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Functional equations and 2×2-matrix valued functions

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Abstract

In this paper, we solve the functional equation

$$f(x+y)+f(x+\sigma(y))=2g_1(x)h_1(y)+2g_2(x)h_2(y), \quad x,y\in G,$$

where G is an abelian topological group, $\sigma : G \longrightarrow G$ is a continuous homomorphism such that $\sigma^2 = I$, and f, g_1, g_2, h_1, h_2 are continuous complex valued functions on G, to be determined.

Keywords: Functional equation, matrix-valued function, vector-valued function, abelian group.

I. INTRODUCTION

Throughout this paper $(G, +)$ is an abelian topological group and $\sigma : G \longrightarrow G$ a continuous homomorphism of G such that $\sigma^2 = I$, where I denotes the identity map. We consider the functional equation

$$f(x+y)+f(x+\sigma(y))=2g_1(x)h_1(y)+2g_2(x)h_2(y), \quad x,y\in G. \tag{1.1}$$

where f, g_1, g_2, h_1, h_2 to be determined, are continuous complex valued functions on G. This equation contains the following functional equations

$$f(x+y)=g_1(x)h_1(y)+g_2(x)h_2(y), \quad x,y\in G, \tag{1.2}$$

and

$$f(x+y)+f(x-y)=2g_1(x)h_1(y)+2g_2(x)h_2(y), \quad x,y\in G. \tag{1.3}$$

The equation (1.2) (the Levi-Civitá's functional equation $n = 2$) corresponds to $\sigma = I$ and (1.3) to $\sigma = -I$. This tow cases have been discussed respectively in [34] and [25], where G is abelian respectively

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G is 2-divisible abelian group. In this work, we give a unified treatment outside the situation of 2-divisible group for general involutive automorphisms σ, so our discussion comprises not only $\sigma = \pm I$, but also examples like the reflexion in a hyperplane of $\mathbf{R^n}$ and the symplectic involution of the additive group of 2×2 matrices. In our solutions formulas we take into account homomorphisms $\gamma : G \longrightarrow \mathbf{C}^*$ for which $\gamma = \gamma \circ \sigma$. When G is 2-divisible and $\sigma = -I$ the only such homomorphism is $\gamma = 1$. Our purpose is to solve the functional equation (1.1) in the case where G is abelian and σ is a general automorphism of G of order 2: We show that the solution of this equation are related to d'Alembert's and Wilson's functional equations for vectors and 2×2 matrix valued functions. Our point of depart are the works by Sinopoulos [24] and Stetkær [31]. There are versions of d'Alembert's functional equation involving matrix or operator valued functions, e.g. Fattorini [11], Kurepa [17], [18], Baker and Davison [5], Kisyński [15], [16], Székelyhidi [34], Chojnacki [6], [7] Sinopolous [25] and Stetkær [27], [30], [31] and [32]. The special case of $\sigma = I$ encompasses the functional equation (1.4) below, studied by O'Connor ($n = 2$) in [19], the Cauchy functional equations (1.5) and (1.6), the functional equation of type Pexider (1.7) and the sine (1.8) and cosine (1.9) addition equations, viz

$$f(x+y) = \overline{h_1(-x)}h_1(y) + \overline{h_2(-x)}h_2(y), \quad x, y \in G, \tag{1.4}$$

$$f(x+y) = g(x) + h(y), \quad x, y \in G, \tag{1.5}$$

$$f(x+y) = f(x)f(y), \quad x, y \in G, \tag{1.6}$$

$$f(x+y) = f(x) + g(x)h(y), \quad x, y \in G, \tag{1.7}$$

$$f(x+y) = f(x)g(y) + g(x)f(y), \quad x, y \in G, \tag{1.8}$$

$$f(x+y) = f(x)f(y) + g(x)g(y), \quad x, y \in G. \tag{1.9}$$

For more information about these equations, we refer to the monographs by Aczél [1] and by Aczél and Dhombres [2].
The equation (1.1) is also a joint generalization of the d'Alembert's functional equation (1.10) ([1], [4], [14], [20], [21], [22], [28], [29]), the Jensen's functional equation (1.11) ([28], [29], [30]), the Gajda's functional equation of type d'Alembert's (1.12) ([13], [32]), the Wilson's functional equation (1.13) ([2], [23], [29], [37]), the quadratic equation (1.14) ([2], [30]), the functional equation of type Pexider (1.15) ([3], [30]) and the cosine (1.16) and the sine (1.17) addition equations ([9], [29], [32]), viz

$$f(x+y) + f(x+\sigma(y)) = 2f(x)f(y), \quad x, y \in G, \tag{1.10}$$

$$f(x+y) + f(x+\sigma(y)) = 2f(x), \quad x, y \in G, \tag{1.11}$$

$$f(x+y) + f(x-y) = 2\overline{h_1(-x)}h_1(y) + 2\overline{h_2(-x)}h_2(y), \quad x, y \in G, \tag{1.12}$$

$$f(x+y) + f(x+\sigma(y)) = 2g(x)h(y), \quad x, y \in G, \tag{1.13}$$

$$f(x+y) + f(x+\sigma(y)) = 2f(x) + 2f(y), \quad x, y \in G, \tag{1.14}$$

$$f(x+y) + f(x+\sigma(y)) = 2f(x) + 2g(x)h(y), \quad x, y \in G, \tag{1.15}$$

$$f(x+y) + f(x+\sigma(y)) = 2f(x)g(y) + 2g(x)f(y), \quad x, y \in G, \tag{1.16}$$

$$f(x+y) + f(x+\sigma(y)) = 2f(x)f(y) + 2g(x)g(y), \quad x, y \in G. \tag{1.17}$$

This paper organized as follows: In the second section, we deal with the solutions of (1.1). In the last section, we use these results to determine the solutions of (1.2), (1.4), (1.8), (1.9), (1.12), (1.16) and (1.17).

In addition to the terminology introduced above we shall need the following notation: We let e denote the neutral element of the abelian topological group $(G,+)$, $M_2(\mathbf{C})$ the set of all 2×2 matrices over $\mathbf{C}$ and $\mathcal{F}$ the set of all vector-valued functions on G with linearly independent components. The algebra of all continuous complex-valued functions on G is denoted $C(G)$. We let $\mathcal{M}(\mathcal{G})$ be the set of all continuous homomorphisms $\gamma : G \longrightarrow (\mathbf{C}\setminus\{\mathbf{0}\},\cdot)$, and put $\mathcal{M}^+(\mathcal{G}) = \{\gamma \in \mathcal{M}(\mathcal{G}) : \gamma\circ\sigma = \gamma\}$. We let $\mathcal{A}(\mathcal{G})$ be the set of all continuous additive maps of G into $(\mathbf{C},+)$, and let $\mathcal{A}^{\pm}(\mathcal{G}) = \{\dashv \in \mathcal{A}(\mathcal{G}) : \dashv\circ\sigma = \pm\dashv\}$. We let $\mathcal{S}(\mathcal{G})$ be the set of maps $S : G \longrightarrow \mathbf{C}$ of the form $S(x) = q(x,x)$, $x \in G$, where $q : G\times G \longrightarrow \mathbf{C}$ ranges over the continuous bi-additive symmetric maps, and let $\mathcal{S}^-(\mathcal{G})$ be the subset for which q satisfies $q(\sigma(x),y) = -q(x,y)$, $x,y \in G$. The transpose of a matrix A is denoted by A^t and the adjoint by A^*.

II. THE MAIN RESULT

In this section, we will solve the functional equation

$$f(x+y)+f(x+\sigma(y)) = 2g(x)h(y) = 2g_1(x)h_1(y)+2g_2(x)h_2(y),\quad x,y\in G, \tag{2.1}$$

where we use the notation $g=(g_1,g_2)$, $h=(h_1,h_2)^t$, and where $f,g_1,g_2,g_1,g_2 \in C(G)$. We will view $g(x)$ as a row vector and $h(x)$ as a column vector for each $x\in G$.
By a solution of (2.1), we understand a triplet $\{f,g,h\}$ of continuous functions satisfying (2.1).

Theorem II.1 *Let $f : G \longrightarrow \mathbf{C}$, $g=\{g_1,g_2\} : G \longrightarrow \mathbf{C^2}$, $h=\{h_1,h_2\}^t : G \longrightarrow \mathbf{C^2}$, with $g,h\in\mathcal{F}$ satisfy the functional equation* (2.1). *Then there exists a solution $\Phi : G \longrightarrow M_2(\mathbf{C})$ of d'Alembert's functional equation*

$$\Phi(x+y)+\Phi(x+\sigma(y)) = 2\Phi(x)\Phi(y),\ x,y\in G, \tag{2.2}$$

such that $\Phi(e)=I$, and there exists a column vector $v_0\in\mathbf{C^2}$ such that

$$h(x)=\Phi(x)v_0,\ \ x\in G, \tag{2.3}$$

$$g(x+y)+g(x+\sigma(y)) = 2\Phi(y)^t g(x),\ x,y\in G, \tag{2.4}$$

$$f(x)=g(x)v_0,\ \ x\in G. \tag{2.5}$$

proof II.2 *Let $V = span\{h(x),\ x\in G\}\subseteq\mathbf{C^2}$. For each fixed $x\in G$, we define a mapping $\Phi(x) : V \longrightarrow V$ by*

$$\Phi(x)v = \sum_{i=1}^{m} c_i\frac{h(x+y_i)+h(x+\sigma(y_i))}{2},\ \ x\in G,$$

where $v=\sum_{i=1}^m c_ih(y_i)$ for all $y_1,...,y_m\in G$ and $c_1,...,c_m\in\mathbf{C}$.

First we will prove that $\Phi(x)$ is well-defined. To show that we suppose that an element $v\in V$ admits two expressions

$$v=\sum_{i=1}^{m}c_ih(y_i)=\sum_{j=1}^{l}d_jh(z_j).$$

Then for all $x,y\in G$,

$$\begin{aligned}
g(x)\sum_{i=1}^{m}c_i\frac{h(y+y_i)+h(y+\sigma(y_i))}{2} &= \frac{1}{4}\sum_{i=1}^{m}c_i\{2g(x)h(y+y_i)+2g(x)h(y+\sigma(y_i))\}\\
&= \frac{1}{4}\sum_{i=1}^{m}c_i\{f(x+y+y_i)+f(x+\sigma(y)+\sigma(y_i))\\
&\quad +f(x+y+\sigma(y_i))+f(x+\sigma(y)+y_i)\}\\
&= \frac{1}{4}\sum_{i=1}^{m}c_i\{2g(x+y)+2g(x+\sigma(y))\}h(y_i)\\
&= \frac{g(x+y)+g(x+\sigma(y))}{2}\sum_{i=1}^{m}c_ih(y_i)\\
&= \frac{g(x+y)+g(x+\sigma(y))}{2}\sum_{j=1}^{l}d_jh(z_j)\\
&= \frac{1}{4}\sum_{j=1}^{l}d_j\{(2g(x+y)h(z_j)+2g(x+\sigma(y))h(z_j)\}\\
&= \frac{1}{4}\sum_{j=1}^{l}d_j\{f(x+y+z_j)+f(x+\sigma(y)+\sigma(z_j))\\
&\quad +f(x+y+\sigma(z_j))+f(x+\sigma(y)+z_j)\}\\
&= g(x)\sum_{j=1}^{l}d_j\frac{h(y+z_j)+h(y+\sigma(z_j))}{2}.
\end{aligned}$$

Since $g\in\mathcal{F}$ it follows that

$$\Phi(x)(\sum_{i=1}^{m}c_ih(y_i))=\Phi(x)(\sum_{j=1}^{l}d_jh(z_j)),\ x\in G.$$

Furthermore since $g\in\mathcal{F}$, we show that $h\circ\sigma=h$. Accordingly we get that $\Phi(e)=I$.
We will now show that Φ satisfies the equation (2.2). *Evidently, for each $x\in G$, $\Phi(x)$ is a linear mapping on V. Furthermore for all $x,y\in G$ and $v\in V$, we have*

$$\begin{aligned}
[\Phi(x+y)+\Phi(x+\sigma(y))]v &= \Phi(x+y)v+\Phi(x+\sigma(y))v\\
&= \frac{1}{2}\sum_{i=1}^{m}c_i\{h(x+y+y_i)+h(x+y+\sigma(y_i))\\
&\quad +h(x+\sigma(y)+y_i)+h(x+\sigma(y)+\sigma(y_i))\}\\
&= 2\Phi(x)(\frac{1}{2}\sum_{i=1}^{m}c_ih(y+y_i)+h(y+\sigma(y_i))\\
&= 2\Phi(x)\Phi(y)v,
\end{aligned}$$

which establishes (2.2). *Furthermore since $h\in\mathcal{F}$, then by ([2, lemma 14.1]), we get that $V=\mathbf{C}^2$.*
Setting $v_0=h(e)$, then we obtain

$$h(x)=\Phi(x)v_0,\ \ x\in G.$$

which establishes (2.3).
Now by taking $y=e$ in (2.1), *we get*

$$\begin{aligned}
f(x) &= g(x)h(e)\\
&= g(x)v_0.
\end{aligned}$$

Finally for all $x,y\in G$ and $v\in V$, we have

$$\begin{aligned}
2\Phi^t(y)g(x)v &= 2g(x)\Phi(y)v \\
&= 2g(x)\frac{1}{2}\sum_{i=1}^{m} c_i\{h(y+y_i)+h(y+\sigma(y_i))\} \\
&= \frac{1}{2}\sum_{i=1}^{m} c_i\{2g(x)h(y+y_i)+2g(x)h(y+\sigma(y_i))\} \\
&= \frac{1}{2}\sum_{i=1}^{m} c_i\{f(x+y+y_i)+f(x+\sigma(y)+\sigma(y_i)) \\
&\quad +f(x+y+\sigma(y_i))+f(x+\sigma(y)+y_i)\} \\
&= \frac{1}{2}\sum_{i=1}^{m} c_i\{2g(x+y)+2g(x+\sigma(y))\}h(y_i) \\
&= \{g(x+y)+g(x+\sigma(y))\}v,
\end{aligned}$$

which establishes that

$$g(x+y)+g(x+\sigma(y)) = 2\Phi(y)^t g(x), \quad x,y \in G.$$

Remark II.3 *a) By using the isomorphism* $(h_1(x),h_2(x))^t \longrightarrow h_1(x)g_1+h_2(x)g_2$ *for all* $x \in G$, *we get that* Φ *is the same matrix as in [32].*
b) If $g \notin \mathcal{F}$ *or* $h \notin \mathcal{F}$, *then* (2.1) *reduces to a Wilson equation (see [23] and [28]). So we will only consider solutions such* $g,h \in \mathcal{F}$.

In the next theorem, we will solve the functional equation (2.1).

Theorem II.4 *Let* $\{f,g_1,g_2,h_1,h_2\} : G \longrightarrow \mathbf{C}$, *with* $g=\{g_1,g_2\}$, $h=\{h_1,h_2\}^t \in \mathcal{F}$ *be a solution of the functional equation* (2.1). *Then there exist* $v_0,u,w \in \mathbf{C}^2$ *and* $A \in Gl(2,\mathbf{C})$ *such that*

$$h(x) = (A^{-1})^t\frac{E(x)^t+E(\sigma(x))^t}{2}A^t v_0, \quad x \in G, \tag{2.6}$$

$$g(x) = A(E(x)u+E(\sigma(x))w)), \quad x \in G, \tag{2.7}$$

$$f(x) = A(E(x)u+E(\sigma(x)w))v_0, \quad x \in G, \tag{2.8}$$

where $E : G \longrightarrow M_2(G)$ *has one of the following forms in which* $\gamma,\gamma_1,\gamma_2 \in \mathcal{M}(\mathcal{G})$, $\gamma^+,\gamma_1^+,\gamma_2^+ \in \mathcal{M}^+(\mathcal{G})$, $a \in \mathcal{A}(\mathcal{G})$, $a^-,a_1^-,a_2^- \in \mathcal{A}^-(\mathcal{G})$, $a^+ \in \mathcal{A}(\mathcal{G})$, $S^- \in \mathcal{S}^-(\mathcal{G})$ *and* $c,c_1 \in \mathbf{C}$

$$E_1 = \begin{pmatrix} \gamma_1 & 0 \\ 0 & \gamma_2 \end{pmatrix}, \quad E_2 = \begin{pmatrix} \gamma_1 & 0 \\ 0 & \gamma_2^+(1+a^-) \end{pmatrix},$$

$$E_3 = \begin{pmatrix} \gamma_1^+(1+a_1^-) & 0 \\ 0 & \gamma_2^+(1+a_2^-) \end{pmatrix}, \quad E_4 = \gamma\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix},$$

$$E_5 = \gamma^+\begin{pmatrix} 1 & a+S^- \\ 0 & 1 \end{pmatrix}, \quad E_6 = \gamma^+\begin{pmatrix} 1+a^- & c(a^-)^3+3c(a^-)^2+a^++a^+a^-+a_1^- \\ 0 & 1+a^- \end{pmatrix}.$$

proof II.5 *Let* $\{f,g,h\}$ *be a solution of* (2.1). *According to Theorem 2.1, there exists a solution* $\Phi : G \longrightarrow M_2(\mathbf{C})$ *of d'Alembert's functional equation* (2.2) *such that* $g(x+y)+g(x+\sigma(y)) = 2\Phi(y)^t g(x)$. *Applying Theorem 3.3 in [31] to this equation there exist* $u,w \in \mathbf{C}^2$ *and* $A \in Gl(2,\mathbf{C})$ *such that* $g(x) = A(E(x)u+E(\sigma(x))w))$ *and* $\Phi^t = A\frac{E+(Eo\sigma)}{2}A^{-1}$, *where* E *has one of the 6 form above. For the remainder we use the formulas* (2.3) *and* (2.5) *in Theorem 2.1.*

Remark II.6 *When* $\sigma = -I$ *and* G *is* 2*-divisible, there are some simplifications and the corresponding matrices* $(E_i)_{1\le i\le 6}$ *are given in [31, Remark 3.4]. This case is considered by Sinopoulos in [25].*

III. APPLICATIONS

By applying our study we illustrate how 2×2 matrix-valued functions Φ can be used to solve special cases ($n = 2$) of Levi-Civitá's functional equation [34], the functional equation studied by O'Connor in [19], the Gajda's functional equation of d'Alembert's type [13] and the sine and cosine addition equations [8], [9], [29], [36]. In a recent paper [32] Stetkær deled with these equations in terms of a matrix-valued spherical function.

Corollary III.1 *Let $\{f, g_1, g_2, h_1, h_2\} : G \longrightarrow \mathbf{C}$ with $g = \{g_1, g_2\}$, $h = \{h_1, h_2\}^t \in \mathcal{F}$ be a solution of the Levi-Civitá's functional equation*

$$f(x+y) = g_1(x)h_1(y) + g_2(x)h_2(y), \quad x, y \in G. \tag{3.1}$$

Then there exist $v_0, w \in \mathbf{C}^2$ and $A \in Gl(2, \mathbf{C})$ such that

$$h(x) = (A^{-1})^t E(x)^t A^t v_0, \quad x \in G, \tag{3.2}$$

$$g(x) = AE(x)w, \quad x \in G, \tag{3.3}$$

$$f(x) = (AE(x)w)v_0, \quad x \in G, \tag{3.4}$$

where $E : G \longrightarrow M_2(G)$ has one of the following forms

$$E_1 = \begin{pmatrix} \gamma_1 & 0 \\ 0 & \gamma_2 \end{pmatrix}, \quad E_2 = \gamma \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad E_3 = \gamma \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix},$$

where $\gamma, \gamma_1, \gamma_2 \in \mathcal{M}(\mathcal{G})$, $a \in \mathcal{A}(\mathcal{G})$.

proof III.2 *Since $\sigma = I$, there are some simplifications : $\mathcal{M}^+(\mathcal{G}) = \mathcal{M}(\mathcal{G})$, $\mathcal{A}^+(\mathcal{G}) = \mathcal{A}(\mathcal{G})$, $\mathcal{A}^-(\mathcal{G}) = \{\prime\}$ and $\mathcal{S}^-(\mathcal{G}) = \{\prime\}$.*

Corollary III.3 *Let $\{f, h_1, h_2\} : G \longrightarrow \mathbf{C}$, with $h = \{h_1, h_2\}^t \in \mathcal{F}$ be a solution of the functional equation*

$$f(x+y) = \overline{h_1(-x)}h_1(y) + \overline{h_2(-x)}h_2(y), \quad x, y \in G. \tag{3.5}$$

Then there exist $v_0 \in \mathbf{C}^2$ and $A \in Gl(2, \mathbf{C})$ such that

$$h(x) = AE(x)A^{-1}v_0, \quad x \in G, \tag{3.6}$$

$$f(x) = v_0^* AE(x)A^{-1}v_0, \quad x \in G, \tag{3.7}$$

where $E : G \longrightarrow M_2(G)$ has the form

$$E = \begin{pmatrix} \gamma_1 & 0 \\ 0 & \gamma_2 \end{pmatrix},$$

where $\gamma_1, \gamma_2 \in \mathcal{M}(\mathcal{G})$, and $\gamma_1 \neq \gamma_2$.

proof III.4 *Let $\langle ., . \rangle$ be the inner product on $V = \mathbf{C}^2$. For all $x \in G$ and $u, v \in V$ we have*

$$\begin{aligned}
\langle \Phi(-x)u, v \rangle &= \langle \sum_{i=1}^{n} c_i h(-x + y_i), \sum_{j=1}^{m} d_j h(z_j) \rangle \\
&= \sum_{i=1}^{n} \sum_{j=1}^{m} c_i \overline{d_j} \langle h(-x + y_i), h(z_j) \rangle \\
&= \sum_{i=1}^{n} \sum_{j=1}^{m} c_i \overline{d_j} f(x - y_i + z_j) \\
&= \sum_{i=1}^{n} \sum_{j=1}^{m} c_i \overline{d_j} \langle h(y_i), h(x + z_j) \rangle \\
&= \langle u, \Phi(x)v \rangle \\
&= \langle \Phi(x)^* u, v \rangle.
\end{aligned}$$

So $\Phi(x)^t = \Phi(-x)$, *for all* $x \in G$.
Since $\Phi(x)\Phi(y) = \Phi(y)\Phi(x)$, *for all* $x, y \in G$ *and* $\Phi(x)^* = \Phi(-x)$ *it follows that* $\Phi(x)$ *can be diagonalized simultaneously. So there exists* $A \in Gl(2, \mathbf{C})$ *such that* $\Phi(x) = AE(x)A^{-1}$ *for all* $x \in G$, *where*

$$E = \begin{pmatrix} \gamma_1 & 0 \\ 0 & \gamma_2 \end{pmatrix},$$

and $\gamma_1, \gamma_2 \in \mathcal{M}(\mathcal{G})$. *Since* $h \in \mathcal{F}$ *it follows that* $\gamma_1 \neq \gamma_2$. *Putting* $y = e$ *in* (3.5), *we get that* $f(x) = \tilde{h(x)}v_0$, *where* $\tilde{h(x)} = h(-x)^*$.

In the next corollary, we solve the Gajda's functional equation of type d'Alembert ([13], [32]) without imposing Gajda's assumption that there exists a point $z_0 \in G$ such that $h(z_0) = 0$.

Corollary III.5 *Let* $\{f, h_1, h_2\} : G \longrightarrow \mathbf{C}$, *with* $\{h_1, h_2\}^t \in \mathcal{F}$ *be a solution of the Gajda's functional equation*

$$f(x+y) + f(x-y) = 2\overline{h_1(-x)}h_1(y) + 2\overline{h_2(-x)}h_2(y), \quad x, y \in G. \tag{3.8}$$

Then there exist $v_0 \in \mathbf{C^2}$ *and* $A \in Gl(2, \mathbf{C})$ *such that*

$$h(x) = A\frac{E(x) + E(-x)}{2}A^{-1}v_0, \quad x \in G, \tag{3.9}$$

$$f(x) = v_0^* A^{-1}\frac{E(x) + E(-x)}{2}A^{-1}v_0, \quad x \in G, \tag{3.10}$$

where $E : G \longrightarrow M_2(G)$ *has the form*

$$E = \begin{pmatrix} \gamma_1 & 0 \\ 0 & \gamma_2 \end{pmatrix},$$

where $\gamma_1, \gamma_2 \in \mathcal{M}(\mathcal{G})$, *and* $\gamma_1 \neq \gamma_2$.

proof III.6 *As in the proof of Corollary 3.2, we show that* $\Phi^*(x) = \Phi(-x)$ *and* $\Phi(x)$ *can be diagonalized simultaneously. So there exists* $A \in Gl(2, \mathbf{C})$ *such that*

$$\Phi(x) = A\begin{pmatrix} \omega_1(x) & 0 \\ 0 & \omega_2(x) \end{pmatrix}A^{-1}.$$

Since $\Phi(x+y) + \Phi(x-y) = 2\Phi(x)\Phi(y)$, $x, y \in G$, *it follows that* $\omega_i(x+y) + \omega_i(x-y) = 2\omega_i(x)\omega_i(y)$, *for all* $x, y \in G$, *and* $i \in \{1, 2\}$. *According to [14], there exist* $\gamma_i \in \mathcal{M}(\mathcal{G})$, $i \in \{1, 2\}$, *such that* $\omega_i(x) = \frac{\gamma_i(x)+\gamma_i(-x)}{2}$, $x \in G$. *So* $\Phi(x) = A\frac{E(x)+E(-x)}{2}A^{-1}$, *where* $E = \begin{pmatrix} \gamma_1 & 0 \\ 0 & \gamma_2 \end{pmatrix}$ *and* $\gamma_1, \gamma_2 \in \mathcal{M}(\mathcal{G})$, *and* $\gamma_1 \neq \gamma_2$. *Finally we get from Theorem 2.1 that* $h(x) = \Phi(x)v_0$, $x \in G$ *and by putting* $y = e$ *in* (3.8), *we get* $f(x) = \tilde{h(x)}v_0$, *for all* $x \in G$. *this ends the proof of the corollary.*

In the next two theorems, we will apply our study to solve certain versions of sine and cosine addition functional equations. It were solved in ([8], [9], [10], [32]). [29] treats the sine addition equation.

Theorem III.7 *Let* $\{f, g\} \in C(G)$ *be a solution of the functional equation*

$$\frac{f(x+y) + f(x+\sigma(y))}{2} = f(x)g(y) + g(x)f(y), \quad x, y \in G. \tag{3.11}$$

Then there are only the following possibilities
i) $f = 0$ *and* g *arbitrary in* $C(G)$.
ii) There exist a $\gamma \in \mathcal{M}(\mathcal{G})$ *and a constant* $c \in \mathbf{C} \setminus \{\mathbf{0}\}$ *such that*

$$f = c\frac{\gamma + \gamma \circ \sigma}{2}, \; g = \frac{\gamma + \gamma \circ \sigma}{4}.$$

iii) There exist $\gamma_1, \gamma_2 \in \mathcal{M}(\mathcal{G})$, $\gamma_1 \neq \gamma_2$ *and a constant* $c \in \mathbf{C} \setminus \{\mathbf{0}\}$ *such that*

$$f = c\left(\frac{\gamma_1 + \gamma_1 \circ \sigma}{2} - \frac{\gamma_1 + \gamma_2 \circ \sigma}{2}\right),$$

$$g = \frac{\gamma_1 + \gamma_1 \circ \sigma}{4} + \frac{\gamma_1 + \gamma_2 \circ \sigma}{4}.$$

4i) $g = \frac{\gamma + \gamma \circ \sigma}{2}$, *where* $\gamma \in \mathcal{M}(\mathcal{G})$ *and* f *is a solution of the equation* (3.11).

Remark III.8 *When* $\sigma = I$, (3.11) *becomes*

$$f(x+y) = f(x)g(y) + g(x)f(y), \quad x, y \in G, \tag{3.12}$$

which generalizes the sine addition formula $\sin(x+y) = \sin(x)\cos(y) + \cos(x)\sin(y)$ *see [8] and [36].*

proof III.9 $f = 0$ *is case (i), we shall in the remainder of the proof assume that* $f \neq 0$. *If* $h = \{g, f\} \notin \mathcal{F}$, *then there exists* $\lambda \in \mathbf{C}$ *such that* $g = \lambda f$, *where* $\lambda \neq 0$, *because* $g = 0$ *implies* $f = 0$. *So* $\frac{f(x+y)+f(x+\sigma(y))}{2} = 2\lambda f(x)f(y)$, *for all* $x, y \in G$, *from which we get (2i).*
iii) Assume that $h = \{g, f\}^t \in \mathcal{F}$. *According to Theorem 6 in [29], there exists a constant* $\alpha \in \mathbf{C}$ *such that* $\frac{g(x+y)+g(x+\sigma(y)}{2} = g(x)g(y) + \alpha f(x)f(y)$, $x, y \in G$.
If $\alpha = 0$, *then* $g = \frac{\gamma+\gamma\circ\sigma}{2}$, *where* $\gamma \in \mathcal{M}(\mathcal{G})$ *and* f *is a solution of* (3.11), *we get the case (4i).*
If $\alpha \neq 0$, *then for all* $y \in G$ *and* $v \in V$, *one has*

$$\begin{aligned}
\Phi(y)v &= \frac{1}{2}\sum_{i=1}^{n} c_i\{h(y+y_i) + h(y+\sigma(y_i))\} \\
&= \sum_{i=1}^{n} c_i\left(\frac{g(y+y_i)+g(y+\sigma(y_i))}{2}, \frac{f(y+y_i)+g(y+\sigma(y_i))}{2}\right)^t \\
&= \sum_{i=1}^{n} c_i(g(y)g(y_i) + \alpha^2 f(y)g(y_i), f(y)g(y_i) + g(y)f(y_i))^t \\
&= \begin{pmatrix} g(y) & \alpha^2 f(y) \\ f(y) & g(y) \end{pmatrix} \sum_{i=1}^{n} c_i(g(y_i), f(y_i))^t \\
&= \begin{pmatrix} g(y) & \alpha^2 f(y) \\ f(y) & g(y) \end{pmatrix} v.
\end{aligned}$$

Then

$$\Phi(y) = \begin{pmatrix} g(y) & \alpha^2 f(y) \\ f(y) & g(y) \end{pmatrix}, \quad y \in G.$$

Since $\alpha \neq 0$ *then*

$$\Phi(y) = \begin{pmatrix} \alpha & -\alpha \\ 1 & 1 \end{pmatrix} \begin{pmatrix} g + \alpha f(y) & 0 \\ & g - \alpha f(y) \end{pmatrix} \begin{pmatrix} \alpha & -\alpha \\ 1 & 1 \end{pmatrix}^{-1}, \ y \in G.$$

Since $\Phi(x+y) + \Phi(x+\sigma(y)) = 2\Phi(x)\Phi(y)$, *for all* $x, y \in G$, *it follows that*

$$g + \alpha f = \frac{\gamma_1 + \gamma_1 \circ \sigma}{2}, \ g - \alpha f = \frac{\gamma_2 + \gamma_2 \circ \sigma}{2},$$

where $\gamma_1, \gamma_2 \in \mathcal{M}(\mathcal{G})$ *and* $\gamma_1 \neq \gamma_2$.

Theorem III.10 *Let $\{f,g\} \in C(G)$ be a solution of the functional equation*

$$\frac{f(x+y)+f(x+\sigma(y))}{2} = f(x)f(y)+g(x)g(y), \quad x,y \in G. \tag{3.13}$$

Then there are only the following possibilities
i) $f = g = 0$.
ii) There exist a $\gamma \in \mathcal{M}(\mathcal{G})$ *and a constant* $\lambda \in \mathbf{C} \setminus \{\pm\mathbf{i}\}$ *such that*

$$f = \frac{1}{1+\lambda^2}\frac{\gamma+\gamma\circ\sigma}{2}, \; g = \frac{\lambda}{1+\lambda^2}\frac{\gamma+\gamma\circ\sigma}{2}.$$

iii) There exist $\gamma_1, \gamma_2 \in \mathcal{M}(\mathcal{G})$, $\gamma_1 \neq \gamma_2$ *and a constant* $\lambda \in \mathbf{C} \setminus \{\mathbf{0}, \mathbf{i}, -\mathbf{i}\}$ *such that*

$$f = \frac{\lambda}{\lambda+\lambda^{-1}}\frac{\gamma_1+\gamma_1\circ\sigma}{2} + \frac{\lambda^{-1}}{\lambda+\lambda^{-1}}\frac{\gamma_1+\gamma_2\circ\sigma}{2},$$

$$g = \frac{1}{\lambda+\lambda^{-1}}\{\frac{\gamma_1+\gamma_1\circ\sigma}{2} - \frac{\gamma_2+\gamma_2\circ\sigma}{2}\}.$$

4i) There exist a $\gamma \in \mathcal{M}(\mathcal{G})$ *such that* g *is a solution of the equation*

$$\frac{g(x+y)+g(x+\sigma(y))}{2} = g(x)l(y)+l(x)g(y), \quad x,y \in G, \tag{3.14}$$

where $f = l + ig$ *or* $f = l - ig$ *and* $l = \frac{\gamma+\gamma\circ\sigma}{2}$.

proof III.11 *If* $f = 0$, *then* $g = 0$ *is case (i). We shall in the remainder of the proof assume that* $f \neq 0$. *If* $h = \{f,g\} \notin \mathcal{F}$, *then there exists* $\lambda \in \mathbf{C}$ *such that* $g = \lambda f$, *then we get the case (2i).*
iii) Assume that $h = \{g,f\}^t \in \mathcal{F}$. *According to Theorem 7 in [29], there exists a constant* $\alpha \in \mathbf{C}$ *such that*
$\frac{g(x+y)+g(x+\sigma(y))}{2} = f(x)g(y)+g(x)f(y)+\alpha g(x)g(y)$, $x,y \in G$. *Then for all* $y \in G$ *and* $v \in V$, *one has*

$$\begin{aligned}
\Phi(y)v &= \frac{1}{2}\sum_{i=1}^{n} c_i\{h(y+y_i)+h(y+\sigma(y_i))\} \\
&= \sum_{i=1}^{n} c_i(\frac{f(y+y_i)+f(y+\sigma(y_i))}{2}, \frac{g(y+y_i)+g(y+\sigma(y_i))}{2})^t \\
&= \sum_{i=1}^{n} c_i(f(y)f(y_i)+g(y)g(y_i), f(y)g(y_i)+g(y)f(y_i)+\alpha g(y)g(y_i))^t \\
&= \begin{pmatrix} f(y) & g(y) \\ g(y) & f(y)+\alpha g(y) \end{pmatrix} \sum_{i=1}^{n} c_i(f(y_i), g(y_i))^t \\
&= \begin{pmatrix} f(y) & g(y) \\ g(y) & f(y)+\alpha g(x) \end{pmatrix} v.
\end{aligned}$$

Then

$$\Phi(y) = \begin{pmatrix} f(y) & g(y) \\ g(y) & f(y)+\alpha g(y) \end{pmatrix}, \quad y \in G.$$

Since $\alpha \neq \pm i$, *then we may diagonalize* $\Phi(x)$, $x \in G$, *as follows*

$$\Phi(y) = A \begin{pmatrix} f(x)-\lambda g(x) & 0 \\ 0 & f(x)+\lambda^{-1}g(x) \end{pmatrix} A^{-1}, \; x \in G, A \in Gl(2,\mathbf{C})$$

Since $\Phi(x+y)+\Phi(x+\sigma(y))=2\Phi(x)\Phi(y)$, *foe all* $x,y\in G$, *it follows that*

$$f-\lambda g=\frac{\gamma_1+\gamma_1\circ\sigma}{2},\quad f+\lambda^{-1}g=\frac{\gamma_2+\gamma_2\circ\sigma}{2},$$

where $\gamma_1,\gamma_2\in\mathcal{M}(\mathcal{G})$ *and* $\gamma_1\neq\gamma_2$, *this is a case (iii). If* $\alpha=\pm i$, *then* $\lambda=\alpha=\lambda^{-1}$. *So* $g=\frac{\gamma+\gamma\circ\sigma}{2}\pm if$, *where* $\gamma\in\mathcal{M}(\mathcal{G})$ *and* g *is a solution of* (3.14).

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