African Journal Of Mathematical Physics Vol **3** *No 1 (2006)121-127*

Planck Scale Variations of the Metric Tensor Leading to a Cosmological Constant

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Abstract

Gravity and quantum physics become inseparable at the Planck scale. This leads to the expectation that the space-time metric itself behaves as a quantum variable. In the present work, this behavior is modeled as conformal variations about the classical background metric. The scalar field which describes these variations represents the quantum fluctuations; it is treated in lowest order by employing the classical variation principle. Variation of the gravitational action leads thus to a modified Einstein equation, where the new extra term is interpreted as a cosmological constant. The resulting equation of motion for the expanding universe is capable of accommodating the current observational data, which indicate an accelerating expansion during the current epoch.

I. INTRODUCTION

Considerations of Quantum Mechanics of General Relativity lead to the concept of the Planck scale. This scale is reached when the energy of a particle becomes high enough so its Schwarzschild radius is comparable to its Compton wavelength. The Planck energy is

$$E_P = c^2\sqrt{\frac{\hbar c}{G}} \ . \tag{1.1}$$

corresponding to the Planck length

$$l_P = \sqrt{G\hbar c}/c^2 \cong 10^{-35}\mathrm{m} \ . \tag{1.2}$$

At this length scale general relativity and quantum mechanics are fundamentally interwoven and a theory of quantum gravity is needed. It is generally believed that the most promising candidates for a consistent theory of quantum gravity are of higher dimensionality, like string theories and brane world scenarios. However a full theory of quantum gravity is still lacking. Therefore a pragmatic approach in 3+1 dimensions is proposed here; it considers the metric tensor g_{mn} as a quantum variable.

The starting point of the present approach is a generalized metric in which an additional scalar field φ is included. This field is to represent the quantum fluctuations of the metric about its classical value. The field φ itself is treated in an approximate way, as a classical field. The generalized metric is inserted

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in the gravitational action. In the usual classical case without the extra field φ, the variation of the action leads to Einstein's equation *without* a cosmological constant[1],

$$R_{mn} - \frac{1}{2} g_{mn} R = \frac{8\pi G}{c^4} T_{mn} \ . \tag{1.3}$$

The variation of the action with the generalized metric yields however additional terms which take the role of a cosmological constant Λ, so the Einstein eq. becomes

$$R_{mn} - \frac{1}{2} g_{mn} R - \Lambda g_{mn} = \frac{8\pi G}{c^4} T_{mn} \ . \tag{1.4}$$

In this way, the cosmological constant is interpreted as stemming from quantum fluctuations of the spacetime metric. The fact that the variation principle is applied in the present work means that the field φ, representing the quantum fluctuations, is treated in classical approximation as to its effect on Einstein's equation.

Originally, the idea of a cosmological constant was introduced by Einstein to allow for a static solution of the gravitational equations describing the universe, a requirement which turned out later to be in contradiction with observation. Nowadays Λ is getting revived interest again in the context of dark energy, driving the expansion of the universe. Recent observations of type Ia supernovae give an ever stronger evidence of an expansion of the universe which is currently accelerating (e.g.[1,2]), contrary to what would be expected when considering the gravitational attraction of energy and matter contents of the universe. The acceleration may explain also the existent discrepancies between the age of the universe and age estimates of stars in globular clusters[3,4].

II. FLUCTUATIONS OF THE METRIC

In the present approach, the simplest kind of metric variation is considered,

$$g_{mn} = (1+\varphi)^2 \bar{g}_{mn} \ , \tag{2.1}$$

where $\bar{g}_{mn}$ denotes the "classical" or "background" metric about which the fluctuations occur and the fluctuation average (vacuum expectation value) is $< \varphi > = < \varphi_{,m} > = 0$, meaning that the fluctuations are centered around the classical value and that there is no drift of φ in spacetime. These requirements guarantee also energy conservation, as expressed in eq. (3.9) below. This purely conformal degree of freedom is of course not the most general one, but it has the advantage to keep the light cone structure of spacetime intact, which is important for not violating causality at any instant of the fluctuation. (The fluctuations lead however to a Planck size fuzziness of the light cone *after averaging* over the fluctuations, as can be gleaned from the behavior of the four-distance eq. (2.6) below).

The scalar field φ represents an additional degree of freedom which is to describe the quantum fluctuations, expressed as the conformal deviations of the metric from its classical value without fluctuations. Einstein's equation can be derived from variation of the action ($\hbar = c = 1$)

$$S = S_{\mathrm{g}} + S_{\mathrm{m}} = \frac{1}{16\pi G} \int \mathrm{d^4x}\sqrt{-\mathrm{g}}\mathrm{R} + \mathrm{S_m} \ , \tag{2.2}$$

where S_{g} is the gravitational (Hilbert) action *without an extra cosmological constant* and S_{m} the matter part, for which the expression

$$\delta S_{\mathrm{m}} = -\frac{1}{2} \int \mathrm{d^4x}\sqrt{-\mathrm{g}}\delta \mathrm{g_{mn}} \mathrm{T^{mn}} \tag{2.3}$$

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[1] In the sign convention used here the Ricci tensor is calculated by the contraction $R_{mn} = R^{l}{}_{mln}$, all Latin indices running from 0 to 3. The signature is $(+ - --)$.

holds. This is the point at which the field φ, representing the quantum fluctuations, is treated in an approximate way, namely in a classical variation instead of evaluating path integrals.

The presence of conformal fluctuations changes the scalar curvature to (see *e.g.*[5])

$$R = \frac{\bar{R}}{(1+\varphi)^2} - \frac{6\bar{g}^{mn}\varphi_{;mn}}{(1+\varphi)^3} \ , \tag{2.4}$$

where the overbar indicates the classical value, i.e. the one obtained without fluctuations. In the special case of flat background spacetime the gravitational action reduces to

$$S_{\mathrm{g}} = \frac{3}{8\pi l_P^2}\int \mathrm{d}^4\mathrm{x}\partial^{\mathrm{i}}\varphi\partial_{\mathrm{i}}\varphi \ . \tag{2.5}$$

The proper distance acquires a lower limit of the order of the Planck length, as a consequence of the vacuum expectation value of φ^2. To be specific, the fluctuation average of the squared four-distance becomes[5]

$$< x^2 >= x^2 + \frac{l_P^2}{3\pi} \ . \tag{2.6}$$

Thus spacetime is coarse grained at the level of the Planck scale. As a consequence, divergences related to Green's functions (propagators) disappear, without the need for regularization or renormalization[6]. On the other hand, when looking at scales larger than the Planck length, the fluctuation average $< x^2 >$ is equal to the classical background value x^2. In the discussion of the expansion of the universe it is therefore the behavior of the background metric which one is interested in studying.

III. EINSTEIN'S EQUATION WITH FLUCTUATIONS

The variation leading to Einstein's equation includes now two parts: the variation of the background metric and the variation of the scalar field describing the quantum fluctuations:

$$\delta S = \frac{\delta S}{\delta g^{ik}}\delta g^{ik} = \frac{\delta S}{\delta \bar{g}^{ik}}\delta \bar{g}^{ik} + \frac{\delta S}{\delta \varphi}\delta\varphi \ . \tag{3.1}$$

The variations are independent and lead to two equations, one proportional to $\delta\bar{g}^{ik}$,

$$\bar{R}_{ik} - \frac{1}{2}\bar{g}_{ik}\bar{R} + \frac{3}{2}\bar{g}_{ik}\frac{\varphi^{;m}_{\ ;m}}{1+\varphi} - 8\pi G T_{ik} = 0 \ , \tag{3.2}$$

and the other proportional to $\delta\varphi$,

$$\bar{R} - 6\frac{\varphi^{;m}_{\ ;m}}{1+\varphi} + 8\pi G\bar{g}^{ik}T_{ik} = 0 \ . \tag{3.3}$$

The second equation is then used to eliminate the φ-dependence from the first and the resulting Einstein equation can now be cast in the form of eq. (1.4) with the extra term $\bar{g}_{ik}\Lambda$ in which

$$\Lambda = -\frac{1}{4}(8\pi G\bar{g}^{mn}T_{mn} + \bar{R}) \ . \tag{3.4}$$

This additional term, which is not present in Einstein's equation without a cosmological constant, is a consequence of the inclusion of φ in the generalized metric. The fluctuations of the metric lead therefore to the appearance of a cosmological constant of the form (3.4).

To proceed further the background metric must be specified. The most symmetric type of universe is described by a Friedmann-Weyl background cosmology. The corresponding line element is

$$\mathrm{ds}^2 = \mathrm{dt}^2 + \bar{\mathrm{g}}_{\alpha\beta}\mathrm{dx}^\alpha\mathrm{dx}^\beta \qquad (\alpha,\beta = 1...3) \tag{3.5}$$

and the cosmological principle of homogeneity and isotropy leads to the Robertson-Walker metric

$$ds^2 = dt^2 - Q^2(t)\left[\frac{dr^2}{1-\kappa r^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right] . \tag{3.6}$$

The quantity $Q(t)$ describes the evolution of the size of the universe and the sign of κ determines whether the universe is classically open or closed. To complete the derivation of the Einstein equation, an assumption on the matter distribution in the universe is needed as well. The simplest and most symmetrical distribution considers homogeneous comoving matter (dust approximation):

$$T_{kl} = \rho u_k u_l \qquad \text{with } \{u_k\} = (1,0,0,0) . \tag{3.7}$$

Energy conservation, the 0-component of energy-momentum conservation ${T^{kl}}_{;l} = 0$, leads to the equation

$$(1+\varphi)[\dot\rho + 3\rho\frac{\dot Q}{Q}] + 5\rho\dot\varphi = 0 . \tag{3.8}$$

With the fluctuation averages $<\varphi>=<\dot\varphi>=0$ discussed before, the vacuum expectation value of the above equation leads to the usual density behavior of the dust universe,

$$\rho(t) = \frac{\rho_0 Q_0^3}{Q^3} . \tag{3.9}$$

Einstein's equation takes finally the form of an equation of motion for Q,

$$\ddot Q Q^2 - \dot Q^2 Q - \kappa Q + 4\pi G\rho_0 Q_0^3 = 0 . \tag{3.10}$$

Adjusting the scale such that the initial conditions are $Q_0 = \dot Q_0 = 1$ (meaning that time is measured in units of H_0^{-1}), the differential equation has the solution

$$t - t_0 = \int_1^Q dq\sqrt{\frac{q}{Aq^3 - \kappa q + \frac{8}{3}\pi G\rho_0}} , \tag{3.11}$$

where

$$A = \kappa + 1 - \frac{8}{3}\pi G\rho_0 . \tag{3.12}$$

From the solution it follows that

$$\dot Q = \sqrt{\frac{AQ^3 - \kappa Q + \frac{8}{3}\pi G\rho_0}{Q}} \tag{3.13}$$

and therefore

$$\ddot Q = \frac{AQ^3 - \frac{4}{3}\pi G\rho_0}{Q^2} . \tag{3.14}$$

IV. DISCUSSION OF THE RESULTS

Fixing the present epoch parameters to the observed values allows us to study the behavior of the expansion of the universe and the cosmological constant. Figure 1 shows the time evolution of the expansion parameter Q (full line) in units of the present epoch value Q_0, assuming a present time density of 0.3 in units of the critical density $\rho_c = 3H_0^2/8\pi G$ (where $H_0 = \dot Q_0/Q_0$ is the Hubble constant which is unity in the chosen scale) and a deceleration parameter

$$d_0 = -\frac{\ddot Q_0 Q_0}{\dot Q_0^2} = -\ddot Q_0 \tag{4.1}$$

of value -0.5 (the negative sign meaning accelerating expansion). The choice of d_0 fixes also the value of κ through eq. (3.10) to

$$\kappa = (4\pi G\rho_0 - (d_0 + 1)) \tag{4.2}$$

with the value -0.017 in the conventional units of $3H_0^2$. The cosmological constant becomes

$$\Lambda = 4\pi G\rho_0 - 3d_0 \tag{4.3}$$

with the value 0.65 in the usual units $3H_0^2$, in agreement with recent observational data[7,8]. The set $\rho_0, d_0, \Lambda, \kappa$ is consistent with data sets in ref.[2,9,10], too. Fig. 1 shows also the classical $\Lambda = 0$ time evolution (dashed line), assuming the same present epoch density. The corresponding expansion is of course decelerating in that case (with the deceleration parameter +0.15). The time axis is drawn in units of the inverse Hubble constant and the present epoch time is at zero. It should be noted that due to the dust approximation the results are not realistic for times in the remote past, where for instance radiation pressure dominated. In addition, an eventual early inflationary stage is not considered here. The two curves show nevertheless that the quantum fluctuations result in an increased age of the universe, in this comparison by more than 15%. Adopting a Hubble parameter (defined as $h = H_0/100$ km s^{-1}Mpc^{-1}) with the typical value $h = 0.7$ results in an age of 13.3×10^9 years, also consistent with data[9,10]. The future expansion rate is visibly influenced by the repulsive effect of the fluctuations as well.

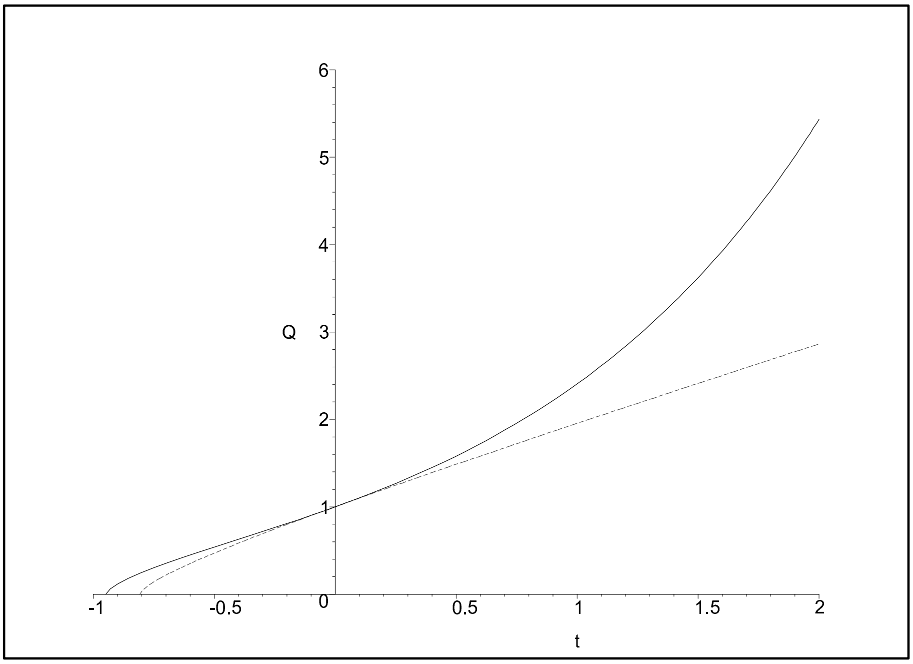

Figure 1 - Q vs. t in relative units (see text), with (full line) and without (dashed) quantum fluctuations.

Figures 2 and 3 show the time dependence of $\dot{Q}$ and $\ddot{Q}$, respectively, from which one can more easily detect that the expansion driven by Λ picks up speed at around 60% of the present age of the universe; at that epoch the acceleration $\ddot{Q}$ becomes positive. In the classical case without Λ, the expansion velocity decreases monotonically, af course.

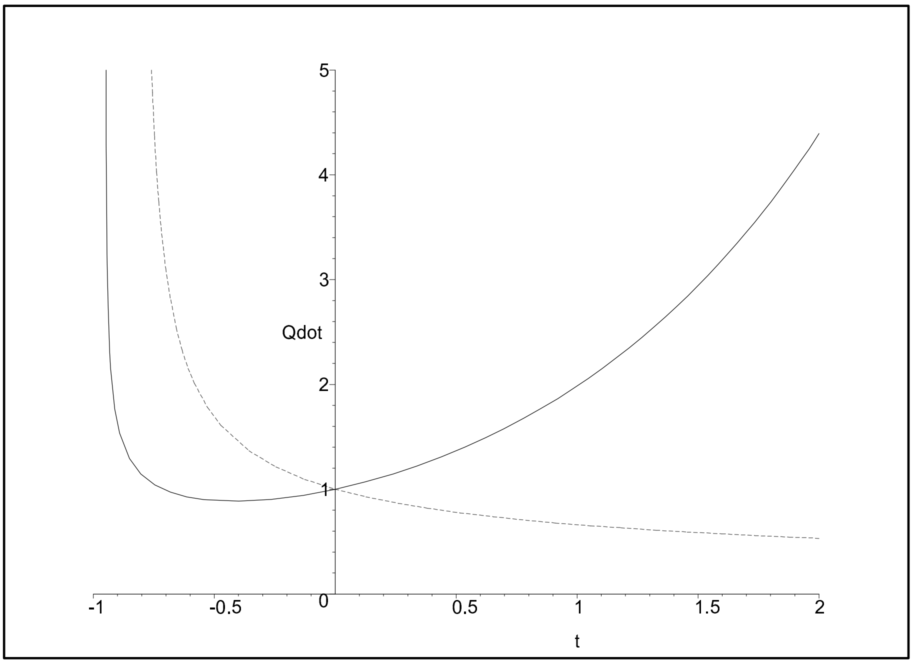

Figure 2 - $\dot{Q}$ vs. t in relative units (see text), with (full line) and without (dashed) quantum fluctuations.

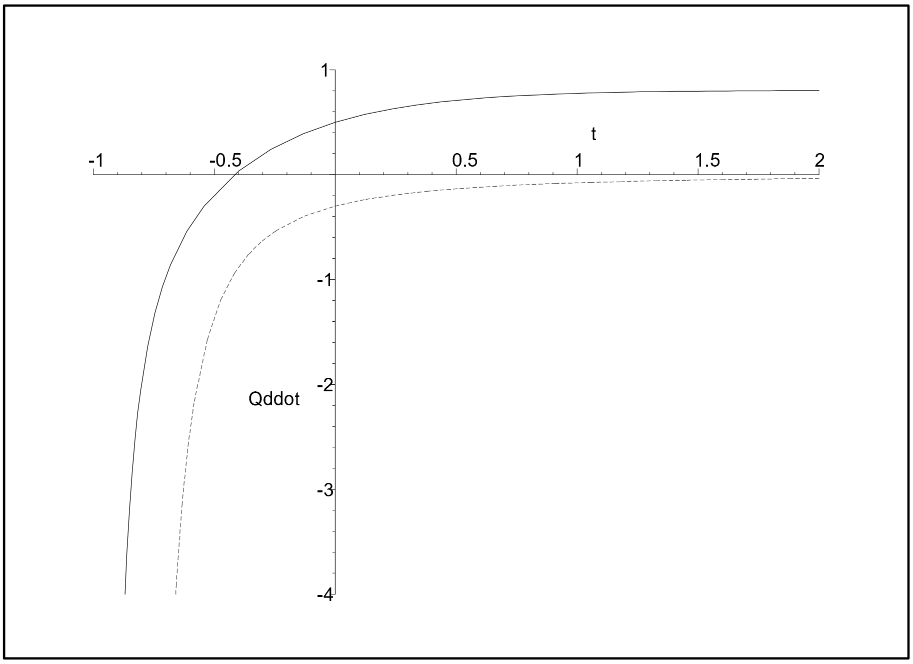

Figure 3 - $\ddot{Q}$ vs. t in relative units (see text), with (full line) and without (dashed) quantum fluctuations.

The present approach does not predict the initial (or present epoch) parameters; these depend on the physics during the big bang, of course. However, the fluctuations of the metric lead to a cosmological constant and the resulting equation of motion is consistent with the observational data. This consistency cannot exist taking only the classical background metric into account. The fact that the correct cosmological constant is predicted using the observational present epoch parameters lends support to the validity of this approach.

To summarize, the effect of quantum gravity at the Planck scale is modeled here by conformal fluctuations of the metric. The scalar field which describes the conformal quantum variations is treated approximately in a variational approach. The fluctuations are shown to result in a cosmological constant, explaining why an accelerated expansion of the universe is observed. It might seem surprising that fluctuations at the level of the Planck size have consequences on the expansion of the universe as a whole. One should keep in mind, however, that these fluctuations pervade the whole of space of the universe and in this way are able to produce a macroscopic effect. The quantum fluctuations make it possible to obtain an equation of motion which is consistent with the observational data. As mentioned in the beginning, degrees of freedom other than the conformal one may have to be considered to reflect more fully the effects of quantum gravity. In addition, no eventual early inflationary phase is considered. Due to this and to the approximate treatment of the fluctuation field, as well as to the dust approximation employed in the present calculations, together with our incomplete knowledge of the present day parameters, the results are to be considered qualitative rather than quantitative, although they are in perfect agreement with the observational data within the observational accuracy. It seems clear that quantum fluctuations have to be taken seriously as they affect macroscopic observables.

ACKNOWLEDGEMENTS

This work was financially supported by ESO, CERN, POCTI and FCT.

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