



Extension of the Poincaré Symmetry for p -Forms

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Abstract

A non-trivial extension of the D -dimensional Poincaré algebra is investigated. It is shown that this extension acts in a natural geometric way on antisymmetric gauge fields or p -forms. Some field theoretical aspects of this symmetry are studied and invariant Lagrangians are explicitly given.

I.

The usual electromagnetic gauge field admits a generalisation in terms of antisymmetric p th order tensors or p -forms¹. In this paper we investigate a new possible symmetry among p -forms. This symmetry turns out to be a non-trivial extension of the Poincaré algebra in D -dimensional space-time, an extension different from supersymmetry. By definition this symmetry is not in contradiction with the no-go theorems² because its underlying algebraic structure is neither a Lie algebra nor a Lie superalgebra, but an F -Lie algebra. The F -Lie algebras were introduced in^{3,4}. They admit a \mathbb{Z}_F -gradation, the zero-graded part being a Lie algebra and the non-zero graded parts being appropriate representations of the zero-graded part. An F -fold symmetric product (playing the role of the anticommutator in the case $F = 2$) expresses the zero graded part in terms of the non-zero graded part. A general study of F -Lie algebras was undertaken in⁴. This study leads to possible non-trivial extensions of the Poincaré algebra. In these extensions, the zero graded part reduces to the D -dimensional Poincaré algebra. Additional generators, in the vector representation of the Poincaré algebra, denoted V , such that $V^F \sim P$, with P the space-time translation generators, are added (see also^{5,6}). In previous papers⁷ a field theoretical realisation of the simplest extension (with $F = 3$) has been realised in four space-time dimensions. In this paper, we will extend to an arbitrary number of spacetime dimensions some of the results obtained in⁷ (some results can also be found in⁸) and show that this new symmetry, named *cubic symmetry* acts naturally on antisymmetric gauge fields or p -forms.

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II.

The cubic symmetry algebra is constructed from the Poincaré generators (L_{MN} and P_M , $M, N = 0, \dots, D - 1$) together with some additional generators V_M in the vector representation of the Poincaré algebra,

$$\begin{aligned}
 [L_{MN}, L_{PQ}] &= \eta_{NP}L_{PM} - \eta_{MP}L_{PN} + \eta_{NP}L_{MQ} - \eta_{MP}L_{NQ}, \\
 [L_{MN}, P_P] &= \eta_{NP}P_M - \eta_{MP}P_N, \\
 [L_{MN}, V_P] &= \eta_{NP}V_M - \eta_{MP}V_N, \quad [P_M, V_N] = 0, \\
 \{V_M, V_N, V_R\} &= \eta_{MN}P_R + \eta_{MR}P_N + \eta_{RN}P_M,
 \end{aligned}
 \tag{2.1}$$

where $\{V_M, V_N, V_P\} = V_MV_NV_R + V_MV_RV_N + V_NV_MV_R + V_NV_RV_M + V_RV_MV_N + V_RV_NV_M$ stands for the symmetric product of order 3 and $\eta_{MN} = \text{diag}(1, -1, \dots, -1)$ is the D -dimensional Minkowski metric. From now on we consider the case where $D = 4n$. Matrix representations of our algebra are given by

$$V_{+M} = \begin{pmatrix} 0 & \Lambda^{1/3}\Sigma_M & 0 \\ 0 & 0 & \Lambda^{1/3}\tilde{\Sigma}_M \\ \Lambda^{-2/3}P_M & 0 & 0 \end{pmatrix}, \quad V_{-M} = \begin{pmatrix} 0 & \Lambda^{1/3}\tilde{\Sigma}_M & 0 \\ 0 & 0 & \Lambda^{1/3}\Sigma_M \\ \Lambda^{-2/3}P_M & 0 & 0 \end{pmatrix}, \tag{2.2}$$

with $\Gamma_M = \begin{pmatrix} 0 & \Sigma_M \\ \tilde{\Sigma}_M & 0 \end{pmatrix}$ the D -dimensional Γ -matrices ($\{\Gamma_M, \Gamma_N\} = 2\eta_{MN}$), $P_M = \partial_M$ and Λ a parameter with mass dimension (that we take equal to 1).

Before exhibiting multiplets associated to (2.2) let us recall two points⁸. Firstly, since this extension is based on some new generators lying in the vector representation of the Poincaré algebra, its representations contain states of a definite statistics, *i.e.* the multiplets are purely bosonic or purely fermionic. Secondly, since P^2 is a Casimir operator, the representations are degenerate in mass. To build a representation of the algebra (2.1) using the matrix representations (2.2) we have also to specify the representation of the vacuum. If the vacuum is in the trivial representation of the Lorentz algebra, the multiplet consists of three spinors of definite chirality (we have two possibilities corresponding to the two choices for the matrices V_{\pm})

$$\Psi_+ = \begin{pmatrix} \Psi_{1+} \\ \tilde{\Psi}_{2-} \\ \Psi_{3+} \end{pmatrix}, \quad \Psi_- = \begin{pmatrix} \tilde{\Psi}_{1-} \\ \Psi_{2+} \\ \tilde{\Psi}_{3-} \end{pmatrix}. \tag{2.3}$$

They transform as $\delta_\varepsilon \Psi_{\pm} = \varepsilon^M V_{\pm M} \Psi_{\pm}$, with ε a commuting Lorentz vector that we take real (we stress that ε is not an anticommuting spinor unlike in supersymmetry), Ψ_+ a left-handed spinor and $\tilde{\Psi}_-$ a right-handed spinor.

Invariant Lagrangian involving the spinor multiplets were given in the first paper of⁷. We concentrate here on the case where the vacua lie in the spinor representations of the Lorentz algebra. In this case we take two copies $\Psi_{\pm}, \Lambda_{\pm}$ (see (2.3)) transforming with V_{\pm} , and two copies of the vacuum in the spinor representation $\Omega_{\pm}, \omega_{\pm}$. From the decomposition of the product of spinors on the set of p -forms

$$\begin{aligned}
 \mathcal{S}_+ \otimes \mathcal{S}_+ &= [0] \oplus [2] \oplus \dots [2n]_+ \\
 \mathcal{S}_- \otimes \mathcal{S}_- &= [0] \oplus [2] \oplus \dots [2n]_- \\
 \mathcal{S}_+ \otimes \mathcal{S}_- &= [1] \oplus [3] \oplus \dots [2n - 1]
 \end{aligned}
 \tag{2.4}$$

with \mathcal{S}_{\pm} a left/right handed spinor, $[p]$ a p -form and $[2n]_{\pm}$ an (anti-)self-dual $2n$ -form, one gets the four multiplets

$$\begin{aligned} \Xi_{++} &= \Psi_+ \otimes \Omega_+ = \begin{pmatrix} \Xi_{1++} \\ \Xi_{2-+} \\ \Xi_{3++} \end{pmatrix} = \begin{pmatrix} A_{[0]} \oplus A_{[2]} \oplus \dots \oplus A_{[2n]_+} \\ \tilde{A}_{[1]} \oplus \tilde{A}_{[3]} \oplus \dots \oplus \tilde{A}_{[2n-1]} \\ \tilde{A}_{[0]} \oplus \tilde{A}_{[2]} \oplus \dots \oplus \tilde{A}_{[2n]_+} \end{pmatrix} \\ \Xi_{-+} &= \Lambda_- \otimes \omega_+ = \begin{pmatrix} \xi_{1-+} \\ \xi_{2++} \\ \xi_{3-+} \end{pmatrix} = \begin{pmatrix} A_{[1]} \oplus A_{[3]} \oplus \dots \oplus A_{[2n-1]} \\ \tilde{A}_{[0]} \oplus \tilde{A}_{[2]} \oplus \dots \oplus \tilde{A}_{[2n]_+} \\ \tilde{A}_{[1]} \oplus \tilde{A}_{[3]} \oplus \dots \oplus \tilde{A}_{[2n-1]} \end{pmatrix} \\ \Xi_{--} &= \Psi_- \otimes \Omega_- = \begin{pmatrix} \bar{\Xi}_{1--} \\ \bar{\Xi}_{2+-} \\ \bar{\Xi}_{3--} \end{pmatrix} = \begin{pmatrix} A'_{[0]} \oplus A'_{[2]} \oplus \dots \oplus A'_{[2n]_-} \\ \tilde{A}'_{[1]} \oplus \tilde{A}'_{[3]} \oplus \dots \oplus \tilde{A}'_{[2n-1]} \\ \tilde{A}'_{[0]} \oplus \tilde{A}'_{[2]} \oplus \dots \oplus \tilde{A}'_{[2n]_-} \end{pmatrix} \\ \Xi_{+-} &= \Lambda_+ \otimes \omega_- = \begin{pmatrix} \bar{\xi}_{1+-} \\ \bar{\xi}_{2--} \\ \bar{\xi}_{3+-} \end{pmatrix} = \begin{pmatrix} A'_{[1]} \oplus A'_{[3]} \oplus \dots \oplus A'_{[2n-1]} \\ \tilde{A}'_{[0]} \oplus \tilde{A}'_{[2]} \oplus \dots \oplus \tilde{A}'_{[2n]_-} \\ \tilde{A}'_{[1]} \oplus \tilde{A}'_{[3]} \oplus \dots \oplus \tilde{A}'_{[2n-1]} \end{pmatrix}. \end{aligned}$$

Thus in each Ξ multiplets we have various set of p -forms, with $0 \leq p \leq 2n$. Due to the property of (anti-)self-duality of $2n$ -forms in $1 + (4n - 1)$ -dimensions, $*A_{[2n]_+} = iA_{[2n]_+}$, $*A'_{[2n]_-} = -iA'_{[2n]_-}$, etc (with $*A_{[2n]_+}$ the Hodge dual of $A_{[2n]_+}$), the $2n$ -forms are complex representations of $SO(1, D - 1)$ and consequently also the other p -forms (see Eq.[2.6] below). The multiplets in (2.5) can be taken complex conjugate of each other ($\Xi_{++}^* = \Xi_{--}$ and $\Xi_{+-}^* = \Xi_{-+}$), that is

$$\begin{aligned} A_{[2p]}^* &= A'_{[2p]}, & A_{[2n]_+}^* &= A'_{[2n]_-} \\ \tilde{A}_{[2p]}^* &= \tilde{A}_{[2p]}, & \tilde{A}_{[2n]_+}^* &= \tilde{A}_{[2n]_-} \\ \tilde{A}_{[2p+1]}^* &= A'_{[2p+1]} \end{aligned} \tag{2.5}$$

(the complex conjugate A^* of A should not to be confused with its dual $*A$) and similarly for the fields coming from Ξ_{-+} and Ξ_{+-} . The underlying algebra (2.1) and its representations (2.2) have a \mathbb{Z}_μ -graded structure. Hence, one can assume that the fields with no tilde symbol are in the (-1) -graded sector, the fields with one tilde are in the 0 -graded sector and the fields with two tilde symbols are in the 1 -graded sector. We now calculate the transformation laws of these various p -forms. For example, for Ξ_{++} , from the transformation $\delta_\varepsilon \Xi_{++} = (\varepsilon^M V_M \Psi_+) \otimes \Omega_+$, using the trace property of the Γ matrices together with the correspondence between product of spinors and p -forms (2.4), (see also⁸ for more details) we get

$$\begin{aligned} \delta_\varepsilon A_{[0]} &= i_\varepsilon \tilde{A}_{[1]} & \delta_\varepsilon \tilde{A}_{[1]} &= i_\varepsilon \tilde{A}_{[2]} + \tilde{A}_{[0]} \wedge \varepsilon \\ &\vdots & &\vdots \\ \delta_\varepsilon A_{[2p]} &= i_\varepsilon \tilde{A}_{[2p+1]} + \tilde{A}_{[2p-1]} \wedge \varepsilon & \delta_\varepsilon \tilde{A}_{[2p+1]} &= i_\varepsilon \tilde{A}_{[2p+2]} + \tilde{A}_{[2p]} \wedge \varepsilon \\ &\vdots & &\vdots \\ \delta_\varepsilon A_{[2n]_+} &= \tilde{A}_{[2n-1]} \wedge \varepsilon - i^* (\tilde{A}_{[2n-1]} \wedge \varepsilon) & \delta_\varepsilon \tilde{A}_{[2n-1]} &= i_\varepsilon \tilde{A}_{[2n]_+} + \tilde{A}_{[2n-2]} \wedge \varepsilon \end{aligned} \tag{2.6}$$

$$\delta_\varepsilon \tilde{A}_{[0]} = \varepsilon A_{[0]}, \quad \dots \quad \delta_\varepsilon \tilde{A}_{[2n]_+} = \varepsilon A_{[2n]_+}.$$

In these relations $i_\varepsilon A$ represents the inner product of ε and A^1 , $A \wedge \varepsilon$ represents the exterior product of A and ε , εA represents the action of the vector field $\varepsilon = \varepsilon^M \partial_M$ on A and $*A$ represents the dual of A^2 . The

¹In this paper, we take $(i_\varepsilon A_{[p]})_{M_1 \dots M_{p-1}} = A_{[p]M_1 \dots M_p} \varepsilon^{M_p}$.

²We remark that the transformation laws (2.6) have a geometrical interpretation in terms of inner and exterior product.

term $-i \star (\tilde{A}_{[2n-1]} \wedge \varepsilon)$ in $\delta_\varepsilon A_{[2n]+}$ preserves the self-dual character of $A_{[2n]+}$. Similar transformation laws can be obtained for the other multiplets. In the case of multiplets involving anti-self-dual $2n$ -forms, the $-i$ becomes a $+i$, in perfect agreement with the complex conjugation (2.5).

The transformations (2.6) suggest that an invariant Lagrangian should contain only zero-graded terms. Thus, one is forced to couple the fields in the (-1) -graded sector to the fields in the 1 -graded sector and the fields in the 0 -graded sector to themselves. Furthermore, if we consider for example the Ξ_{++} multiplet, in order to have a real Lagrangian, one has also to take into consideration the conjugate multiplet Ξ_{--} (see (2.5)). To construct a Lagrangian invariant under (2.6), we write $\mathcal{L} = \mathcal{L}(\Xi_{++}) + \mathcal{L}(\Xi_{--})$ with $\mathcal{L}(\Xi_{++}) = \mathcal{L}_{[0]} + \dots + \mathcal{L}_{[2n]}$ constructed from the Ξ_{++} multiplet and $\mathcal{L}_{[p]}$ involving only p -forms, and similar notations for $\mathcal{L}(\Xi_{--})$. Then we notice that $\delta_\varepsilon \mathcal{L}(\Xi_{++})$ and $\delta_\varepsilon \mathcal{L}(\Xi_{--})$ do not mix. It is then enough to check separately their invariance, hence we do it for $\mathcal{L}(\Xi_{++})$. Starting from a specific normalisation for $\mathcal{L}_{[0]}$, its variation fixes the normalisation for $\mathcal{L}_{[1]}$. By a step-by-step process, the normalisations for $\mathcal{L}_{[p]}, 0 \leq p \leq 2n$ are also fixed. At the very end, all the terms of $\delta_\varepsilon \mathcal{L}$ compensate each others, up to a total derivative and the Lagrangian involving the Ξ_{++} and Ξ_{--} multiplets writes

$$\begin{aligned} \mathcal{L} &= \mathcal{L}(\Xi_{++}) + \mathcal{L}(\Xi_{--}) = \mathcal{L}_{[0]} + \dots + \mathcal{L}_{[2n]} + \mathcal{L}'_{[0]} + \dots + \mathcal{L}'_{[2n]} \\ &= dA_{[0]} d\tilde{A}_{[0]} + \dots + \\ &\quad - \frac{1}{2} \frac{1}{(2p+2)!} d\tilde{A}_{[2p+1]} d\tilde{A}_{[2p+1]} - \frac{1}{2} \frac{1}{(2p)!} d^\dagger \tilde{A}_{[2p+1]} d^\dagger \tilde{A}_{[2p+1]} \\ &\quad + \frac{1}{(2p+3)!} dA_{[2p+2]} d\tilde{A}_{[2p+3]} + \frac{1}{(2p+1)!} d^\dagger A_{[2p+2]} d^\dagger \tilde{A}_{[2p+2]} \\ &\quad + \dots + \\ &\quad + \frac{1}{2} \frac{1}{(2n+1)!} dA_{[2n]+} d\tilde{A}_{[2n]+} + \frac{1}{2} \frac{1}{(2n-1)!} d^\dagger A_{[2n]+} d^\dagger \tilde{A}_{[2n]+} \\ &\quad + dA'_{[0]} d\tilde{A}'_{[0]} + \dots + \\ &\quad - \frac{1}{2} \frac{1}{(2p+2)!} d\tilde{A}'_{[2p+1]} d\tilde{A}'_{[2p+1]} - \frac{1}{2} \frac{1}{(2p)!} d^\dagger \tilde{A}'_{[2p+1]} d^\dagger \tilde{A}'_{[2p+1]} \\ &\quad + \frac{1}{(2p+3)!} dA'_{[2p+2]} d\tilde{A}'_{[2p+3]} + \frac{1}{(2p+1)!} d^\dagger A'_{[2p+2]} d^\dagger \tilde{A}'_{[2p+2]} \\ &\quad + \dots + \\ &\quad + \frac{1}{2} \frac{1}{(2n+1)!} dA'_{[2n]-} d\tilde{A}'_{[2n]-} + \frac{1}{2} \frac{1}{(2n-1)!} d^\dagger A'_{[2n]-} d^\dagger \tilde{A}'_{[2n]-}. \end{aligned} \tag{2.7}$$

Here $\omega_{[p]} \omega'_{[p]}$ stands for $\omega_{[p]M_1 \dots M_p} \omega'_{[p]M_1 \dots M_p}$, where $\omega_{[p]}$ and $\omega'_{[p]}$ are two p -forms³. In the Lagrangian (2.7), $dA_{[p]}$ is the exterior derivative of $A_{[p]}$ ($(dA_{[p]})^2$ is the usual kinetic term) and $d^\dagger A_{[p]}$ its adjoint ($(d^\dagger A_{[p]})^2$ is a gauge fixing term see⁸) $d^\dagger A_{[p]} = (-1)^{pD+D} d^\star A_{[p]} = \star d^\star A_{[p]}$ for even D . Due to the presence of the $(d^\dagger A_{[p]})^2$ terms imposed by cubic symmetry, it is important to emphasize that one symmetry (cubic symmetry) partially fixes another (gauge symmetry). Consequence of this gauge fixing is partially analysed in^{7,8}.

After diagonalising the Lagrangian and introducing real p -forms (remember that $A_{[p]'} = A_{[p]}^\star$ etc.), it turns out that some of the fields have a wrong sign for their kinetic term (see⁸). In Ref⁸ a possible way to avoid this problem was given, based on Hodge duality and on the simple constatation

$$\begin{aligned} &(-1)^p \left(\frac{1}{(p+1)!} d\omega_{[p]} d\omega_{[p]} + \frac{1}{(p-1)!} d^\dagger \omega_{[p]} d^\dagger \omega_{[p]} \right) = \\ &-(-1)^p \left(\frac{1}{(D-p-1)!} d^\dagger \rho_{[D-p]} d^\dagger \rho_{[D-p]} + \frac{1}{(D-p+1)!} d\rho_{[D-p]} d\rho_{[D-p]} \right) \end{aligned} \tag{2.8}$$

³In p -form notation we can rewrite $\int d^D x \omega_{[p]} \omega'_{[p]}$ as $\int \omega_{[p]} \wedge \star \omega'_{[p]}$.

with $\rho = *\omega$ the Hodge dual of ω . This substitution is due to the special form of our Lagrangian: through the substitution $\omega \rightarrow \rho$, the kinetic term of ω becomes the gauge fixing term of ρ and *vice versa*. Performing such a substitution for the fields with a wrong sign for their kinetic term, at the very end, we have one one-form, one three-form, . . . , one $(D - 1)$ -form in the zero-graded sector and two zero-forms, two two-forms, . . . and two D -form in the mixture of the sectors of gradation (-1) and 1 .

The final Lagrangian obtained after diagonalisation and the substitutions (2.8) can be found in⁸. Some analysis of the interplay of this new symmetry with the gauge invariance can also be found in⁷ and⁸ (cubic symmetry is compatible with gauge invariance if the gauge is partially fixed). Finally let us conclude that we also have proved in⁷ that cubic symmetry forbid self interacting terms for the multiplets involving p -forms in $D = 4$. Since p -forms couple naturally to extended objects one may wonder whether or not cubic symmetry could play a role in physics involving extended objects or branes.

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