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Anharmonic Oscillator Hamiltonian and Multiple Scale Techniques

M. Capdequi Peyranère

Laboratoire de Physique Théorique et Astroparticules, CNRS-UMR 5207
Université Montpellier II, F–34095 Montpellier Cedex 5, France

Abstract

A multiple scale technique is applied to the quantum anharmonic oscillator in the Heisenberg picture, leading to interesting features of the solution of the Heisenberg equations of motion and of the Hamiltonian spectrum. Remarks on the classical case are also sketched.

1. INTRODUCTION

Multiple scale techniques (MST) are methods to deal with secularities or resonances arising in solving differential equations in physics such as newtonian celestial mechanics or fluid dynamics. In this work, we apply one of the various MST to the quantum anharmonic oscillator, initiated by Bender and Bettencourt (B&B)[1,2], and we extend their studies in two apparent different directions, which turn out to be equivalent.

In order to introduce the problem, we begin with the classical case. The classical anharmonic oscillator (CAO) is probably one of the most popular examples where the conventional and MST perturbative theories lead to obvious differences. Starting from the CAO Lagrangian (in convenient units)

$$L(q,\dot{q}) = \dot{q}^2/2 - q^2/2 - gq^4$$

one readily gets from the Euler-Lagrange equation:

$$(e) : \ddot{q} + q + 4gq^3 = 0.$$

The usual formal perturbation expansion reads

$$q(t) = \sum_{n=0}^{\infty} g^n q_n(t),$$

(with some initial conditions, say $q(0) = Q$ and $\dot{q}(0) = 0$), and the first equations one obtains from (e) are :

$$(e_0) : \ddot{q}_0 + q_0 = 0,$$

$$(e_1) : \ddot{q}_1 + q_1 = -4q_0^3.$$

Then the frequency of the solution of the homogeneous part of (e_1) coincides with the frequency of $q_0(t) = Qcost$, which generates a resonance in the solution $q_1(t)$ of the full equation (e_1):

$$q_1(t) = \frac{Q^3}{8}(cos3t - cost - 12tsint).$$

Hence $q_1(t)$ is unbounded and the truncated expansion $q_0(t) + gq_1(t)$ cannot be an acceptable approximation of $q(t)$ for times t larger than $1/Q^2g$, however small g may be. The flaw is even worse at the higher orders. It is obvious on this simple example that the perturbative solution develops spurious behaviour which is absent in the exact solution, which is a bounded elliptic function.

The main idea of MST for dealing with this problem is the introduction of new variables, independent and appropriate, and we refer to textbooks for an extensive review of the various possibilities. Here we concentrate on the anharmonic oscillator. Some methods take into account ab initio that the circular functions play a major role. Rather we will use a method without prerequisite, the Derivative Expansion Method. It promotes the time variable to be a function of the coupling constant, namely $t_n = g^n t$. Actually, the method is not so rough and one first extends the function depending on t to an "extended" function depending on all the variables t_n $(n = 0, 1, 2, ...)$ assumed to be independent. So, one introduces a new position function $Q(T, g)$ depending on the collection $T = \{t_0, t_1, t_2, ...\}$ of independent variables t_n. This function is considered as an extension of the true position in the Lagrange formalism, which is recovered by restricting Q to the section $t_n = g^n t$ of the $T-$space: $q(t, g) = Q(T, g)|_{t_n = g^n t}$.

Then, forgetting temporarily any reference to the coupling constant in these t_n variables, one expands in power of g the new position function Q :

$$Q(T, g) = \sum_{n=0}^{\infty} g^n Q_n(T).$$

One obtains from (e) the first two equations,

$$D_0^2 Q_0(T) + \omega^2 Q_0(T) = 0\,,$$

$$D_0^2 Q_1(T) + \omega^2 Q_1(T) = -2D_0 D_1 Q_0(T) - 4Q_0^3(T)\,,$$

using $d/dt = \Sigma_n g^n D_n$, $D_n = \partial/\partial t_n$ ..
The basic principle of the method now consists in adjusting the t_1 dependence of $Q(T)$ so as to eliminate the secularity in the second equation, and so on for the higher orders not considered here. We get from the first equation

$$Q_0[t_0, t_1, t_2, ...] = A_0[t_1, t_2, ...]exp(-it_0) + c.c.$$

Inserting this solution in the second equation, one collects terms proportional to $exp(-it_0)$ which must be removed by an adequate choice of $A_0[t_1, t_2, ...]$:

$$2iD_1 A_0[t_1, t_2, ..] + 12A_0[t_1, t_2, ..]^2 A_0[t_1, t_2, ..]^* = 0$$

Then the second equation is solved free of secularity and one gets:

$$Q_1[t_0, t_1, t_2, ...] = A_1[t_1, t_2, ...]exp(-it_0) + A_0[t_1, t_2, ..]^3 exp(-3it_0)/2 + c.c.$$

Finally the solution of the equation of motion for our CAO, up to first order, reads as:

$$q(t, g) = \frac{a}{2}[exp(-i(\Omega t + b)) + \frac{ga^2}{8} exp(-3i(\Omega t + b))] + c.c.,$$

where

$$\Omega = 1 + \frac{3ga^2}{2},$$

and a and b are real integration constants fixed by certain initial conditions.
Thanks to the time invariance, the energy is conserved and reads :

$$E = \frac{a^2}{2}(1 + \frac{9ga^2}{4}) + O(g^2). \tag{0.1}$$

We conclude this first section by a few comments. As far as we know, all the multiple scale techniques dealing with the secularities of the classical anharmonic oscillator are successful. However this is not a general feature, and some methods are not suitable for certain problems. Moreover it is absolutely not our purpose to discuss on a rigorous basis the mathematical aspects of the secular or non secular perturbative expansions. Lastly, the frequency, proportional to Ω here, is no longer independent of the boundary conditions in non linear equations.

2. THE QUANTUM CASE : QAO_1

The quantum anharmonic oscillator (QAO) has been studied in the paper of B&B through the Heisenberg equation of motions for the relevant operators and we will follow this method. The difference between the work of B&B and ours is that we will use the creation and annihilation operators to manage the problem of removing the secularities. Also it appears a variation in the status of the initial conditions: we do not use these conditions as in the classical case, which is the way taken by B&B.

We start with the QAO Hamiltonian H written in terms of the momentum p and position q operators in convenient units : $H = p^2/2 + q^2/2 + gq^4$, where g is assumed to be a "small" (positive) coupling constant. Within the Heisenberg picture, the dynamics is governed by the equations :

$$\dot{q} = i[H, q]\,,\, \dot{p} = i[H, p],$$

supplemented by the canonical commutation relation $[q, p] = i$, valid at all times. The Heisenberg equations give : $\dot{q} = p$ and $\dot{p} = -q - 4gq^3$. Writing as usual $q = (a + a^\dagger)/\sqrt{2}$ and $p = -i(a - a^\dagger)/\sqrt{2}$, the Hamiltonian becomes:

$$H(a, a^\dagger, g) = 1/2 + a^\dagger a + g(a + a^\dagger)^4/4 \tag{0.2}$$

together with :

$$[a(t, g),\, a(t, g)] = 1\,,\ \forall t, \tag{0.3}$$

where, to avoid possible confusion later on, we have kept track of the variables t and g. The Heisenberg equation for the annihilation operator :

$$\dot{a}(t, g) = i[H(a(t, g), a^\dagger(t, g), g), a(t, g)],$$

reads, in our case :

$$\dot{a}(t, g) = -i(a(t, g) + g(a(t, g) + a^\dagger(t, g))^3). \tag{0.4}$$

Since the Hamiltonian is conserved, its formal solution is:

$$a(t, g) = exp(iH(a(0), a^\dagger(0), g)t)a(0)exp(-iH(a(0), a^\dagger(0), g)t),$$

with $a(0) \equiv a(0, g)$.
We now turn on the formal series of the multi-time perturbative expansion, similar to that used in the classical case. First one introduces an operator valued function $A(T, g)$ depending on the collection T of independent variables t_j. This function is considered as an extension of the true annihilation operator in the Heisenberg picture, which is recovered through the restriction:

$$a(t,g) = A(T,g)|_{t_j=g^jt}. \tag{0.5}$$

Then the time derivative becomes :

$$\dot{a}(t,g) = \sum_{n\geq 0} g^n D_n A(T,g)|_{t_j=g^jt}.$$

Secondly, $A(T,g)$ is expanded as :

$$A(T,g) = \sum_{n\geq 0} g^n A_n(T). \tag{0.6}$$

As for the initial conditions to be associated with the equation of motion (4), one notices that

$$a(0,g) = \sum_{n\geq 0} g^n A_n(0). \tag{0.7}$$

This forces us to choose between two possible starting viewpoints :
either a) : $a(0,g)$ is taken as independent of g , which implies

$$A_n(0) = 0, \forall n \geq 1, \tag{0.8}$$

or b) : the previous condition is not imposed, in which case the initial values of $a(t,g)$ must be considered as a function of g.

In this paper, we will follow the procedure b), which we found much more convenient, and in a sense, more natural. The equation of motion for $a(t,g)$ gives us the following infinite system for the $A_n(T)$'s :

$$D_0 A_n + iA_n = -\sum_{m=0}^{n-1} D_{n-m} A_m - i \sum_{\substack{m,r,s,>,0 \\ m+r+s=n-1}} Q_m Q_r Q_s \quad (n = 0,1,2..) \tag{0.9}$$

where $Q_n = A_n + A_n^\dagger$, or explicitly :

$$D_0 A_0 + iA_0 = 0, \tag{9.a}$$
$$D_0 A_1 + iA_1 = -D_1 A_0 - iQ_0^3, \tag{9.b}$$
$$D_0 A_2 + iA_2 = -(D_2 A_0 + D_1 A_1) - i(Q_0^2 Q_1 + Q_0 Q_1 Q_0 + Q_1 Q_0^2), \tag{9.c}$$

etc...

A simple check shows us that **any** formal solution of (9) generates via (6) and (7) a formal solution $a(t,g)$ of (4). In particular, this implies that, for such a solution, $[A(T,g), A^\dagger(T,g)]|_{t_j=g^jt}$ is independent of t. Of course, this does no mean yet that $[A(T,g), A^\dagger(T,g)]$ is independent of T, allowing us to impose :

$$[A(T,g), A^\dagger(T,g)] = 1\,, \; \forall T\,, \tag{0.10}$$

in order to insure the canonical commutation relation (3) . However, one can look for **those** solutions of (9) which are subjected to the stronger condition (10), if such solutions do exist indeed, i.e. if no inconsistencies or obstructions arise in their iterative construction. Together with (6), this entails :

$$\left\{ \begin{array}{l} [A_0(T), A_0^\dagger(T)] = 1 \\ \sum_{m=0}^{n} [A_m(T), A_{n-m}^\dagger(T)] = 0, \; n \geq 1 \end{array} \right. \quad \forall T \tag{0.11}$$

, which has to be considered as a **working hypothesis**.

We are now ready to construct step by step the resonance-free solution of the problem. To zeroth order, the equation (9.a) and the first equation (11) yield :

$$A_0(T) = A_{01}(T_1) exp(-it_0) \tag{0.12}$$

with

$$[A_{01}(T_1), A_{01}^\dagger(T_1)] = 1, \forall T_1, \tag{0.13}$$

and the notation : $T_k = \{t_k, t_{k+1}, ...\}, (k = 1, 2, ...)$.

Then, one can proceed to the first order step by inserting eq (12) into eq (9.b) :

$$D_0 A_1 + iA_1 = -(D_1 A_{01} + i(A_{01}^2 A_{01}^\dagger + A_{01} A_{01}^\dagger A_{01} + A_{01}^\dagger A_{01}^2))exp(-it_0) - i(A_{01}^3 exp(-3it_0)$$

$$+A_{01}^{\dagger^3} exp(+3it_0) + (A_{01}^{\dagger 2} A_{01} + A_{01}^\dagger A_{01} A_{01}^\dagger + A_{01} A_{01}^{\dagger 2})exp(+it_0)). \tag{0.14}$$

Before integrating this equation, one has to get rid of the first resonant term on the right hand side, which would produce a contribution growing linearly with $t_0(= t)$. This leads to the condition :

$$D_1 A_{01} = -i(A_{01}^2 A_{01}^\dagger + A_{01} A_{01}^\dagger A_{01} + A_{01}^\dagger A_{01}^2) \tag{0.15}$$

which will fix the t_1 dependence of A_{01}.

To do that, let us first introduce the self-adjoint operator $N(T) = A_0^\dagger(T) A_0(T)$. Thanks to (12) and its creator version, $N(T)$ is only T_1 dependent: $N(T) = A_{01}^\dagger(T_1) A_{01}(T_1)$. Moreover as a consequence of (13), $A_{01}(T_1) N(T_1) = (N(T_1) + 1) A_{01}(T_1)$. Lastly, from (15) and its conjugate version, one observes that $D_1 N(T_1) = 0$. Thus N is also independent of t_1 and (15) can be now written in the tractable form :

$$D_1 A_{01} = -3i A_{01}(T_1) N(T_2),$$

which produces :

$$A_{01}(T_1) = A_{02}(T_2) exp(-3iN(T_2)t_1).$$

This allows us to write down the first order annihilation operator, which is basically the result of B&B , written differently :

$$A_0(T) = A_{02}(T_2) exp(-i(t_0 + 3N(T_2)t_1)). \tag{0.16}$$

At the same time, (13) becomes :

$$[A_{02}(T_2), A_{02}^\dagger(T_2)] = 1, \forall T_2. \tag{0.17}$$

One can now come back to the form of (14) exempted of secularity to obtain its general solution :

$$A_1(T) = A_{01}^3(T_1) exp(-3it_0)/2 - A_{01}^{\dagger 3}(T_1) exp(+3it_0)/4 - 3N(T_2) A_{01}^\dagger(T_1) exp(+it_0)/2$$

$$+C_1(T_1) exp(-it_0). \tag{0.18}$$

where the operator $C_1(T_1)$ is an integration "constant". The latter must be so adjusted, if possible, as to insure that the second equation (11) ,

$$[A_0(T), A_1^\dagger(T)] + [A_1(T), A_0^\dagger(T)] = 0, \tag{0.19}$$

be fulfilled at all times T. Here, it turns out that (19) is satisfied by taking simply $C_1(T_1) = 0$. One ends up with :

$$A_1(T) = A_0^3(T)/2 - A_0^{\dagger 3}(T)/4 - 3N(T_2) A_0^\dagger(T)/2 \tag{0.20}$$

and the first order step is complete.

Clearly, one can go iteratively through the higher order steps by similar (although rapidly tedious) calculations as long as the integration "constants" analogous to $C_1(T_1)$ can be properly adjusted and as long as the **working hypothesis** seems valid. It turns out that the second order can be successfully obtained, but we will not report it in this short note.

So far, the perturbative expression of the energy levels of the QAO (which was not our main goal) did not show up in full within our MST procedure. Yet, it can be found (without appealing to other perturbative methods) by inserting $a(t, g)$ as given by equations (12) and (20) in the Hamiltonian (2). Obviously, we are waiting for an expansion in powers of g polynomially dependent on A_0 and $A_0^\dagger$, up to the fist order in g :

$$H = H_0 + gH_1 + O(g^2),$$

The result is that the H_j's are function of $N = A_0^\dagger A_0$, not of A_0 and $A_0^\dagger$ separately :

$$H = 1/2 + N + 3g(1 + 2N + 2N^2)/4 + O(g^2), \tag{0.21}$$

in perfect agreement with the standard computations (e.g. Rayleigh-Schrodinger method) of the eigenvalues of the QAO in any textbook. The second order not written in this note shows the same feature. Let us stress : one gets an operator expansion and not the eigenvalue expansion.

3. THE QUANTUM CASE : QAO_2

In the previous section, in particular on eqs (12) and (20), one observes that the construction is made with two elementary bricks $A_0(T)$ and $A_0^\dagger(T)$, where $A_0(T)$ is the first term in the MST expansion of the annihilation operator. We have also pointed out the simple connection between the operator N and the Hamiltonian. More precisely, the previous perturbative results suggest the following features :

1) $N = A_0^\dagger(T)A_0(T)$ is independent of t_0, t_1, t_2..., i.e. of T.

2)The "nonhomogeneous" parts of $A_n(T)$ depend on T through the basic operators $A_0(T)$ and $A_0^\dagger(T)$. The same is true for the "homogeneous" parts $C_n(T_1)exp(-it_0)$ which, after determination of $C_n(T_1)$, can be recast in the form of functions of $A_0(T)$ and $A_0^\dagger(T)$ only.

3) The operators H and N commute.

If these features are really true at all orders, we could, upside down, require two properties, forgetting any multiple scale reminiscences:

1) The Hamiltonian H must be (perturbatively) diagonal together with $N = a^\dagger a$, which means the Hamiltonian is cast in its Birkhoff "normal" form[3]

2) The canonical commutation relations are (perturbatively) enforced : $[a, a^\dagger] = 1$.

It appears that one gets, as far as we have computed with the mustiple scale techniques (order two) and with this "new" algebraic way, exactly the same results for the expansions of the annihilation/creation operators and of the Hamiltonian.

Therefore the obvious question is the following: are these two methods equivalent? It turns out that the answer is yes. We skipp all the proof in this short contribution to ICHEMP05, just mentioning that it involves a Von Neumann theorem on the unitary transformations and the canonical commutation relations. You could find some similarities with the well known Folfy-Wouthuysen transformation on the non relativistic limit of the Dirac equation for spin half particle.

It is now natural to look for other naive questions. First of all, what about the **working hypothesis**? The answer is simple: since the pure algebraic method do not contain any assumption, one must consider that the **working hypothesis** will be true at any perturbative order of the MST, no obstruction at all will arise in the procedure.

The second question is the technical ability to obtain higher order, in using the algebraic way. We do not conceal to the reader that we used the computer to go beyond the second order. Actually, even with the help of symbolic software, it is not easy to "play" with operators. In the appendix, the results are displayed until the sixth order.

The last question concerns the classical case. Is it possible to find an algebraic (or algorithmic) method for the CAO, inspired by our QAO knowledge?

4. BACK TO CAO

Essentially, the algorithmic way to get quantum results involves two requirements which lead in the classical case to demand both:

i) the classical Hamiltonian is in a normal form (here an expansion in powers of a free classical harmonic Hamiltonian),

ii) the Poisson brackets are mandatory at each order.

Of course the proof of the equivalence between the derivative expansion method and the algoritmic method is slightly different in this classical context, and involves tools of differential and symplectic geometries.

Anyway, discarding mathematical subtleties and using this iterative method, we have obtained the Hamiltonian until the order 10 in power of g:

$$H = H_0(1 + \frac{3gH_0}{2} - \frac{17g^2H_0^2}{4} + \frac{375g^3H_0^3}{16} - \frac{10689g^4H_0^4}{64} + \frac{87549g^5H_0^5}{64} - \frac{3132399g^6H_0^6}{256} + \frac{238225977g^7H_0^7}{2048} - \frac{18945961925g^8H_0^8}{16384} + \frac{194904116847g^9H_0^9}{16384} - \frac{8240234242929g^{10}H_0^{10}}{65536}) + O(g^{11}),$$

where H_0 is a free "normalized" Hamiltonian $(p^2+q^2)/2$. We have taken into account the boundary conditions as described before in the quantum case, that indeed explains the apparent discrepancy with (1)

At this point, one can check the correspondence principle between quantum and classical worlds: when the eigenvalue of N is large (so if the integer n such as $N|n >= n|n >$ is large), the quantum case must recover the classical case; this is clear from the formula 21, for the two lowest orders in g. Owing to the appendix, this is also clear until the sixth order.

5. CONCLUSION

The arguments presented there are quite general, not specific of the QAO. If one considers an Hamiltonian which is the sum of an harmonic oscillator one and a "potential" represented by a self-adjoint operator function of the position and the momentum, such an analysis can be repeated. ¿From a technical point of view, we do not hide the difficulties inherent to the non commutative operators, therefore limiting the method to polynomial functions, probably the price to pay in using the Heisenberg picture.

Moreover, the equivalence between MST and unitary transformation diagonalizing the Hamiltonian is likely to be a rather general feature. In particular, the previous discussion can be extended in a rather straightforward way to systems with more than one degree of freedom. So we believe that this work strengthens the status of the Multiple Scale Techniques in quantum mechanics.

6.REFERENCES

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7. APPENDIX

We give below, up to order 6:
i) the coefficients a_n of the expansion of the annihilator $a(g)$ in terms of $N = a_0^\dagger a_0$,
ii) the coefficients $E_k(n)$ of the expansion of the energy levels:

$$E_n(g) = \frac{1}{2} + n + \sum_{n=1}^{\infty} g^k E_k(n).$$

Both have been computed by applying the algorithm described in section 3.

i) To be simpler, a and $a^\dagger$ stand for a_0 and $a_0^\dagger$.

$a_1 = (2a^3 - a^{\dagger 3} - 6Na^\dagger)/4$;

$a_2 = (-9a + 120a^3 + 6a^5 - 120a^3N + 27aN^2 + 84a^\dagger - 42a^{\dagger 3} - 4a^{\dagger 5} + 42Na^{\dagger 3} + 276N^2a^\dagger)/32$;

$$\begin{aligned} a_3 = (&6092a^3 + 756a^5 + 8a^7 - 8844a^3N + 4422a^3N^2 - 378a^5N + 1278aN - 1062aN^3 \\ &-1708a^{\dagger 3} - 464a^{\dagger 5} - 6a^{\dagger 7} - 9282Na^\dagger + 2406Na^{\dagger 3} + 232Na^{\dagger 5} - 1203N^2a^{\dagger 3} \\ &-9042N^3a^\dagger)/128 \ ; \end{aligned}$$

$$\begin{aligned} a_4 = (&-200645a + 1546416a^3 + 380868a^5 + 9264a^7 + 40a^9 - 2975280a^3N + 2143296a^3N^2 \\ &-714432a^3N^3 - 322896a^5N + 80724a^5N^2 - 3088a^7N - 500298aN^2 + 162755aN^4 \\ &+506760a^\dagger - 358200a^{\dagger 3} - 221832a^{\dagger 5} - 6696a^{\dagger 7} - 32a^{\dagger 9} + 673392Na^{\dagger 3} + 186464Na^{\dagger 5} \\ &+2232Na^{\dagger 7} + 3040992N^2a^\dagger - 472788N^2a^{\dagger 3} - 46616N^2a^{\dagger 5} + 157596N^3a^{\dagger 3} + 1365240N^4a^\dagger)/2048 \ ; \end{aligned}$$

$$\begin{aligned} a_5 = (&116798776a^3 + 51228696a^5 + 2189520a^7 + 20832a^9 + 48a^{11} - 266946576a^3N \\ &+255315936a^3N^2 - 121842648a^3N^3 + 30460662a^3N^4 - 58614828a^5N + 24750360a^5N^2 \\ &-4125060a^5N^3 - 1300128a^7N + 216688a^7N^2 - 5208a^9N + 52602092aN + 41073824aN^3 \\ &-6417388aN^5 - 21539684a^{\dagger 3} - 28787584a^{\dagger 5} - 1542096a^{\dagger 7} - 16320a^{\dagger 9} \\ &-40a^{\dagger 11} - 121625250Na^\dagger + 50282976Na^{\dagger 3} + 32551232Na^{\dagger 5} + 912600Na^{\dagger 7} \\ &+4080Na^{\dagger 9} - 47389884N^2a^{\dagger 3} - 13618080N^2a^{\dagger 5} - 152100N^2a^{\dagger 7} - 219914676N^3a^\dagger \\ &+22248396N^3a^{\dagger 3} + 2269680N^3a^{\dagger 5} - 5562099N^4a^{\dagger 3} - 55675938N^5a^\dagger)/8192 \ ; \end{aligned}$$

$$\begin{aligned} a_6 = (&-2649077789a + 19854323040a^3 + 14799326898a^5 + 1000498176a^7 + 15874840a^9 \\ &+78720a^{11} + 112a^{13} - 15744a^{11}N - 52410470592a^3N + 59605775856a^3N^2 - 37821182832a^3N^3 \\ &+13464443160a^3N^4 - 2692888632a^3N^5 - 20808622800a^5N + 11852140500a^5N^2 \\ &-3324992400a^5N^3 + 415624050a^5N^4 - 817924896a^7N + 242212752a^7N^2 - 26912528a^7N^3 \\ &-7253184a^9N + 906648a^9N^2 - 16271788323aN^2 - 6097875991aN^4 + 521267535aN^6 \\ &+4255953324a^\dagger - 2581523304a^{\dagger 3} - 8101045372a^{\dagger 5} - 691648560a^{\dagger 7} - 12242240a^{\dagger 9} \\ &-64720a^{\dagger 11} - 96a^{\dagger 13} + 12944Na^{\dagger 11} + 7571823000Na^{\dagger 3} + 11245609120Na^{\dagger 5} \\ &+562441728Na^{\dagger 7} + 5584128Na^{\dagger 9} + 35458238196N^2a^\dagger - 9129805056N^2a^{\dagger 3} \\ &-6326409960N^2a^{\dagger 5} - 165946104N^2a^{\dagger 7} - 698016N^2a^{\dagger 9} + 5783860872N^3a^{\dagger 3} \\ &+1757503840N^3a^{\dagger 5} + 18438456N^3a^{\dagger 7} + 29695249188N^4a^\dagger - 2055444390N^4a^{\dagger 3} \\ &-219687980N^4a^{\dagger 5} + 411088878N^5a^{\dagger 3} + 4768483548N^6a^\dagger)/65536. \end{aligned}$$

ii)

$E_1(n) = 3(1+2n+2n^2)/4$;

$E_2(n) = -(1+2n)(21+17n+17n^2)/8$;

$E_3(n) = 3(111+347n+472n^2+250n^3+125n^4)/16$;

$E_4(n) = -(1+2n)(30885+49927n+60616n^2+21378n^3+10689n^4)/128$;

$E_5(n) = 3(305577+1189893n+2060462n^2+1857870n^3+1220765n^4+350196n^5$
$+116732n^6)/256$;

$E_6(n) = -(1+2n)(65518401+146338895n+213172430n^2+139931868n^3+85627929n^4$
$+18794394n^5+6264798n^6)/1024.$

The main remark consists in the correspondence between classical and quantum cases, as aforesaid. Further, several number theoretic properties of the $E_k(n)$'s are worth pointing out. First, all the coefficients c_{kp} of n^p in $E_k(n)$ are rational and positive, and the signs of the $E_k(n)$'s alternate, as it should be. Perhaps new are the following observations: whereas the denominator in the expression of $E_k(n)$ is a power of 2, the numerator is always a multiple of 3 (for integer n). This peculiarity was already noticed by Bender and Wu[4] for the ground state ($n=0$). It thus turns out to hold for the excited levels too. Also, the sum of the numerators of the coefficients c_{kp} in each $E_k(n)$ is a multiple of 5. Finally, if one expresses the $E_k(n)$'s in terms of the variable $m = n+\frac{1}{2}$, one observes that they are even polynomials with positive coefficients (multiplied by $-m$ if k is even). More than that, all the zeroes of these polynomials are pure imaginary. This means that all the zeroes of $E_k(n)$ lie on the line $n = -\frac{1}{2}+iy$!

Now, if an acute reader would like to make a conjecture " la Riemann", the author would be not responsible for it ...