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A QFT Realisation of Kac-Moody Theory of Lie Algebras

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Abstract

We construct the structure of a graded q-differential algebra on a $\mathbf{Z}_N$-graded algebra by means of a graded q-commutator. We apply this construction to a reduced quantum plane and study the first order differential calculus on a reduced quantum plane induced by the N-differential of the graded q-differential algebra.

I. INTRODUCTION

Kac-Moody (KM) algebras g and their representations have been playing a central role in many areas of classical and quantum physics. These KM algebras, which are classified into three main classes (ordinary, affine and indefinite), are behind the derivation of many physical results including exact quantum field theoretic ones[1]-[9].

However, despite almost four decades since their discovery in 1968, these KM extensions of simple Lie algebras[12,13] and their representations have not been fully explored and are far away from full control. Only partial results have been obtained for the so called KM hyperbolic subset and the indefinite sector of KM algebras and their representations is still an open Lie algebra problem. Building mathematical physics models, in particular field theoretic ones, dealing with such Lie algebra symmetries would be then of great interest in the understanding of indefinite KM invariance. This may also shed light on hyperbolic symmetries encountered in the framework of $10d$ type IIB superstring and $11d$ M-theory compactifications[14]-[18].

In this paper, we propose a quantum field theory (QFT) realization of theory KM algebras and their underlying Vinberg theorem. Our interest into this quantum field realization has been motivated by a set of observations; in particular the following three ones:

(**1**) Dynkin diagrams of KM algebras have a remarkable similarity with the QFT Feynman graphs. For instance, Dynkin diagram of $A_n \simeq su\,(n+1)$ semi simple Lie algebra looks like a scalar QFT propagator. A naive correspondence shows that the remaining known Dynkin diagrams are associated with a special class of QFT Green functions. Note also that Dynkin diagrams of less familiar KM algebras such as $T_{p,q,r}$ hyperbolic algebras, with p,q and r positive integers greater than 2, have also a QFT counter part. The $T_{p,q,r}$s (resp. T_{p_1,p_2,p_3,p_4}) are formally analogous to the three (four) points tree vertex of scalar quantum field theory with a cubic (quartic) interaction.

(**2**) Cartan matrix A of generic $su(n+1)$ algebras, with its very particular entries $A_{ij} = 2\delta_{ij} - \delta_{i,j+1} - \delta_{i,j-1}$, admits a special factorisation, $A = P^{\dagger}P$; its properties are quite similar to those of the $(1+1)$ dimensional Laplacian $\Delta = \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} = \partial_+\partial_-$ of QFT_{1+1}. As we will see later, the (A_{ij}) operator is noting but the discrete version of the Laplacian Δ.

(**3**) The basis of classification of KM theory rests on Vinberg theorem relations namely $K_{ij}^{(+)}u_j > 0$, $K_{ij}^{(0)}u_j = 0$, $K_{ij}^{(-)}u_j < 0$ where the K_{ij}s are the KM generalized Cartan matrices. These relations, which can be also put in the compact form $K(z_i, z_j)\,u(z_j) = v(z_i)$, have all the good properties to be interpreted as quantum field equations of motion following from an action principle. Moreover, in a continuous scalar field $\Phi(t, x)$ interpretation, the right term $v(z_i)$ of above equation would be associated with $\frac{\partial W(\Phi)}{\partial \Phi(t,x)}$ evaluated at point z_i. Here $W(\Phi)$ is the interacting field potential $W(\Phi)$. In this continuous QFT limit of Vinberg equations, one also sees that KM affine sector is associated with the critical points of the field potential $W(\Phi)$; in agreement with the general picture we have about realization of KM affine symmetries and conformal invariance à la Sugawara.

Motivated by these observations, see also[19] for others correspondences, we develop in the present study a quantum field theoretic model realising Vinberg theorem and KM theory. Our field representation, which takes advantages of the power of QFT method, has direct consequences on the following points. (**a**) Clarifying the striking link between Dynkin diagrams of KM extensions of semi simple Lie algebras and Feynman graphs of quantum field theory. (**b**) Giving a new way to think about the theory of Lie algebras and their KM classification. (**c**) Offering a new method to deal with the KM classification of Dynkin diagrams of indefinite sector of KM algebras which is still a black box. Recall in passing that semi simple Lie algebras were classified by Cartan and affine extension have been classified by Kac-Moody. Partial results have been obtained for hyperbolic subset of indefinite class.

The organization of this paper is as follows: In section 2, we review Vinberg theorem of classification of KM algebras. We give the basic steps of the standard way to approach KM theory. This is an algebraic way whose details can be found in[12]. In section 3, we study the QFT realization of Vinberg theorem and KM theory. In section 4, we give the physical representation of Vinberg condition requiring positivity of Vinberg vectors (u_i). Last section is devoted to conclusion.

II. STANDARD KM THEORY

What we mean by standard KM theory is just the usual theory describing the Kac-Moody extensions of semi-simple Lie algebras. The basis of this algebraic construction relies on the three following: (**i**) Vinberg theorem of classification of square matrices K; in particular KM generalized Cartan matrices, (**ii**) Minimal realization of Vinberg matrices in terms of a triplet and (**iii**) Serre construction of Lie algebras using Chevalley generators. Let us comment briefly these three algebraic steps in order to make the reader in the general picture.

Roughly speaking, Vinberg theorem is a linear algebra theorem which applies to KM theory and beyond such as Borcherds algebras; it states that generalized Cartan matrices K_{ij} (Cartan matrices for short) are of three kinds as shown below,

$$
\begin{aligned}
K_{ij}^{+}u_j &> 0, \\
K_{ij}^{0}u_j &= 0, \\
K_{ij}^{-}u_j &< 0,
\end{aligned}
$$

where u_j are the positive numbers we refer to above, and which will be discussed in section 4. The three upper indices $+$, $-$ and 0 are conventional notations introduced in order to distinguish the three KM sectors. The rigorous statement of Vinberg theorem, as used in KM formulation, is as follows,

Theorem 1 *A generalized indecomposable Cartan matrix* $\mathbf{K}$ *obey one and only one of the following three statements:*
(1) Finite type ($\det \mathbf{K} > \mathbf{0}$ *): There exist a real positive definite vector* $\mathbf{u}$ *(* $u_i > 0$; $i = 1, 2, ...$*) such that*

$\mathbf{K_{ij}u_j} = \mathbf{v_j} > \mathbf{0}$.
(2) Affine type, corank$(\mathbf{K}) = 1$, $\det \mathbf{K} = \mathbf{0}$*: There exist a unique, up to a multiplicative factor, positive integer definite vector* $\mathbf{n}$ *(* $n_i > 0$; $i = 1, 2, ...$*) such that* $\mathbf{K_{ij}n_j} = \mathbf{0}$.
(3) Indefinite type ($\det \mathbf{K} \leq \mathbf{0}$ *), corank*$(\mathbf{K}) \neq 1$*: There exist a real positive definite vector* $\mathbf{u}$ *(*$u_i > 0$; $i = 1, 2, ...$*) such that* $\mathbf{K_{ij}u_j} = -\mathbf{v_i} < \mathbf{0}$.

From the physical point of view, the first sector (ordinary class) of this KM classification deals with the ordinary semi simple Lie algebras. These algebras, which are familiar symmetries for model builders of elementary particle physics, are just the usual finite dimensional algebras classified many decades ago by Cartan, figure 1.

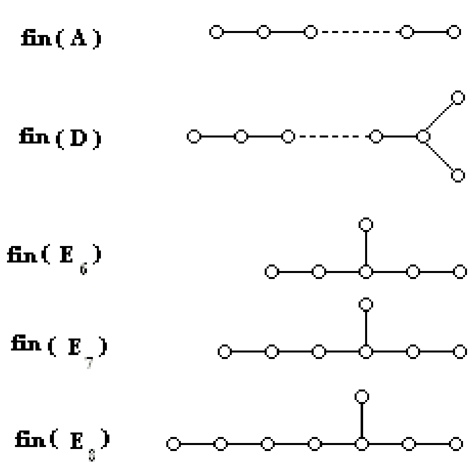

FIG. 1. *Dynkin diagrams of finite dimensional Lie algebras as classified by Cartan. These are graphs representing the usual* $A_n \sim su(n+1)$ *and* $D_n \sim so(2n)$ *classical simple Lie algebras as well as the ordinary exceptional ones. All of them have symmetric Cartan matrix K*

The second class (affine class) of KM theory concerns affine Kac-Moody algebras; they play a basic role in $2d$ conformal field theory (CFT_2) and underlying current algebras. These infinite dimensional algebras were classified by Kac-Moody; see also figure 2.

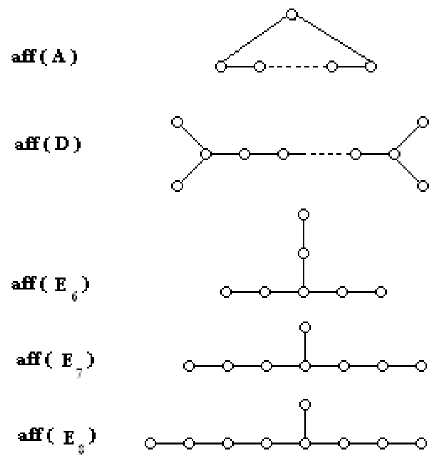

FIG. 2. *These are Dynkin diagrams of affine Kac Moody extension of the corresponding ordinary ones given by figure (1). Like ordinary graphs, these diagrams are simply laced and symmetric Cartan matrix K All these matrices have a vanishing determinant.*

The third class (indefinite class) is the so called KM indefinite class. In this sector, we dispose of partial results only; in particular for hyperbolic subset. No complete classification of the indefinite sector is availiable yet.

Before going ahead, let us make two comments regarding the Vinberg relations (**??**). First, note that Vinberg relations as shown on *theorem 1*, are given by inequalities. However, they can be formulated as equations by introducing positive quantities v_i (vectors) as follows:

$$K_{ij}^{(q)} u_j = q v_i, \qquad q = +1, 0, -1, \tag{2.1}$$

where the (u_i)s and (v_i)s are positive vectors. The second comment we want to make is that, because of the fact that any irreducible generalized Cartan matrix $K_{ij}^{(q)}$ can be decomposed as $A_{ij} - \delta A_{ij}^{(q)}$ with $\delta A_{ij}^{(q)} > 0$, i.e

$$K_{ij}^{(q)} = A_{ij} - \delta A_{ij}^{(q)}, \tag{2.2}$$

the above system of Vinberg equations may be also put in the following equivalent form,

$$A_{ij} u_j = w_i(u), \qquad A_{ij} = (2\delta_{ij} - \delta_{i,j+1} - \delta_{i,j-1}), \tag{2.3}$$

where appears, on the left side, the ordinary $su(n)$ Cartan matrix A_{ij} and where w_i are some numbers whose physical meaning will be given when we consider our QFT realization.

Concerning the two other points **(ii)** and **(iii)** dealing with the algebraic construction of KM theory, the key idea of their content may be summarized as follows. Given a generalized Cartan matrix K, one can associate to it a KM algebra $g(K)$. This is achieved in two steps: First by using the minimal realization of Cartan matrix K based on the usual triplet

$$(\hbar, \Pi, \Pi^v). \tag{2.4}$$

This triplet involves the following familiar objects: (1) Cartan subspace $\hbar$ with a bilinear form $< .,. >$ and a dual space $\hbar^*$, (2) the root basis $\Pi = \{a_i, \quad 1 \leq i \leq n\} \subset \hbar^*$ and (3) the coroot basis $\Pi^v = \{a_i^v, \quad 1 \leq i \leq n\} \subset \hbar$. In terms of these quantities, the Cartan matrix reads as,

$$K_{ij} =< a_i^v, a_j >, \tag{2.5}$$

which reads generally as $K_{ij} = 2a_i a_j / a_i^2$; or more conveniently like $K_{ij} = a_i a_j$ for simply laced KM algebras in which we will be interested in wht follows. Note in passing that this algebraic formulation is not specific for Kac-Moody extension of semi simple algebras requiring,

$$\begin{aligned} K_{ii} &= 2, \\ K_{ij} &< 0, \qquad i \neq j, \\ K_{ij} = 0 \qquad &\Rightarrow \qquad K_{ji} = 0. \end{aligned} \tag{2.6}$$

It is also valid for matrices beyond KM generalized Cartan ones. For instance, this above analysis applies as well for the case of Borcherds algebras using *real* matrices (B_{ij}) constrained as,

$$2\frac{B_{ij}B_{ji}}{B_{ii}} \in \mathbf{Z}, \qquad \mathbf{B_{ii} \neq 0}, \qquad \mathbf{B_{ij} \in R}, \tag{2.7}$$

where $\mathbf{Z}$ is the set of integers. The third step in building KM algebra $g(K)$ is by using Chevalley generators $\{e_i\}$ and $\{f_i\}$, $i = 1,..,n$. Commutation relations of KM algebra $g(K)$ associated with a generalized Cartan matrix K reads as follows,

$$\begin{aligned} &[e_i, f_j] = \delta_{ij} a_i^\nu, \qquad 1 \leq i,j \leq n \\ &[h, h'] = 0, \qquad h, h' \in \hbar \\ &[a_i^\nu, e_j] = K_{ij} e_i, \\ &[a_i^\nu, f_j] = -K_{ij} f_i, \end{aligned} \tag{2.8}$$

together with Serre relations. For more technical details on this last step see for instance[13] and refs therein. In what follows, we shall develop an other way to approach Vinberg theorem and KM theory describing the extension of semi simple Lie algebras.

III. QFT REALIZATION OF KM THEORY

To start note that a quantum field realization of Vinberg theorem can be naturally built by thinking about eq(2.3) as a $(1+1)$ dimensional field equation of motion resulting from the variation of the following discrete field action,

$$\mathcal{S}[u] = \sum_{i,j\in\mathbf{Z}} \frac{1}{2} u_i A_{ij} u_j + \sum_{i\in\mathbf{Z}} W(u_i). \tag{3.1}$$

In this relation u_i is as before, $A_{ij} = (2\delta_{ij} - \delta_{i,j+1} - \delta_{i,j-1})$ and $W(u)$ is an interacting polynomial potential whose variation with respect to u_i reads as follows,

$$\frac{\partial W(u)}{\partial u_i} = w_i(u), \tag{3.2}$$

in agreement with eq(2.3). With this discrete field action at hand, one can go ahead and study quantization of this QFT by computing the generating functional $\mathcal{Z}[J]$ of Green functions of this theory.

$$\mathcal{Z}[J] = \int [Du] \exp\left(-\mathcal{S}[u] - \sum_i u_i J_i\right). \tag{3.3}$$

In this relation $\mathcal{S}[u]$ is as in eq(3.1) and the J_is are the discrete values of an external source dual to the u_is. In this formulation, two points Green function (propagator) $G_{ij} =< u_i, u_j >$ with $|i-j| = n$, is interpreted as the Dynkin diagram of the $su(n+1)$ semi simple Lie algebra; see also figure 3.

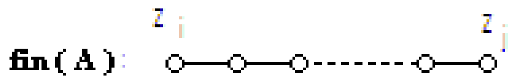

FIG. 3. A generic *Dynkin diagram of semi simple* A_n Lie algebra realized as the two point function $< \Phi(z_i, \bar{z}_i)\, \Phi(z_j, \bar{z}_j) >$ with $n = 1 + |i-j|$.

More generally, Feynman graphs of the QFT eq(3.3) should be associated with Dynkin diagrams. We will not develop here the study of Green functions; for details see[?]. What we want to do now is to establish the general setting of the QFT realization of KM theory and it relationship with $(1+1)$ dimension continuous quantum scalar field theory.

Theorem 2 *The Cartan matrix operator* $A_{ij} = (2\delta_{ij} - \delta_{i,j+1} - \delta_{i,j-1})$ *of* $su(n)$ *semi simple Lie algebra is, up a multiplicative constant, exactly equal to the discrete version of the one dimensional Laplacian operator* $\Delta = \frac{d^2}{dx^2}$.

$$\Delta \leftrightarrow \frac{1}{a^2} A_{ij}, \tag{3.4}$$

where a is period length of the discretised one dimensional lattice.
Vinberg theorem has a $(1+1)$ *QFT realization; and Vinberg relations* ($K_{ij}^{(+)} u_j > 0$, $K_{ij}^{(0)} u_j = 0$, $K_{ij}^{(-)} u_j < 0$) *are given by the discretisation of interacting field equations of motion,* $A_{ij} u_j = \frac{\partial W(u)}{\partial u_i}$, *with* $\partial_i W(u) > 0$, $\partial_i W(u) = 0$ *and* $\partial_i W(u) < 0$ *respectively*

Before proving this theorem, let us introduce some tools and useful convention notations for our QFT realization of KM theory. First, let $\Psi(t,x)$ be a $(1+1)$ real scalar field of kinetic energy density,

$$\mathcal{E}_c = \frac{\partial^2 \Psi}{\partial t^2} - \frac{\partial^2 \Psi}{\partial x^2} = \partial_- \partial_+ \Psi. \tag{3.5}$$

Let also $\mathcal{R}(x)$ be a static *real positive definite* scalar field ($\mathcal{R} > 0$ and $\frac{\partial \mathcal{R}}{\partial t} = 0$) varying on the one dimensional real line $\mathbf{R}$. Because of stationarity, its kinetic energy density, given by a relation similar to the above one, reduces now to $\mathcal{E}_c = -\frac{d^2\mathcal{R}}{dx^2}$. In presence of field interactions $W(\mathcal{R})$, the action $S = S[\mathcal{R}]$ of the scalar field model is given by,

$$S[\mathcal{R}] = -\int_{\mathbf{R}} dx \left(\frac{1}{2}\left(\frac{d\mathcal{R}}{dx}\right)^2 + W(\mathcal{R})\right). \tag{3.6}$$

The continuous equation of motion of the real positive scalar field $\mathcal{R}$ reads as,

$$\frac{d^2\mathcal{R}}{dx^2} = \frac{dW}{d\mathcal{R}}, \qquad W(\mathcal{R}) = \sum_{m=1}^{n} \lambda_m \mathcal{R}^m, \tag{3.7}$$

where λ_m are coupling constants. To get the discrete version of this field equation, we use the correspondence $x \to x_i$ and $x + dx \to x_i + a$ and denote by

$$\mathcal{R}_k = \mathcal{R}(x)\,|_{x=x_k}, \qquad k \in \mathbf{Z}, \tag{3.8}$$

which is nothing but the field value at the node $x_k = ka$ of the one dimensional lattice $\mathbf{Z}$ with a being the lattice period length. After this digression, we are now in position to prove our theorem.

First, consider the discrete version of energy density $\left(\frac{d\mathcal{R}}{dx}\right)^2$. This is obtained by help of the usual definition of differentiation namely $\frac{d\mathcal{R}(x)}{dx} = \frac{\mathcal{R}(x+dx)-\mathcal{R}(x)}{dx}$ and by making the following substitutions,

$$\mathcal{R}(x) \to \mathcal{R}_i \qquad \mathcal{R}(x+dx) \to \mathcal{R}_{i+1}. \tag{3.9}$$

Putting these expressions back into the continuous integral $\int_{\mathbf{R}} dx \left(\frac{d\mathcal{R}}{dx}\right)^2$, we get the discrete sum $\sum_{i\in\mathbf{Z}} \left(\mathcal{R}_{i+1} - \mathcal{R}_i\right)^2$ which expands as,

$$\sum_{i\in\mathbf{Z}} \left(\mathcal{R}_{i+1}^2 - \mathcal{R}_{i+1}\mathcal{R}_{)}\right) + \sum_{i\in\mathbf{Z}} \left(\mathcal{R}_i^2 - \mathcal{R}_{i+1}\mathcal{R}_i\right). \tag{3.10}$$

Using translation invariance of the one dimensional lattice $\mathbf{Z}$, we can rewrite the first term of above equation $\sum_{i\in\mathbf{Z}} \left(\mathcal{R}_{i+1}^2 - \mathcal{R}_{i+1}\mathcal{R}_i\right)$ as $\sum_{i\in\mathbf{Z}} \left(\mathcal{R}_i^2 - \mathcal{R}_i\mathcal{R}_{i-1}\right)$. This is achieved by shifting the indices as $(i+1) \to i$. The term $\sum_{i\in\mathbf{Z}} \left(\mathcal{R}_{i+1} - \mathcal{R}_i\right)^2$ reads then as $\sum_{i\in\mathbf{Z}} \left(2\mathcal{R}_i^2 - \mathcal{R}_i\mathcal{R}_{i-1} - \mathcal{R}_{i+1}\mathcal{R}_i\right)$ and consequently we have the following continuous-discrete correspondence,

$$\frac{1}{2}\int_{\mathbf{R}} dx \left(\frac{d\mathcal{R}}{dx}\right)^2 \to \frac{1}{2a}\sum_{i,j\in\mathbf{Z}} \mathcal{R}_i A_{ij} \mathcal{R}_j, \tag{3.11}$$

where A_{ij} is exactly as given in theorem 2. Note that the apparition of the global factor $\frac{1}{a}$ in front of the discrete sum may be also predicted by using the following scaling properties of the scalar QFT under change $x \to ax$ We have,

$$\mathcal{R}(x) \to \mathcal{R}(ax) = \mathcal{R}(x), W(\mathcal{R}(ax)) = \frac{1}{a^2} W(\mathcal{R}(x)) \tag{3.12}$$

This completes the proof of our theorem. What remains to do is to find the physical interpretation of the positivity condition of the u_is in Vinberg theorem. This will be done in the next section.

IV. SOLVING VINBERG CONDITION

In Vinberg classification theorem of KM algebras (theorem 1), the (u_i) variables are required to be positive numbers. From physical point of view, such kind of conditions are familiar in the study of constrained systems; in particular in gauge theories. In the problem at hand, Vinberg condition may implemented by considering a static complex scalar QFT with a $U(1)$ gauge invariance. To do so consider a QFT system composed by a static one dimensional gauge field $\mathcal{A}(x)$ (a pure gauge field) and a complex scalar field Φ,

$$\Phi(x) = \frac{1}{\sqrt{2}} [\Phi_1(x) + i\Phi_2(x)] . \tag{4.1}$$

For convenience, it is interesting to rewrite the field Φ by using Euler representation $\mathcal{R}(x) \exp i\vartheta(x)$ where the field $\mathcal{R}$ is same as before. Using the following $U(1)$ gauge transformations,

$$\mathcal{R}(x) \to \mathcal{R}(x) ,$$

$$\vartheta(x) \to \vartheta(x) - \lambda(x) , \tag{4.2}$$

$$\mathcal{A}(x) \to \mathcal{A}(x) - i\frac{d\lambda(x)}{dx}$$

where $\lambda(x)$ is the gauge parameter, and the gauge covariant derivative $\mathcal{D} = \left(\frac{d}{dx} + i\mathcal{A}(x)\right)$, one can write down the static one dimensional action $S[\Phi]$ describing the complex scalar field dynamics. It reads as,

$$S[\Phi] = -\int_{\mathbf{R}} dx \left[(\mathcal{D}\Phi)^* (\mathcal{D}\Phi) + W(|\Phi|)\right] , \tag{4.3}$$

where $W(|\Phi|) = W(\mathcal{R})$ is gauge invariant interacting potential, the same as in eq(3.6). Using gauge symmetry of this action, one can make the gauge choice

$$\vartheta(x) = \lambda(x) , \qquad \mathcal{A}(x) = i\frac{d\lambda(x)}{dx}, \qquad \mathcal{D}\Phi = \frac{d\mathcal{R}}{dx}, \tag{4.4}$$

to kill the local phase $\vartheta(x)$ of the complex field $\Phi(x)$ which reduces then to $\mathcal{R}(x)$. Vinberg condition corresponds then to fixing the gauge field.

V. CONCLUSION

In this paper, we have given a quantum field realization of Vinberg theorem and KM theory. In the case of simply laced Dynkin diagrams, we have shown that Vinberg theorem, behind the classification of KM algebras and Borcherds theory, is in fact just the discrete version of the static field equation, $\frac{d^2\mathcal{R}}{dx^2} = \frac{dW}{d\mathcal{R}}$ of a complex scalar $U(1)$ gauge invariant theory. Vinberg condition requiring positivity of the u_is is interpreted as $U(1)$ gauge fixing. According to the sign of $\frac{dW}{d\mathcal{R}}$, one distinguishes three sectors ($\frac{dW}{d\mathcal{R}} > 0$, $\frac{dW}{d\mathcal{R}} = 0$, $\frac{dW}{d\mathcal{R}} < 0$). In this representation, one sees that affine KM sector is associated with the critical point of the interacting field potential $W(R)$. Semi simple algebras are associated with stable fluctuations ($\frac{dW}{d\mathcal{R}} > 0$) around the critical point while indefinite symmetries related with unstable deformation ($\frac{dW}{d\mathcal{R}} < 0$).

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