African Journal Of Mathematical Physics Vol **3** *No 1 (2006)21-27*

Local Currents for a Stable Algebra of Quantum Mechanics

Gerald A. Goldin[1,*] and Sarben Sarkar[2,†]
1. Departments of Mathematics and Physics, Rutgers University SERC Bldg. Rm. 239, 118 Frelinghuysen Road Piscataway, New Jersey 08854, USA
2. Department of Physics, King's College London, The Strand, London WC2R 2LS, UK *
gagoldin@dimacs.rutgers.edu † *sarben.sarkar@kcl.ac.uk*

Abstract

Beginning with nonrelativistic local current algebra in two space dimensions, we show how to recover a certain deformed algebra of quantum mechanics in one space dimension discussed by Vilela Mendes, having a fundamental length scale ℓ. The extra dimension becomes an unobservable internal degree of freedom in the $\ell \to 0$ limit. This hints at a new possible interpretation of representations of the local current algebra, not as describing conventional bosonic, fermionic, or anyonic particles in two-space, but as describing systems having such features in the deformed one-dimensional quantum mechanics. We also look at discretized local currents within irreducible representations of the deformed algebra.

I. INTRODUCTION

In the last few years, some researchers have again focused attention on the combined Heisenberg and Poincaré Lie algebras, as a possible kinematical algebra for relativistic quantum mechanics[1,2]. The nontrivial second cohomology of this Lie algebra[3] means it is "unstable", but there is a parameterized family of nontrivial deformations that are "stable"—in the sense that all the Lie algebras in an open neighborhood in the space of structure constants are mutually isomorphic. This means that small changes in the values of physical constants do not affect the Lie-algebraic structure. The algebra proposed by Vilela Mendes is a particular choice among several possibilities. It is a deformation by two fundamental length parameters ℓ and R; taking $\ell \to 0$ and $R \to \infty$ leads to recovery of the original Lie algebra.

Here we seek to define an equal-time, local current algebra compatible with the nonrelativistic quantum kinematics that follows from Vilela Mendes' proposal. Like him we consider the case where the large length $R \to \infty$, but $\ell \neq 0$. Our main idea is to extend the usual nonrelativistic local current algebra of scalar functions and vector fields by introducing an abstract single-particle configuration space—a manifold having one dimension more than that of the spatial manifold for the limiting situation with $\ell \to 0$. We focus on the case where the spatial dimension $d = 1$. Then the many interesting, inequivalent representations of the local current algebra in two-space, including representations describing N conventional quantum particles obeying the statistics of bosons, fermions, or anyons, might possibly be reinterpreted as descriptive of local currents for a deformed algebra of quantum mechanics. This is a topic of continuing investigation by the authors.

We also comment here on the relation of the irreducible representations of the deformed algebra to those of the limiting Heisenberg algebra.

II. NONRELATIVISTIC LOCAL CURRENT ALGEBRA

Operator-valued distributions for the local, fixed-time mass density $\rho(\mathbf{x})$ and momentum density $J(\mathbf{x})$ are defined formally in terms of underlying canonical fields $\widehat{\psi}, \widehat{\psi}^*$ by[4]:

$$\rho(\mathbf{x}) = m\widehat{\psi}^*(\mathbf{x})\,\widehat{\psi}(\mathbf{x})\,,\ \mathbf{J}(\mathbf{x}) = \frac{\hbar}{\mathbf{2i}}\left\{\widehat{\psi}^*(\mathbf{x})\,\nabla\widehat{\psi}(\mathbf{x}) - \left[\nabla\widehat{\psi}^*(\mathbf{x})\right]\widehat{\psi}(\mathbf{x})\right\}, \tag{2.1}$$

where m is the particle mass. We let $\rho(f) = \int \rho(\mathbf{x})\,f(\mathbf{x})\,d\mathbf{x}$ and $J(\mathbf{g}) = \sum_{k=1}^{d} J_k(\mathbf{x})\,g_k(\mathbf{x})\,d\mathbf{x}$, where f and the components g_k of the vector field $\mathbf{g}$ are real-valued C^∞ test functions on $\mathbf{R^d}$ with compact support. Then, independent of whether the original field $\widehat{\psi}$ is bosonic, fermionic, or (in the case $d = 2$) anyonic, one obtains the local current algebra[5,6]

$$[\rho(f_1), \rho(f_2)] = 0\,, \quad [\rho(f), J(\mathbf{g})] = i\hbar\rho(\mathbf{g}\cdot\nabla\mathbf{f})\,,$$

$$[J(\mathbf{g_1}), J(\mathbf{g_2})] = -i\hbar J([\mathbf{g_1}, \mathbf{g_2}])\,; \tag{2.2}$$

where $[\mathbf{g_1}, \mathbf{g_2}] = \mathbf{g_2}\cdot\nabla\mathbf{g_1} - \mathbf{g_1}\cdot\nabla\mathbf{g_2}$ is the usual Lie bracket of vector fields. In the 1-particle Hilbert space $L^2_{d\mathbf{x}}(\mathbf{R}^d)$, we have the self-adjoint representation

$$\rho(f)\,\Psi = mf(\mathbf{x})\Psi,\ J(\mathbf{g})\,\Psi = \frac{\hbar}{2i}\{\,\mathbf{g}(\mathbf{x})\cdot\nabla\Psi + \nabla\cdot[\mathbf{g}(\mathbf{x})\,\Psi]\,\}, \tag{2.3}$$

where $\Psi(\mathbf{x})$ is a square-integrable function. Now as the test function $f(\mathbf{x})$ is taken to approximate an indicator function $\chi_B(\mathbf{x})$ for a Borel set $B \subseteq \mathbf{R^d}$, the expectation value $(\Psi, \rho(f)\,\Psi)$ with respect to the single-particle wave function Ψ approximates $m\int \chi_B(\mathbf{x})\,|\psi(\mathbf{x})|^2\,d\mathbf{x}$, which is the mass times the usual probability for finding the particle in the region B. If $f(\mathbf{x})$ is, on the other hand, taken as a succession of increasingly sharply peaked gaussian functions as an approximating series to $\delta(\mathbf{x}-\mathbf{x_0})$ for a fixed point $\mathbf{x}_0 \in \mathbf{R^d}$, then $(\Psi, \rho(f)\,\Psi)$ approaches $m\,|\Psi(\mathbf{x_0})|^2$. We also see how to recover the Heisenberg algebra: if $f(\mathbf{x})$ approximates the coordinate function x_j, then $\rho(f)$ approximates the moment operator mq_j acting in $L^2_{d\mathbf{x}}(\mathbf{R^d})$ *via* multiplication by mx_j. Similarly, if $\mathbf{g}(\mathbf{x})$ is taken to approximate a constant vector field in the j-direction, so that (let us say) $g_j(\mathbf{x}) \sim 1$ with $g_k(\mathbf{x}) = 0$ for $k \neq j$, then $J(\mathbf{g}) \sim -i\hbar\partial/\partial x_j$, which is the action of the momentum operator p_j in $L^2_{d\mathbf{x}}(\mathbf{R^d})$. Then of course, if I is the identity operator,

$$[\,q_j, p_k\,] = i\hbar\delta_{jk}I\,. \tag{2.4}$$

Finally, generators of spatial rotations may be recovered—e.g., in three space dimensions, the operator for orbital angular momentum about the x_3-axis is approximated by choosing $g_1(\mathbf{x}) = -x_2$, $g_2(\mathbf{x}) = x_1$, and $g_3(\mathbf{x}) = 0$ inside a large compact region $|\mathbf{x}| \leq R$; outside this region, $\mathbf{g}(\mathbf{x})$ falls smoothly to 0. Then $J(\mathbf{g})$ approximates the operator $\hbar M_{12} = L_3 = (\mathbf{q}\times\mathbf{p})\cdot\mathbf{e_3}$ acting in $L^2_{d\mathbf{x}}(\mathbf{R^3})$, where $\mathbf{e_3}$ is the unit vector in the x_3-direction. Introduce the self-adjoint operator-valued distribution $Q(f,\mathbf{g})$ in $L^2_{d\mathbf{x}}(\mathbf{R}^d)$,

$$Q(f,\mathbf{g})\Psi \;=\; f(\mathbf{x})\Psi \;+\frac{1}{2i}\{\,\mathbf{g}(\mathbf{x})\cdot\nabla\Psi(\mathbf{x}) + \nabla\cdot[\mathbf{g}(\mathbf{x})\,\Psi(\mathbf{x})]\}\,, \tag{2.5}$$

which is the 1-particle representation of the natural semidirect sum of the commutative Lie algebra of compactly-supported, real-valued C^∞ functions f on $\mathbf{R^d}$, with the Lie algebra of vector fields $\mathbf{g}$ on $\mathbf{R^d}$; i.e.,

$$[(f_1,\mathbf{g_1}),(\mathbf{f_2},\mathbf{g_2})] = (\mathbf{g_2}\cdot\nabla\mathbf{f_1} - \mathbf{g_1}\cdot\nabla\mathbf{f_2}, -[\mathbf{g_1},\mathbf{g_2}])\,. \tag{2.6}$$

Eqs. (2.3) follow when we set $\rho(f) = m\,Q(f,0)$ and $J(\mathbf{g}) = \hbar\,\mathbf{Q}(\mathbf{0},\mathbf{g})$. The (infinite-dimensional) group corresponding to Eq. (2.6) is the natural semidirect product of the Abelian group of real-valued, compactly supported C^∞ functions on $\mathbf{R^d}$ (under pointwise addition) with the group of compactly supported

C^∞ diffeomorphisms of $\mathbf{R}^d$ under composition.

The construction and classification of inequivalent unitary representations of this group, and corresponding self-adjoint representations of the local current algebra, allows the classification and prediction of kinematical possibilities for quantum systems. We have the usual N-particle representations, $N = 1, 2, 3, \ldots$, satisfying bosonic or fermionic statistics for $N \geq 2$ in more than one space dimension. We have particle systems obeying anyonic statistics in two-dimensional space[7,8,9,10,11] or other exotic statistics. We have also particles with spin, composite systems having dipole or higher multipole moments, and infinite-particle or extended systems with infinite-dimensional configuration spaces; for a recent review, see[12].

III. DEFORMED ALGEBRAS FOR QUANTUM MECHANICS

We now turn to obtaining a certain deformed algebra from the above local currents. Suppose we interpret Eq. (2.5) as applying in a two-dimensional Euclidean space with coordinates (x, w); where x parameterizes a traditional 1-dimensional spatial manifold, and w extends that spatial manifold by an additional dimension. To be concrete, we then have $Q\,(h, g_x, g_w)$ acting in $L^2_{dxdw}(\mathbf{R^2})$, where h is drawn from the space of compactly-supported, real-valued C^∞ test functions on (x, w)-space, and g_x, g_w are the components of a compactly-supported, C^∞ vector field on (x, w)-space:

$$Q(h, g_x, g_w) = h\,(x, w) + \frac{1}{2i}\,\{\, g_x\,(x, w)\,\frac{\partial}{\partial x} + \frac{\partial}{\partial x}\,g_x\,(x, w)\}$$

$$+\frac{1}{2i}\,\{\, g_w\,(x, w)\,\frac{\partial}{\partial w} + \frac{\partial}{\partial w}\,g_w\,(x, w)\}\,. \tag{3.1}$$

Let $\ell > 0$, and define $Q_\ell\,(h, g_x, g_w) = Q(h, \ell g_x, \ell g_w)$. Then we have a family of Lie algebras parameterized by ℓ and represented by self-adjoint operators. In the $\ell \to 0$ limit, $Q_\ell(h, g_x, g_w)$ reduces to the multiplication operator $Q(h, 0, 0)$. We may then write

$$\rho(f) \;=\; \lim_{h\to f} m\,Q(h, 0, 0) = \lim_{h\to f}\,\lim_{\ell\to 0} m\,Q_\ell\,(h, g_x, g_w) \tag{3.2}$$

to recover the mass density operator in one space dimension. The limit here pertains to the fact that f depends only on x and is independent of w, while h is compactly supported in (x, w)-space. Likewise we have

$$J(g) \;=\; \lim_{g_x\to g} \hbar\,Q(0, g_x, 0) = \lim_{g_x\to g}\,\lim_{\ell\to 0}\frac{\hbar}{\ell}\,Q_\ell\,(0, g_x, 0)\,. \tag{3.3}$$

The motivation for writing this is the following. In the limit where $h(x, w)$ approaches the coordinate function x, and the vector field $(g_x(x, w), g_w(x, w))$ has components that approach the coordinate functions $(-w, x)$, we recover from Q_ℓ the operator q, given by

$$q = x + i\ell\left(w\frac{\partial}{\partial x} - x\frac{\partial}{\partial w}\right)\,. \tag{3.4}$$

With $p = (\hbar/\ell)\lim Q_\ell\,(0, g_x, 0)$, taken in the limit where g_x approaches the constant vector field of magnitude 1, and $\mathcal{J}$=$\lim Q_\ell\,(h, 0, g_w)$, taken in the limit where both h and g_w become identically 1, we have

$$p = -i\hbar\frac{\partial}{\partial x}\,,\;\; \mathcal{J} = I - i\ell\frac{\partial}{\partial w}\,. \tag{3.5}$$

These operators represent the following deformed Heisenberg brackets:

$$[\,q, p\,] = i\hbar\,\mathcal{J}\;,\;\; [\,q, \mathcal{J}\,] = -i\frac{\ell^2}{\hbar}p\;,\;\; [\,p, \mathcal{J}\,] = 0\;. \tag{3.6}$$

In this representation, the operators smoothly go over to a representation of the standard Heisenberg representation as $\ell \to 0$. In that limit, the w coordinate survives, but becomes (in our interpretation) unobservable.

Eqs. (3.4) and (3.5) provide a *reducible* representation of (3.6) in the Hilbert space $L^2_{dxdw}(\mathbf{R}^2)$. In the next section, we shall see what happens within an *irreducible* representation of Eqs. (3.6).

One natural choice of local currents corresponding to p and $\mathcal{J}$ is,

$$J(g) = \frac{\hbar}{2i}\{g(x)\frac{\partial}{\partial x} + \frac{\partial}{\partial x}g(x)\}, \tag{3.7}$$

$$\mathcal{J}(k) = k(w) + \frac{\ell}{2i}\{k(w)\frac{\partial}{\partial w} + \frac{\partial}{\partial w}k(w)\}, \tag{3.8}$$

where $g(x)$ and $k(w)$ are compactly-supported C^∞ functions on $\mathbf{R}$. These incorporate the intuitive idea of local flows in the two coordinate directions. To express them in terms of Q or Q_ℓ (again recalling that the arguments of Q are compactly supported in both the x and w coordinates), we use Eq. (3.3) for $J(g)$ together with the equation, $\mathcal{J}(k) = \lim_{h\to k}\lim_{g_w\to k} Q(h, 0, \ell g_w) = \lim_{h\to k}\lim_{g_w\to k} Q_\ell(h, 0, g_w)$. Because of the way the operator q mixes the x and the w directions, it is necessary to incorporate the full (x, w)-dependence in the test functions h, g_x and g_w that appear as arguments of Q.

The Lie algebra of Eq. (3.6), is a special case (with $d = 1$) of the deformation discussed by Vilela Mendes, to which we now turn our attention. Taking the 4-vectors q_μ and p_ν, $\mu, \nu = 0, 1, 2, 3$, and the Lorentz generators $M_{\mu\nu}$, we have the canonical brackets,

$$[p_\mu, q_\nu] = i\hbar\eta_{\mu\nu}\mathcal{J}, \quad [q_\mu, q_\nu] = [p_\mu, p_\nu] = [q_\mu, \mathcal{J}] = [p_\mu, \mathcal{J}] = 0, \tag{3.9}$$

and the Lorentz brackets,

$$[M_{\mu\nu}, M_{\rho\sigma}] = i(M_{\mu\sigma}\eta_{\nu\rho} + M_{\nu\rho}\eta_{\mu\sigma} - M_{\nu\sigma}\eta_{\mu\rho} - M_{\mu\rho}\eta_{\nu\sigma}), \tag{3.10}$$

together with the additional brackets,

$$[M_{\mu\nu}, p_\lambda] = i(p_\mu\eta_{\nu\lambda} - p_\nu\eta_{\mu\lambda}), \ [M_{\mu\nu}, q_\lambda] = i(q_\mu\eta_{\nu\lambda} - q_\nu\eta_{\mu\lambda}), \tag{3.11}$$

$$[M_{\mu\nu}, \mathcal{J}] = 0;$$

where $\eta_{\mu\nu} = \mathrm{diag}[1, -1, -1, -1]$ in units with $c = 1$. To describe the kinematics of a nonrelativistic quantum particle, one may represent a subalgebra of this Lie algebra (replacing μ, ν by $j, k = 1, 2, 3$) by self-adjoint operators in Hilbert space.

Although the Lie algebras of Eqs. (3.9) and Eqs. (3.10) are separately stable, the combined Lie algebra of Eqs. (3.9)-(3.11) is not. Vilela Mendes considers a stable deformation labeled by fundamental lengths R and ℓ, with brackets where Eqs. (3.10) and (3.11) are unchanged, but Eqs. (3.9) are replaced by,

$$[p_\mu, q_\nu] = i\hbar\eta_{\mu\nu}\mathcal{J}, \ [q_\mu, q_\nu] = -i\varepsilon\ell^2 M_{\mu\nu},$$

$$[p_\mu, p_\nu] = -i\frac{\varepsilon'\hbar^2}{R^2}M_{\mu\nu}, \ [q_\mu, \mathcal{J}] = i\varepsilon\frac{\ell^2}{\hbar}p_\mu, \ [p_\mu, \mathcal{J}] = -i\frac{\varepsilon'\hbar}{R^2}q_\mu, \tag{3.12}$$

where ε and ε' are ± 1. Evidently as $\ell \to 0$ and $R \to \infty$, we recover Eqs. (3.9)-(3.11). This Lie algebra is isomorphic to the Lie algebra of the orthogonal group in six dimensions, with metric $\eta_{ab} = \mathrm{diag}[1, -1, -1, -1, \varepsilon', \varepsilon]$. In a self-adjoint representation, the q_μ no longer commute with each other.

As the parameter ℓ is relevant locally, we shall follow Vilela Mendes in focusing on the algebra obtained by taking $R \to \infty$. Then the brackets involving R in Eqs. (3.12) become zero.

We focus on self-adjoint representations of the Heisenberg-like subalgebra, with $j, k = 1, 2, 3$, and with $\varepsilon = -1$, given by the spatial components of Eqs. (3.10)-(3.11), together with the brackets,

$$[q_j, q_k] = i\ell^2 M_{jk}, \ [q_j, p_k] = i\delta_{jk}\hbar\mathcal{J}, \ [q_j, \mathcal{J}] = -i\frac{\ell^2}{\hbar}p_j,$$

$$[p_j, p_k] = [p_j, \mathcal{J}\,] = 0\,. \tag{3.13}$$

(The case $\varepsilon = +1$ does not reduce to the Heisenberg algebra.) The algebra of Eqs. (3.6) corresponds to the $d = 1$ case of Eqs. (3.13).

Now the Lie algebra of Eqs. (3.10)-(3.11) and (3.13) represents the *global* symmetry of the deformed quantum theory. We have seen that one way to obtain a description of *local* symmetry is to turn to the local current algebra on a space augmented by the coordinate w.

IV. THE KINETIC ENERGY HAMILTONIAN

In[1] it was suggested that for a particle of mass m, we should use $H_0 = p^2/2m$, where p is the generator appearing in the algebra of Eqs. (3.6), and that the oscillator Hamiltonian should then be $H_{\rm osc} = p^2/2m + m\omega^2 q^2/2$. But one reasonable criterion for determining the choice of H_0 is the physical condition that the time-derivative of the particle position should be the particle velocity. That is, we should expect H_0 and $H_{\rm osc}$ to satisfy,

$$\dot{q} = \frac{1}{i\hbar}\,[q, H_0] = \frac{1}{i\hbar}\,[q, H_{\rm osc}] = \frac{p}{m}\,. \tag{4.1}$$

However, we have from Eqs. (3.6) that $(1/i\hbar)\,[q, p^2/2m] = (p\,\mathcal{J} + \mathcal{J} p\,)\,/2m$, which becomes p/m when $\mathcal{J}$ is the identity operator but not otherwise. To fulfill Eq. (4.1) we propose to modify the form of the kinetic energy term in the Hamiltonian, so that

$$H_0 = \frac{1}{2m}\left\{p^2 + \frac{\hbar^2}{\ell^2}\,(\mathcal{J} - I)^2\right\}\,, \tag{4.2}$$

where I is the identity operator in a representation of the algebra. Note that the coefficient $\hbar^2/\ell^2$ in (4.2) is required for the correct commutator with q.

V. CURRENTS IN IRREDUCIBLE REPRESENTATIONS

The Casimir operator

$$C = \frac{1}{\hbar^2}\,p^2 + \frac{1}{\ell^2}\,\mathcal{J}^2 \tag{5.1}$$

commutes with all of the generators in Eqs. (3.6). In an irreducible representation, C takes the value ρ_0^2. The spectrum of the self-adjoint operator representing the generator q is discrete, given by $n\ell$ for $n \in \mathbf{Z}$; write the corresponding eigenvector as $|n\ell\rangle$, and $q\,|n\ell\rangle = n\ell|n\ell\rangle$, so that

$$q = \sum_{n=-\infty}^{\infty} n\ell|n\ell\rangle\langle n\ell|\,, \quad I = \sum_{n=-\infty}^{\infty} |n\ell\rangle\langle n\ell|\,. \tag{5.2}$$

The corresponding local mass density operator $\mathbf{J}_q$ takes the form

$$\mathbf{J}_q(g) = m \sum_{n=-\infty}^{\infty} g(n\ell)\,|n\ell\rangle\,\langle n\ell|\,, \tag{5.3}$$

where in analogy with the continuum case, the real-valued function g has compact support; i.e., for some $N > 0$, $g(n\ell) = 0$ whenever $|n\ell| > N\ell$. When $g(n\ell)$ approximates the function $n\ell$, $\mathbf{J}_q(g)$ approximates the moment operator mq. When $g(n\ell) \geq 0$ ($\forall n \in \mathbf{Z}$), $\mathbf{J}_q(g)$ is a positive operator. When $g(n\ell)$ approximates the constant function 1, $\mathbf{J}_q(g)$ approximates the mass times the identity operator. As we are in a representation of the local currents describing a single particle, we can interpret $(1/m)\mathbf{J}_q(g)$ as a spatial probability density operator averaged with $g(n\ell)$.

In the irreducible representation labeled by ρ_0, we have matrix elements

$$\langle n\ell|\,p\,|m\ell\rangle = \frac{\hbar\rho_0}{2}\,(\delta_{n+1,m}+\delta_{n-1,m})\,,\ \langle n\ell|\,\mathcal{J}|m\ell\rangle = \frac{i\ell\rho_0}{2}\,(\delta_{n+1,m}-\delta_{n-1,m}).$$

Now we can introduce the local currents $\mathbf{J}_p(h) = \frac{1}{2}\sum_{n=-\infty}^{\infty}\widetilde{h}(n\ell)\{\,p\,|n\ell\rangle\,\langle n\ell| + |n\ell\rangle\,\langle n\ell|\,p\,\}$, and $\mathbf{J}_{\mathcal{J}}\,(r) = \frac{1}{2}\sum_{n=-\infty}^{\infty}\widetilde{r}\,(n\ell)\,\{\,\mathcal{J}|n\ell\rangle\,\langle n\ell| + |n\ell\rangle\,\langle n\ell|\mathcal{J}\,\}$, where $h\,(n\ell) \equiv (1/2)\,[\,\widetilde{h}\,(n\ell) + \widetilde{h}\,((n+1)\,\ell)\,]$ and $r\,(n\ell) \equiv (1/2)\,[\,\widetilde{r}\,(n\ell)+\widetilde{r}\,((n+1)\,\ell)\,]$ are also compactly supported. As $\widetilde{h}(n\ell)$ and $\widetilde{r}(n\ell)$ approximate the function that is identically 1, so do $h(n\ell)$ and $r(n\ell)$; then $\mathbf{J}_p(h)$ approximates p, $\mathbf{J}_{\mathcal{J}}\,(r)$ approximates $\mathcal{J}$, and the global algebra is recovered. ¿From the above,

$$\mathbf{J}_p\,(h) \;=\; \frac{\hbar\rho_0}{2}\sum_{n=-\infty}^{\infty} h\,(n\ell)\,\{\,|n\ell\rangle\,\langle (n+1)\,\ell| + |(n+1)\,\ell\rangle\,\langle n\ell|\,\}\,, \tag{5.4}$$

$$\mathbf{J}_{\mathcal{J}}\,(r) \;=\; \frac{i\ell\rho_0}{2}\sum_{n=-\infty}^{\infty} r\,(n\ell)\,\{\,|n\ell\rangle\,\langle (n+1)\,\ell| - |(n+1)\,\ell\rangle\,\langle n\ell|\,\}\,. \tag{5.5}$$

For the Lie algebra of currents generated by these operators to be *local,* we would need the commutator brackets of the operators $\mathbf{J}_q\,(g)$, $\mathbf{J}_p\,(h)$, and $\mathbf{J}_{\mathcal{J}}\,(r)$ given by Eqs. (5.3), (5.4), and (5.5) to yield similarly local expressions. These expressions are all linear combinations of operators of the form $|n\ell\rangle\,\langle n\ell|$, $|n\ell\rangle\,\langle (n+1)\,\ell|$, and $|(n+1)\,\ell\rangle\,\langle n\ell|$. However, commutators such as $[\mathbf{J}_p\,(h)\,,\mathbf{J}_{\mathcal{J}}\,(r)]$ generate terms of the form $|(n+1)\,\ell\rangle\,\langle (n-1)\,\ell|$ and $|(n-1)\,\ell\rangle\,\langle (n+1)\,\ell|$. Successive commutators give $|(n-m)\,\ell\rangle\,\langle (n+m)\,\ell|$ and $|(n+m)\,\ell\rangle\,\langle (n-m)\,\ell|$, for arbitrary $m \in \mathbf{Z}$. Therefore, to close the Lie algebra of these currents, one is forced to include new basis elements taking more general forms; e.g.,

$$\sum_{n,m=-\infty}^{\infty} s(n\ell, m\ell)\{\,|(n+m)\,\ell\rangle\,\langle (n-m)\,\ell| + |(n-m)\,\ell\rangle\,\langle (n+m)\,\ell|\,\}\,,$$

where s is a compactly supported function on the square lattice of points $(n\ell, m\ell)$. Such currents are *nonlocal* in the positional eigenvalues, since the points $(n-m)\ell$ and $(n+m)\ell$ become arbitrarily far apart. This sort of behavior by the commutation relations of discretized local derivatives is well-known in the context of lattice models.

It is worth remarking that we do have within this framework an *equation of continuity* for the deformed quantum theory, relating the time-derivative of $\mathbf{J_q}$ to the spatial divergence of $\mathbf{J_p}$. Taking the Hamiltonian H to be $H_0 + V(q)$, with H_0 given by Eq. (4.2), a rather lengthy calculation leads to the result,

$$\dot{\mathbf{J}}_q(g) = \frac{1}{i\hbar}\,[\,\mathbf{J}_{\mathbf{q}}(\mathbf{g}), \mathbf{H}\,] = \mathbf{J}_p\,(Dg)\ , \tag{5.6}$$

where

$$Dg\,(n\ell) \equiv \frac{g\,((n+1)\ell) - g\,(n\ell)}{\ell} \tag{5.7}$$

is the discretized derivative. The density $\mathbf{J}_q$ and current $\mathbf{J}_p$ that appear in this equation of continuity are local, but they belong to a Lie algebra that necessarily includes currents that are nonlocal with respect to eigenvalues of the positional operator q.

VI. ACKNOWLEDGEMENTS

G. Goldin wishes to acknowledge the Leverhulme Trust for support of this research, and King's College London for hospitality during his 2004-2005 sabbatical leave. He also thanks the organizers of ICHEMP '05 in Marrakech for the opportunity to present this work.

VII. REFERENCES

[1] R. Vilela Mendes, *J. Math. Phys.* **41** 156-186 (2000).

[2] C. Chryssomalakos and E. Okon, *Int. J. Mod. Phys. D* **13**, 2003-2034 (2004); hep-th/0410212

[3] J. A. de Azćarraga and J. M. Izquierdo, *Lie Groups, Lie Algebras, Cohomology and Some Applications in Physics,* Cambridge, UK: Cambridge University Press (1995).

[4] R. Dashen and D. H. Sharp, *Phys. Rev.* **165**, 1857 (1968).

[5] G. A. Goldin, *J. Math. Phys.* **12**, 462-487 (1971).

[6] G. A. Goldin and D. H. Sharp, *Phys. Rev. Lett.* **76**, 1183-1187 (1996).

[7] J. M. Leinaas and J. Myrheim, *Nuovo Cimento* **37B**, 1 (1977).

[8] G. A. Goldin, R. Menikoff and D. H. Sharp, *J. Math. Phys.* **21**, 650-664 (1980); **22**, 1664-1668 (1981).

[9] F. Wilczek, *Phys. Rev. Lett.* **48**, 1144-1146 (1982).

[10] G. A. Goldin and D. H. Sharp, *Phys. Rev. D* **28**, 830-832 (1983).

[11] G. A. Goldin, R. Menikoff and D. H. Sharp, *Phys. Rev. Lett.* **51**, 2246-2249 (1983).

[12] G. A. Goldin, "Lectures on diffeomorphism groups in quantum physics." In *Contemporary Problems in Mathematical Physics: Proceedings of the Third International Conference, Cotonou, Benin.* Singapore: World Scientific, 3-93 (2004).