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Spectral Geometry: Case of Quantum Spheres

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Abstract

We review the ideas of noncommutative spectral geometry a la Connes and illustrate it with an example of quantum two-sphere.
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I. INTRODUCTION

Noncommutative geometry studies the geometry of *quantum spaces* - or, being more explicit - geometry of noncommutative algebras. Clearly, the word *quantum*, although at first only superficially related to *quantum mechanics* or *quantum field theory* might be the right one - both physics and mathematics are involved in these directions and there is a huge interplay between them. Indeed, many basic examples come from physics: like the phase space in quantum mechanics or the Brillouin zone in the Quantum Hall Effect [1]. Why noncommutative geometry? First of all it seems to be a natural and rich extension of the concept of spaces, on which one can investigate the notion of *geometry* in its various aspects. Moreover, within noncommutative geometry one has on the same footing various objects: algebras of continuous (or smooth) functions, algebras of pseudodifferential operators, algebras of differential forms etc. In this note we shall review the *machinery* of noncommutative geometry, illustrating it with the example of the Podles quantum sphere.

II. SPECTRAL GEOMETRY A LA CONNES

Spectral triples did not fall from heaven: to understand their axioms and many basic features one shall start with Clifford algebras, go through elements of K-homology and Hochschild homology, not mentioning the technical issues of operator algebras like the Tomita-Takesaki theory or the ideals of operators and the traces on them. We shall outline here the basic ingredients of the theory, concentrating mostly on the *algebraic* data, trying to show the background and the underlying "classical" picture behind the definitions. There are not many examples of spectral triples: so far all known are classical (commutative geometry), finite dimensional (over finite algebras) or based upon the noncommutative tori (including the recently introduced "isospectral deformations"). The later discussed example of quantum geometry is one of the first "truly" noncommutative examples. The geometric object, which gave rise to spectral geometry is the Fredholm module, given by the sign of an elliptic differential operator over a compact manifold M with the Hilbert space of the sections of the vector bundle over M, on which the operator acts. It was realized quite early that this construction is a significant one and is one of the crucial steps of the renowned index theorem. The passage from the "differential geometry" data to K-homology motivated the need for objects, which would correspond directly, in an abstract noncommutative setting, to the classical picture. This led to the notion of K-cycles or unbounded Fredholm modules, later extended to spectral triples. The basic data of a spectral triple is almost the same as we are used to in the *commutative* situation, though expressed in an algebraic way. The structures, which appear in the definition (see Connes' book for a detailed account and references [2]) were motivated by the need for description of spin structure on commutative and noncommutative manifolds, however, the physical motivation played also an important role. The current formulation was established by Connes in [4, 5], a detailed account could be found in his reviews [7] as well as in the book [3]. We begin by the presentation of the algebraic framework.

Definition II.1. Let $\mathcal{A}$ be an involutive algebra represented faithfully on a Hilbert space $\mathcal{H}$ as bounded operators, D an selfadjoint operator such that $[D, a]$ is bounded for every $a \in \mathcal{A}$ (we shall omit, unless necessary, writing explicitly representation map π). We assume that there exists as well an antilinear isometry J of $\mathcal{H}$ such that:

- JaJ^{-1} commutes with b, for every $a, b \in \mathcal{A}$,
- JaJ^{-1} commutes with $[D, \mathcal{A}]$ for every $a, b \in \mathcal{A}$ (order-one axiom).

To present further algebraic restrictions we must introduce the notion of a *dimension* of a spectral triple. If n is an algebraic dimension of the real spectral triple then:

$$DJ + \epsilon JD = 0, \quad J^2 = \epsilon''. \tag{II.2.1}$$

If n is even, then there must exists a $\mathbb{Z}_{\not{2}}$ grading $\gamma = \gamma^\dagger$, such that:

$$D\gamma + \gamma D = 0, \quad J\gamma + \epsilon'\gamma J = 0, \tag{II.2.2}$$

where the constants ϵ, ϵ' and ϵ'' are ± 1 according to the table, where $n' = n \text{mod } 8$:

n'	0	1	2	3	4	5	6	7
ϵ	+	-	+	+	+	-	+	+
ϵ'	+		-		+		-	
ϵ''	+	+	-	-	-	-	+	+

Before we further discuss the contents of these *axioms* let us observe that the signs, which are summarized in the table, follow directly from the possible real, complex or quaternionic structures on Clifford algebras. It is important to observe that if M is a compact Riemannian spin manifold of dimension n, $\mathcal{H} = \mathcal{L}^{\in}(\mathcal{S})$ - space of square-integrable sections of the spinor bundle, D the Dirac operator γ the section of the Clifford bundle giving the volume form (pointwise) and J the Clifford antilinear isometry C - then $(C^\infty(M), L^2(S), D, \gamma, J)$ is a spectral triple of dimension n - and the signs $\epsilon, \epsilon', \epsilon''$ are fixed from the relations in the Clifford algebra. Thus, in the case of manifolds we have only rewritten the already known facts using a different language. We shall see, however, that this language is particularly suited to some more general examples and is sufficient to study geometry and fundamental physical theories. Now, let us briefly discuss the contents of these *algebraic axioms* within the definition. The formulation of the order-one condition tells us that the Hilbert space is in fact a left module over $\mathcal{A}$ and a right module over the representation of differential forms - the latter in the classical case corresponding to the sections of the Clifford algebra: this establishes the Morita equivalence between the Clifford algebra bundle and the algebra of functions on the manifold. Of course another role of the order-one condition is to guarantee that D is a differential operator of order one. The differential calculus is usually constructed by taking the commutators with the Dirac operator, $[D, \pi(a)]$, however, since D^2 is neither 1 nor central, one needs to quotient out some nonempty kernels. Although the bimodule of one-forms is isomorphic to $\pi(\Omega^1_u(\mathcal{A}))$ the higher order forms are obtained by taking the quotient of the universal forms of a given order by the ideal $\Omega^n_u \mathcal{A} \cap (\ker \pi \cup \lceil \ker \pi)$. So far we have abstained from setting up the necessary *analytic* requirements for the spectra triples. We shall not present them here, referring the reader to the book [3]), where details and examples could be found. It is nice to have a well-motivated theory with a nice set of axioms describing some objects - however, it is important that the theory shall be not empty. There are, however, not that many examples of spectral triples, and almost all are, in fact, rather close to the classical geometries and consist of all classical spin-manifolds, discrete geometries, noncommutative tori [5] and isospectral deformations [6].

III. THE STANDARD PODLES QUANTUM SPHERE

The recent construction of isospectral noncommutative spin geometry on the equatorial quantum Podleś sphere and on $SU_q(2)$ [9, 10] raised an expectation that it may be a characteristic feature of q-deformed spaces. Another feature was the modification of the spectral geometry axioms by allowing the *infinitesimal* deviations from the exactness. According to the well-known dictionary of Connes, in the case of operator algebras we use the notion of *infinitesimal* to denote compact operators. We shall review here the first construction of the spectral geometry on the standard quantum Podleś sphere and to see how the spectral triple of [8] with the exponentially growing spectrum of D fits to this scheme.

A. Algebra of the standard Podles sphere

We recall the definition of the algebra $\mathcal{A}(\mathcal{S}^{\in}_{\rm II})$ of the standard Podleś quantum sphere [12]. Let q be a real number $0 < q \leq 1$ and $\mathcal{A}(\mathcal{S}^{\in}_{\rm II})$ be a $*$-algebra generated by A, B, B^*, obeying the following relations:

$$\begin{aligned} AB = q^2 BA, \qquad & AB^* = q^{-2} B^* A, \\ BB^* = q^{-2} A(1-A), \qquad & B^*B = A(1-q^2A). \end{aligned} \tag{III.3.1}$$

It appears that the above algebra has a nontrivial Hopf-algebraic symmetry. It is, as expected, the quantized algebra $\mathcal{U}_{\rm II}(\int\sqcap(\in))$, which is generated by e, f, k, k^{-1}, satisfying relations:

$$ek = qke, \qquad kf = qfk, \qquad k^2 - k^{-2} = (q - q^{-1})(fe - ef). \tag{III.3.2}$$

and the coproduct given by:

$$\Delta k = k \otimes k, \qquad \Delta e = e \otimes k + k^{-1} \otimes e, \qquad \Delta f = f \otimes k + k^{-1} \otimes f. \tag{III.3.3}$$

the counit ϵ, antipode S, and star structure:

$$\begin{array}{lll} \epsilon(k) = 1, & \epsilon(e) = 0, & \epsilon(f) = 0, \\ Sk = k^{-1}, & Sf = -qf, & Se = -q^{-1}e, \\ k^* = k, & e^* = f, & f^* = e. \end{array} \tag{III.3.4}$$

From the canonical pairing between $\mathbb{CU}_q(su(2))$ and $(\mathbb{CS}\,\mathbb{U}_q(2))$ one obtains an action of $\mathbb{CU}_q(su(2))$ on the generators of the standard Podleś sphere:

$$\begin{aligned} e \triangleright B &= -(q^{\frac{1}{2}} + q^{-\frac{3}{2}})A + q^{-\frac{3}{2}}, \\ e \triangleright B^* &= 0, \\ e \triangleright A &= q^{-\frac{1}{2}} B^*, \\ k \triangleright B &= qB, \\ k \triangleright B^* &= q^{-1} B^*, \\ k \triangleright A &= A, && \text{(III.3.5)} \\ f \triangleright B &= 0, \\ f \triangleright B^* &= (q^{\frac{3}{2}} + q^{-\frac{1}{2}})A - q^{-\frac{1}{2}}, && \text{(III.3.6)} \\ f \triangleright A &= -q^{\frac{1}{2}} B. && \text{(III.3.7)} \end{aligned}$$

We remind here the notation of q-numbers: $[x]$ denotes $\frac{q^x - q^{-x}}{q - q^{-1}}$.

B. Equivariant representation of $\mathcal{A}(\mathcal{S}^{\in}_{\amalg})$.

Using the symmetry show above we shall construct an equivariant representation of the Standard Podles' sphere algebra. For details and definitions of equivariance with respect to Hopf algebra actions we refer to [11], whereas details concerning the case of the Podles sphere are discussed in [8]. It appears that there exist two inequivalent equivariant representations of $\mathcal{A}(\mathcal{S}^{\in}_{\amalg})$, $\pi_{\pm}$, on a Hilbert space constructed as the completion of a direct sum of half-integer irreducible representation spaces of $\mathcal{U}_{\amalg}(\int\sqcap(\in))$.

$$\mathcal{H} = \mathcal{V}_{\frac{\infty}{\in}} \oplus \mathcal{V}_{\frac{\ni}{\in}} \oplus \dots ,$$

They have the following form (we use here the obvious notation denoting the vectors from the Hilbert space as $|l,m\rangle$, since the representation theory of $\mathcal{U}_{\amalg}(\int\sqcap(\in))$ and $su(2)$ are the same):

$$\begin{aligned} B|l,m\rangle &= B^{+}_{l,m}|l+1,m+1\rangle + B^{0}_{l,m}|l,m+1\rangle + B^{-}_{l,m}|l-1,m+1\rangle, \\ B^{*}|l,m\rangle &= \widetilde{B}^{+}_{l,m}|l+1,m-1\rangle + \widetilde{B}^{0}_{l,m}|l,m-1\rangle + \widetilde{B}^{-}_{l,m}|l-1,m-1\rangle, \\ A|l,m\rangle &= A^{+}_{l,m}|l+1,m\rangle + A^{0}_{l,m}|l,m\rangle + A^{-}_{l,m}|l-1,m\rangle, \end{aligned} \tag{III.3.8}$$

where $A^{j}_{l,m}, B^{j}_{l,m}, \tilde{B}^{j}_{l,m}$ are:

$$\begin{aligned} B^{+}_{l,m} &= q^{m}\sqrt{[l+m+1][l+m+2]}\ \alpha^{+}_{l}, \\ B^{0}_{l,m} &= q^{m}\sqrt{[l+m+1][l-m]}\ \alpha^{0}_{l}, \\ B^{-}_{l,m} &= q^{m}\sqrt{[l-m][l-m-1]}\ \alpha^{-}_{l}, \\ \widetilde{B}^{+}_{l,m} &= q^{m-1}\sqrt{[l-m+2][l-m+1]}\ \alpha^{-}_{l+1}, \\ \widetilde{B}^{0}_{l,m} &= q^{m-1}\sqrt{[l+m][l-m+1]}\ \alpha^{0}_{l}, \\ \widetilde{B}^{-}_{l,m} &= q^{m-1}\sqrt{[l+m][l+m-1]}\ \alpha^{+}_{l-1}, \\ A^{+}_{l,m} &= -q^{m+l+\frac{1}{2}}\sqrt{[l-m+1][l+m+1]}\ \alpha^{+}_{l} \\ A^{0}_{l,m} &= q^{-\frac{1}{2}}\frac{1}{1+q^2}\left([l-m+1][l+m] - q^2[l-m][l+m+1]\right)\alpha^{0}_{l} + \frac{1}{1+q^2}, \\ A^{-}_{l,m} &= q^{m-l-\frac{1}{2}}\sqrt{[l-m][l+m]}\ \alpha^{-}_{l}\ . \end{aligned} \tag{III.3.9}$$

with:

$$\alpha^{-}_{l+1} = -q^{2l+2}\alpha^{+}_{l}\ , \quad \text{for} \quad l = \tfrac{1}{2}, \tfrac{3}{2}, \dots\ . \tag{III.3.10}$$

The difference between the representation appears only in the form of functions $\alpha^{+}_{l}, \alpha^{0}_{l}$, where we have:

- π_{+}:

$$\alpha^{0}_{l} = \frac{1}{\sqrt{q}}\frac{(q-\frac{1}{2})[l-\frac{1}{2}][l+\frac{3}{2}] + q}{[2l][2l+2]}\ , \tag{III.3.11}$$

$$\alpha^{+}_{l} = q^{-l-2}\frac{1}{[2l+2]}\sqrt{\frac{[l+\frac{1}{2}][l+\frac{3}{2}]}{[2l+1][2l+3]}}\ . \tag{III.3.12}$$

- π_{-}:

$$\alpha^{0}_{l} = \frac{1}{\sqrt{q}}\frac{(q-\frac{1}{q})[l-\frac{1}{2}][l+\frac{3}{2}] - q^{-1}}{[2l][2l+2]}\ , \tag{III.3.13}$$

$$\alpha^{+}_{l} = q^{-l-1}\frac{1}{[2l+2]}\sqrt{\frac{[l+\frac{1}{2}][l+\frac{3}{2}]}{[2l+1][2l+3]}}\ . \tag{III.3.14}$$

Although the form of the representation appears to be rather complicated, we shall see that it simplifies considerably when we shall neglect contributions, which are *small*.

C. Neglecting infinitesimals

The idea to look at the leading parts of an expression and neglect small contributions is a physical one. Here, we need only a suitable notion of objects, which are small. This role in operator algebras is played by compact operators: sparing the reader a formal definition, let us say that a prototype of a compact operator is a diagonal operator with discrete spectrum such that the sequence of eigenvalues leads converges to 0. In our case we shall consider the ideal generated by the following operator:

$$T_q|l,m\rangle = q^l|l,m\rangle. \tag{III.3.15}$$

Clearly, since $q < 1$ this is a compact operator. Now, if we rewrite the equivariant representation constructed above, getting rid of all terms, which are in the ideal generated by T, we obtain:

$$\begin{aligned} \pi(B)|l,m\rangle_\pm &\sim -q^{l+m}\sqrt{1-q^{2l+2m+2}}|l,m+1\rangle_\pm, \\ \pi(B^*)|l,m\rangle_\pm &\sim -q^{l+m-1}\sqrt{1-q^{2l+2m}}|l,m-1\rangle_\pm, \\ \pi(A)|l,m\rangle_\pm &\sim q^{2l+2m}|l,m\rangle_\pm. \end{aligned} \tag{III.3.16}$$

As we can see, the representations are in the leading term identical and diagonal in l. We shall immediately see that this leads to surprising conclusions in the construction of spectral triples. Having established the representation up to infinitesimals we can start studying spectral geometries, still keeping in mind that we might neglect the operators from the ideal generated by T_q. Thus, we can expect that the modification of the spectral triple axioms by allowing that they do not hold exactly but only up to elements in the ideal of compact operators is justified.

IV. CONSTRUCTING THE SPECTRAL TRIPLE DATA

The original equivariant spectral triple, constructed in [8] had the following data: The grading γ:

$$\gamma|l,m\rangle\pm = \pm|l,m\rangle_\pm. \tag{IV.4.1}$$

the reality operator J:

$$J|l,m\rangle_\pm = i^{2m}|l,-m\rangle_\mp, \tag{IV.4.2}$$

and the Dirac operator:

$$D|l,m\rangle_\pm = [l+\tfrac{1}{2}]|l,m\rangle_\mp, \tag{IV.4.3}$$

Now, having a look at the represention up to infinitesimals, we see that this is, to some extent, the extreme case. In fact, all operators of the form:

$$D_a|l,m\rangle_\pm = d_l|l,m\rangle_\mp,$$

such that $\{t_l d_l\}$ is a bounded sequence for any spectral sequence $\{t_l\}$ of an operator from the ideal generated by T_q, would satisfy the axioms "up to infinitesimals". Indeed, since the approximate representation was diagonal, such operator do commute with it, then we must only make sure that the commutator with the neglected part (which is from the ideal generated by T) is bounded at most. Similarly, order one condition, is satisfied almost automatically, though only up to infinitesimals from the ideal. The *isospectral* case with $d_l = l + \frac{1}{2}$ is, of course, also possible. Note that only in the "extreme" case the commutators are bounded and not compact. The specific form of the algebra opens new possibilities and raises many questions. For instance, it appears that almost any Dirac operator fits into the picture of spectral geometry. Then, which one is the correct one? Furthermore, one might attempt to consider some other constructions, with not only the Dirac operator but also the grading γ and the reality structure J deformed. One particular example, motivated by a recent work on projection of spectral triples [13] is the following:

$$\begin{aligned} D|l,m\rangle_\pm &= \pm(l+\tfrac{1}{2})|l,m\rangle_\pm, \\ \gamma|l,m\rangle_\pm &= |l,m\rangle_\mp, \\ J|l,m\rangle_\pm &= i^{2m}|l,m\rangle_\mp, \end{aligned} \tag{IV.4.4}$$

This yields another spectral triple. While some of the relations are evident, let us check directly:

$$(J\gamma + \gamma J)|l,m\rangle_\pm = (i^{2(2l\pm m)} + i^{2(2j\mp m)})|l,m\rangle_\pm = 0,$$

Since the approximate representation was the same for $\pi_\pm$, γ clearly commutes with the algebra up to the ideal of compact operators though it does not commute with it exactly. Therefore, this spectral data is a different one than considered earlier. The use of the axioms "up to infinitesimals" allows also for the generalisations of quantum spheres and taking into account a quantum ellipsoid, that is, an object with the topology of the quantum sphere but a different metric. For instance, by introducing (in the first spectral geometry) a term in D depending linearly on m:

$$D|l,m\rangle_\pm = \pm\left(l + \tfrac{1}{2} + im\right)|l,m\rangle_\pm,$$

we see that such operator still satisfies the axioms (up to infinitesimals from the ideal of compact operators, of course) but is no longer spherically symmetric. Thus, it shall correspond to a metric deviation from the sphere.

V. CONCLUSION

We have reviewed here the basic notions of spectral geometry and its most recent example of Standard quantum Podles sphere. This example is one of the most striking cases and it shows us that the noncommutative world is much richer than expected. We have that shown several approaches might give more construction than originally expected. We hope that by studying this we shall learn to appropriate notions of geometry, which is applicable to noncommutative object, q-deformations in particular. The indicated results are a good starting points for further studies, with the aim to construct fundamental geometrical background for physically relevant theories of fundamental interactions.

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