spin object  $p^{\frac{1}{2}} \star u_0 \in \widehat{\Sigma}^{\left(\frac{1}{2}, \frac{1}{2}\right)}$ , for example, may leads us to interpret naively this object as being of fermionic character. The same judgment is made for the fractional object  $p^{\frac{a}{b}} \star u_0 \in \widehat{\Sigma}^{\left(\frac{a}{b}, \frac{a}{b}\right)}$  having fractional conformal spin.

The obviousness of this correspondence with statistics is only impression. In fact, for example for objects like  $p^{\frac{1}{2}} \star u_0$ , even though they present half conformal spins, they are not fermionic objects because  $p^{\frac{1}{2}} \star p^{\frac{1}{2}} = p$  and a fermionic object should be nilpotent. In order to attribute a fermionic behavior to our momentum operators, one have to consider a graded phase space parametrized by the set of variables  $(x, p, \eta$  and  $p_\eta)$  where  $\eta$  is the Grassmann variable and  $p_\eta$  the corresponding nilpotent conjugate momenta given by  $p_\eta^2 = 0$  [5].

Nevertheless, it would be very interesting to look for the contribution and the meaning of these fractional powers of momenta in the framework of KdV hierarchy equations and also in the building of the GD second Hamiltonian structure.

**P4:** The space  $\widehat{\Sigma}(\theta)$  may decomposes into the underlying sub-algebras as

$$\begin{aligned} \widehat{\Sigma}(\theta) &= \oplus_{r \leq s} \widehat{\Sigma}^{(r,s)} \\ &= \oplus_{r \leq s} \oplus_{m \in \mathbb{Z}} \widehat{\Sigma}_m^{(r,s)}(\theta) \\ &= \oplus_{r \leq s} \oplus_{m \in \mathbb{Z}} \oplus_{k=r}^s \widehat{\Sigma}_m^{(k,k)}(\theta) \end{aligned} \tag{13}$$

where  $\widehat{\Sigma}_m^{(k,k)}(\theta)$  is generated by elements type  $u_{m-k} \star p^k$  or  $p^k \star v_{m-k}$ .

**P5:** Using the  $\theta$ -Leibniz rule, we can write, for fixed value of k:

$$\widehat{\Sigma}_m^{(k,k)} \equiv \Sigma_m^{(k,k)} \oplus \theta \Sigma_m^{(k-1,k-1)} \oplus \theta^2 \Sigma_m^{(k-2,k-2)} \oplus \dots \tag{14}$$

where  $\Sigma_m^{(k,k)}$  is the standard one dimensional subspace, containing the prototype objects  $u_{m-k} p^k$  that we consider as the  $(\theta = 0)$ -limit of  $\widehat{\Sigma}_m^{(k,k)}$ .

**P6:** The space  $\widehat{\Sigma}_m^{(0,0)} \equiv \Sigma_m^{(0,0)}$  is nothing but the ring of analytic fields  $u_m$  of conformal spin  $m \in \mathbb{Z}$ . With respect to this definition, the subspace  $\widehat{\Sigma}_m^{(k,k)}$  can be written formally as:

$$\widehat{\Sigma}_m^{(k,k)} \equiv p^k \star \Sigma_{m-k}^{(0,0)}. \tag{15}$$

**P7:** We can easily check that, in general, the space  $\widehat{\Sigma}_m^{(k,k)}$  is not closed under the action of Moyal bracket (6) since we have:

$$\{.,.\}_\theta : \widehat{\Sigma}_m^{(r,s)} \star \widehat{\Sigma}_m^{(r,s)} \rightarrow \widehat{\Sigma}_{2m}^{(r,2s-1)} \quad (16)$$

Imposing the closure, one gets strong constraints on the integers m, r and s namely

$$\begin{aligned} m &= 0 \\ r &\leq s \leq 1 \end{aligned} \quad (17)$$

Under these constraint equations, the sub-spaces  $\widehat{\Sigma}_m^{(r,s)}$  exhibit then a Lie algebra structure since the  $\star$ -product is associative.

**P8:** The sub-space  $\widehat{\Sigma}_m^{(r,s)}$  is characterized by the existence of a residue operation that we denote as  $\widehat{Res}$  and which acts as follows

$$\begin{aligned} \widehat{Res}(u_k \star p^{-k}) &= (u_k \star p^{-k})\delta_{k-1,0} \\ &= u_1\delta_{k-1,0} \end{aligned} \quad (18)$$

This result coincides with the standard residue:  $Res$ , acting on the sub-space  $\Sigma_m^{(r,s)}$ :

$$Res(u_1.p^{-1}) = u_1 \quad (19)$$

We thus have two type of residues  $\widehat{Res}$  and  $Res$  acting on two different spaces  $\widehat{\Sigma}_m^{(r,s)}$  and  $\Sigma_m^{(r,s)}$  but with value on the same ring  $\Sigma_{m+1}^{(0,0)}$ . This Property is summarized as follows:

$$\begin{array}{ccc} \widehat{\Sigma}_m^{(r,s)} & \xrightarrow{\theta=0} & \Sigma_m^{(r,s)} \\ \widehat{Res} \searrow & & \swarrow Res \\ & \Sigma_{m+1}^{(0,0)} & \end{array} \quad (20)$$

We learn from this diagram that the residue operation exhibits a conformal spin quantum number equal to 1.

**P9:** With respect to the previous residue operation, we define on  $\widehat{\Sigma}$  the following degrees pairing

$$(\cdot, \cdot) : \widehat{\Sigma}_m^{(r,s)} \star \widehat{\Sigma}_n^{(-s-1, -r-1)} \rightarrow \Sigma_{m+n+1}^{(0,0)} \quad (21)$$

such that

$$\binom{(r,s)}{m}(u), \binom{(\alpha,\beta)}{n}(v) = \delta_{\alpha+s+1,0}\delta_{\beta+r+1,0}Res_m^{(r,s)}(u) \star_n^{(\alpha,\beta)}(v), \quad (22)$$

showing that the spaces  $\widehat{\Sigma}_m^{(r,s)}$  and  $\widehat{\Sigma}_n^{(-s-1, -r-1)}$  are  $\widehat{Res}$ -dual as  $\Sigma_m^{(r,s)}$  and  $\Sigma_n^{(-s-1, -r-1)}$  are dual with respect to the  $Res$ -operation.

Other important properties with possible applications will be considered in a forthcoming work.

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