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Charged Open Strings in a Background Field and Euler-Heisenberg Effective Action

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Abstract

This talk is based on work made with L. Magnea and R. Russo [1]. We give an explicit expression of the multiloop partition function of open bosonic string theory in the presence of a constant gauge field strength. The Schottky parameterization allows to perform the field theory limit, which at two-loop level reproduces the Euler-Heisenberg effective action for adjoint scalars minimally coupled to the background gauge field.

String theory achievements are very impressive: maybe the most important is that the consistency between quantum mechanics and special relativity in the description of the motion of a string implies the presence of gravitons (and of their supersymmetric partners) and therefore yields a consistent quantum theory of (super)gravity. But the richness (and the problems) of string theory as a fundamental theory cannot be discussed now; here we will take a much more modest attitude just looking to string theory as a natural and very fruitful embedding for Quantum Field Theory; open strings, in particular, represent a natural generalization of charged particles since they couple, through their endpoints, to gauge fields. A peculiar feature of this embedding is **open-closed string duality**: exchanging the role of the worldsheet coordinates τ and σ, an annulus spanned by the time evolution of an open string can also be seen as a cylinder described by free propagation of a closed string. This property has far reaching consequences as it relates gravity (described by closed strings) with Yang Mills theories (open strings); here we will use it only as a technical device for computing open string amplitudes not otherwise accessible.

We will focus on the case of open bosonic strings and will study the multiloop partition function in the presence of a constant Yang-Mills field strength F. We will then use the string formula in the low-energy limit to recover the Euler-Heisenberg [2] effective action for a gauge field coupled to adjoint scalars; we have performed the calculation at one and two loops, but in principle it could be done at any perturbative order.

From the mathematical point of view, the open string diagram is represented as usual by a Riemann surface with $g+1$ boundaries; however the presence of external fields F introduces twisted boundary conditions along some of the boundaries. As a consequence, the basic geometric building blocks of the string amplitude, such as the determinant of the Laplace operator on the Riemann surface, are deformed by F. The necessary ingredients to derive the multiloop partition function for charged open strings were assembled in [4, 5], developing earlier studies [6, 7].

We will now outline the derivation of the partition function for *charged* open strings attached to $g+1$ (stacks of N) D-branes, *i.e.* open strings with mixed boundary conditions $\left[\partial_\sigma X^i + \mathrm{i}\,\partial_\tau X^j F_j^{\;i\,(A)}\right]_{\sigma=0} = 0$, where F^A is the constant gauge field strength on the Ath D-brane ($A = 0, \ldots, g$). A direct computation of the charged partition function in the open string channel is difficult, mainly because for $F \neq 0$ the string coordinates have a non-integer mode expansion. However the open-closed string duality under the exchange $\tau \leftrightarrow -\sigma$ allows to go to the closed string channel by using boundary states satisfying $\left[\partial_\tau X^i - \mathrm{i}\,\partial_\sigma X^j F_j^{\;i\,(A)}\right]_{\tau=0} |B\rangle_{F_A} = 0$. The field F can always be put in a block-diagonal form, so for the sake of simplicity we will take the space-time indices to be in the plane $i, j = 1, 2$ and we will write $F_{12}^{(A)} = -F_{21}^{(A)} = \tan(\pi\epsilon^A)$. The computation of vacuum diagrams can be done along the lines of [8]; to be specific, we take the external boundary to have Neumann boundary conditions ($F^{(0)} = 0$), so that $\vec{\epsilon}$ is a vector with g components, denoted by ϵ_μ, encoding the values of the gauge field on the remaining g boundaries. The result is

$$Z_F^c(g) = \left(\prod_{\mu=1}^{g} \frac{1}{\cos\pi\epsilon_\mu}\right) \int [dZ]_g^c \; \mathcal{R}_g\left(q_\alpha, \vec{\epsilon}\right) \quad , \tag{1}$$

where[1]

$$[dZ]_g^c = \frac{1}{dV_{abc}} \prod_{\mu=1}^{g} \left[\frac{dq_\mu\, d^2\eta_\mu^c\, (1-q_\mu)^2}{q_\mu^2\, (\eta_\mu^c - \bar{\eta}_\mu^c)^2}\right] \prod_\alpha{}' \left(\prod_{n=1}^{\infty}(1-q_\alpha^n)^{-d} \prod_{n=2}^{\infty}(1-q_\alpha^n)^2\right) \quad , \tag{2}$$

represents the $F = 0$ result[2], while the $\vec{\epsilon}$ dependence is encoded in the factor

$$\mathcal{R}_g\left(q_\alpha, \vec{\epsilon}\right) = \frac{\prod_\alpha{}' \prod_{n=1}^{\infty}(1-q_\alpha^n)^2}{\prod_\alpha{}' \prod_{n=1}^{\infty}\left(1-\mathrm{e}^{-2\pi\mathrm{i}\vec{\epsilon}\cdot\vec{N}_\alpha} q_\alpha^n\right)\left(1-\mathrm{e}^{2\pi\mathrm{i}\vec{\epsilon}\cdot\vec{N}_\alpha} q_\alpha^n\right)} \quad . \tag{3}$$

Here the μ^{th} entry of $\vec{N}_\alpha$ counts how many times the Schottky generator S_μ enters in the element of the Schottky group T_α, whose multiplier is q_α. The factors of $1/\cos(\pi\epsilon)$ in Eq. (1), are nothing but a rewriting of the Born-Infeld contribution to the boundary state normalization (see for instance [9]).

Eq. (1) contains all the information about the interaction among charged open strings, but is written in the closed string representation; as it stands, its low energy limit would describe scattering of gravitons. To get instead Yang Mills theory, we must go back to the open string channel; this can be achieved by performing the modular transformation $\tau^c = -\tau^{-1}$, where τ^c and τ are the period matrices in the closed and open channel. To do this, at first, we rewrite the products over the Schottky group in terms of geometrical objects with simple transformation properties under the modular group, like θ functions, differentials and the prime form; then, we perform the modular transformation; as a last step, we go back to the Schottky parameterization, which is the most appropriate for performing the low energy limit. The technical tool needed in this derivation is the higher-genus generalization of the Jacobi formulae expressing θ functions as products. These formulae can be derived by exploiting bosonization identities in two dimensions in presence of the twists $\vec{\epsilon}$ [4, 5]. The final result for the string effective action in the open string channel is

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[1]We refer to the Appendices of [4] for a short explanation of the Schottky parameterization used in (2); here we just recall that for a Riemann surface of genus g the Schottky group is freely generated by g projective transformations S_μ, each of them characterized by one multiplier q_μ and two fixed points η_μ and ξ_μ; for closed string surfaces related by a modular transformation to open string world surfaces one has $\xi_\mu = \bar{\eta}_\mu$.

[2]dV_{abc} signals that we have to fix three real variable among the η's to take into account the overall projective invariance; the superscript c reminds that the parameters describe closed string exchanges among the various boundaries; $\prod_\alpha{}'$ means the product over all the primitive elements T_α of the Schottky group (*i.e.* those which can not be written as powers of other elements) ; finally d is the dimensionality of space-time.

$$Z_F(g) = \left(\frac{e^{2\pi i \epsilon_g} - 1}{\prod_{\mu=1}^{g} \cos \pi \epsilon_\mu} \right) \int [dZ]_g \left[e^{-i\pi \vec{\epsilon}\cdot\tau\cdot\vec{\epsilon}} \frac{\det(\tau)}{\det(\tau_{\vec{\epsilon}})} \mathcal{R}_g(k_\alpha, \vec{\epsilon}\cdot\tau) \right] , \tag{4}$$

where

$$\begin{aligned} [dZ]_g &= \frac{1}{dV_{abc}} \prod_{\mu=1}^{g} \left[dk_\mu d\eta_\mu d\xi_\mu \frac{(1-k_\mu)^2}{k_\mu^2(\xi_\mu - \eta_\mu)^2} \right] [\det(\operatorname{Im}\tau)]^{-\frac{d}{2}} \\ &\times \prod_\alpha{}' \left[\frac{\prod_{n=2}^{\infty}(1-k_\alpha^n)^2}{\prod_{n=1}^{\infty}(1-k_\alpha^n)^d} \right] \end{aligned} \tag{5}$$

is the open string channel transcription of eq.(2); the crucial new object in eq.(4) is $\tau_{\vec{\epsilon}}$, which is built by means of $g-1$ Prym differentials (generalization of the usual g abelian differentials to the case of *twisted* boundary conditions) and which reduces to the usual period matrix τ for $\vec{\epsilon} = 0$; the explicit expression of $\tau_{\vec{\epsilon}}$ in terms of the Schottky parameterization can be found in [1].

Now we want to show that the low energy limit of this string configuration reproduces a theory of adjoint scalars coupled to a background Yang Mills field. To be more precise let us consider a classical background $U(N)$ gauge field $\mathcal{A}_\mu$, represented as a hermitian $N \times N$ matrix $\mathcal{A}_\mu = \sum_a A_\mu^a T_a$, where T_a ($a = 0, \ldots, N^2-1$) are $U(N)$ generators, coupled to a quantum massive scalar field, also in the adjoint representation, $\Phi = \sum_a \varphi^a T_a$. The gauge field configuration corresponding to a single charged brane is a diagonal $\mathcal{A}_\mu$ matrix, with all eigenvalues vanishing except one, $(\mathcal{A}_\mu)_{AB} = A_\mu \, \delta_{A,N}\delta_{B,N}$. This choice of background breaks the symmetry in color space, so that the matter "multiplet" Φ will have both neutral (σ and Π) and charged ($\boldsymbol{\xi}$ and $\boldsymbol{\xi}^\dagger$) components with respect to $\mathcal{A}_\mu$. The lagrangian is given by

$$\begin{aligned} \mathcal{L} = & \operatorname{Tr}[\partial_\mu \Pi \, \partial^\mu \Pi] + \frac{1}{2}\partial_\mu\sigma\,\partial^\mu\sigma + D_\mu\boldsymbol{\xi}^\dagger D^\mu\boldsymbol{\xi} - m^2 \operatorname{Tr}(\Pi^2) - \frac{1}{2}m^2\sigma^2 \\ & - m^2\boldsymbol{\xi}^\dagger\boldsymbol{\xi} + \frac{2}{3}\lambda \operatorname{Tr}(\Pi^3) + \sqrt{2}\,\frac{\lambda}{6}\sigma^3 + \frac{\lambda}{\sqrt{2}}\sigma\boldsymbol{\xi}^\dagger\boldsymbol{\xi} + \lambda\,\boldsymbol{\xi}^\dagger\Pi\,\boldsymbol{\xi} \; , \end{aligned} \tag{6}$$

where Π is a hermitian $(N-1)\times(N-1)$ matrix representing a field in the adjoint representation of $U(N-1)$, $\boldsymbol{\xi}$ is a complex vector in the fundamental representation of $U(N-1)$, while σ is a singlet real field, and the abelian covariant derivative is defined by $D_\mu\boldsymbol{\xi} = \partial_\mu\boldsymbol{\xi} + iA_\mu\boldsymbol{\xi}$.

Now we perform the low-energy limit of the string partition function in Eq. (4) sending the typical length of the string $\sqrt{\alpha'}$ to zero and taking into account that both the moduli describing the shape of the Riemann surface and the physical magnetic field B (which must be kept fixed) are dimensionful. The logarithms of the multipliers of Schottky transformations, for example, are associated with the length of the corresponding loops by setting $\log k_\mu = -T_\mu/\alpha'$, where T_μ is the sum of the Schwinger parameters associated with the propagators forming the μ^{th} loop. Moreover, taking only one boundary charged and setting $\epsilon_g \equiv \epsilon \neq 0$, the field theory limit is defined by $\tan(\pi\epsilon) = 2\pi\alpha' B$, which implies $\epsilon = 2\alpha' B + \mathcal{O}(\alpha'^3)$. Other dimensionful quantities are an overall normalization constant and the scalar self-coupling λ, which must be matched with the string coupling g_S. In order to isolate the contribution of charged scalars circulating in the loops we have to look to the powers of the multiplier in a Taylor expansion of the integrand for small k_μ. For the bosonic string, this expansion starts with k_μ^{-2}, a sign of the tachyonic instability; this singularity can however be readily regularized by recalling that the tachyon mass squared is $m^2 = -1/\alpha'$ and setting [10] . We skip the easiest one loop case, where Eq. (4) reproduces the results of Ref. [3], for the magnetic case.

At two loops, we will take advantage of the projective invariance by setting $\eta_1 = 0$, $\xi_1 = \infty$ and $\xi_2 = 1$. Looking only for the 1PI vacuum graphs, we set [11] $k_1 = \exp(-(t_1+t_3)/\alpha')$, $k_2 = \exp(-(t_2+t_3)/\alpha')$, $\eta_2 = \exp(-t_3/\alpha')$, where t_i are the Schwinger parameters associated with the three propagators in the diagram (t_2 and t_3 to the loop with a charged boundary and t_1 to the neutral particle). After the expansion in powers of k_μ of the infinite series and products over the Schottky group, one finally gets the effective action in the low energy limit:

$$W_{\xi\Pi}^{(2)}(m,B) \;=\; -i\,V_d\,\frac{\lambda^2}{(4\pi)^d}\,\frac{(N-1)^2}{4}\int_0^\infty dt_1 dt_2 dt_3\, e^{-m^2(t_1+t_2+t_3)}\,\Delta_0^{-\frac{d}{2}+1}\,\Delta_B^{-1} \; , \tag{7}$$

where $\Delta_B = \sinh(Bt_2)\sinh(Bt_3)/B^2 + t_1 \sinh\left[B\left(t_2+t_3\right)\right]/B$ and $\Delta_0 = \lim_{B\to 0}\Delta_B = t_1t_2 + t_1t_3 + t_2t_3$, for the diagram with the exchange of the neutral particle Π; the only change when the exchanged particle is σ is in the color factor, with the replacement $(N-1)^2 \to N-1$. One can check that this result, obtained by the low energy limit of the vacuum string amplitude with twisted boundary conditions, exactly agrees with the field theoretical result that one can compute starting from the lagrangian (6).

The procedure could be generalized to derive the Euler-Heisenberg effective action for pure Yang-Mills theory; the starting point is always Eq. (4), but one has to isolate the contributions to the loop integrals of the first excited state in the spectrum of the open bosonic string, which is a massless vector. In practice, this means that the expansion in multipliers of the various geometrical objects appearing in Eq. (4) has to be pushed one order higher. Finally, the extension to superstrings would allow to study also the coupling with charged fermions.

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