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Carnot-Carathéodory Metric vs Gauge Fluctuation in Noncommutative Geometry

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Abstract

In this talk we give an overview on the metric aspect of noncommutative geometry, especially the metric interpretation of gauge fields via the process of *fluctuation of the metric*. Connes' distance formula associates to a gauge field on a bundle P equipped with a connection H a metric that coincides with the horizontal metric as soon as the holonomy is trivial. Otherwise we exhibit an elementary example on a 2-torus in which the noncommutative metric d is somehow more interesting than the horizontal one since d preserves the S^1-structure of the fiber and also guarantees the smoothness of the length function at the cut-locus. In this sense the fiber appears as an object "smoother than a circle". As a consequence, from a intrinsic metric point of view any observer on the fiber can equally pretend to be "the center of the world".

I. INTRODUCTION

In Noncommutative geometry[2] the metric information is encoded within the Dirac operator. Specifically via Connes' distance formula (2.1) one is able to recover from purely algebraic datas the geodesic distance on a riemannian compact smooth spin manifold M,

$$-i\gamma^\mu\partial_\mu \Longleftrightarrow \text{Riemannian geodesic distance.}$$

Physics not only deal with spin manifold but also with gauge theories, that is to say bundles P equipped with connections H (and associated 1-form A_μ). Therefore it is natural to wonder what distance d is encoded within the covariant Dirac operator,

$$-i\gamma^\mu(\partial_\mu + A_\mu) \Longleftrightarrow ?$$

Note that gauge fields already have a well known metric interpretation in terms of horizontal distance d_H. The latest, also called *Carnot-Carathéodory* or subriemannian distance[11], is by definition the length of the shortest path whose tangent vector is everywhere horizontal with respect to the connection H [1](see figure 1.1),

$$d_H(p,q) = \operatorname*{Inf}_{\dot{c}(t)\in H_{c(t)}P} \int_0^1 \|\dot{c}(t)\|\, dt \qquad (1.1)$$

[1]Let us recall that a connection H is the splitting of the tangent space TP in an horizontal subspace (the kernel of the connection 1-form) and a vertical subspace, $T_pP = V_pP \oplus H_pP$.

with $c(0) = p$, $c(1) = q$ a smooth curve in P.

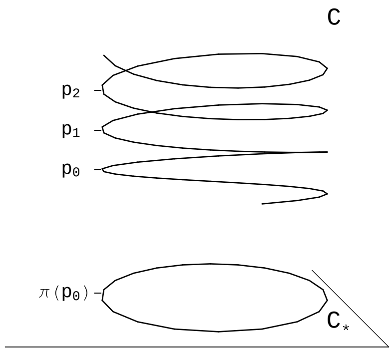

FIG. 1. With shortest horizontal path an helix of radius 1, $d_H(p_0, p_2) = 4\pi$.

In [3] Connes has pointed out the link between d_H and d. In [9] we have examined this link in details, showing the importance of the holonomy of the connection on that matter. In this talk we give a brief and comprehensive account of these results.

II. THE DISTANCE FORMULA IN NONCOMMUTATIVE GEOMETRY

Noncommutative geometry aims at understanding the geometry of a space whose algebra of functions (defined on it) is noncommutative. Such objects are well described in terms of spectral triples $(\mathcal{A}, \mathcal{H}, D)$ where $\mathcal{A}$ is an associative $*$-algebra (commutative or not) represented by π on a Hilbert space $\mathcal{H}$ and D is an operator on $\mathcal{H}$. Together with a chirality Γ and a real structure J also acting on $\mathcal{H}$, these three elements satisfies a set of 7 conditions[3] providing the necessary and sufficient conditions for
1) an axiomatic definition of riemannian spin geometry in terms of commutative algebra,
2) its natural extension to the noncommutative framework.

Explicitly given a spin manifold M one checks that

$$(C^\infty(M), L_2(M, S), -i\gamma^\mu\partial_\mu)$$

is a spectral triple, with $L_2(M, S)$ the space of square integrable spinors on M. Conversely starting from a spectral triple $(\mathcal{A}, \mathcal{H}, D)$ with $\mathcal{A}$ the algebra of smooth functions over a compact[2] riemannian manifold N, one obtains that N is indeed a spin manifold with corresponding Dirac operator D (modulo a torsion term). Moreover the geodesic distance corresponding to the riemannian structure of N is given by[3]

$$d_{geo}(x, y) = \sup_{f \in C^\infty(N)} \{|f(x) - f(y)| \,/\, \|[D, f]\| \leq 1\}. \tag{2.1}$$

Extension to the noncommutative framework is obtained by dropping the commutativity of $\mathcal{A}$. "Points" are recovered as pure states[4] $\mathcal{P}(\mathcal{A})$ of $\mathcal{A}$, in analogy with the commutative case where, Gelfand duality, $\mathcal{P}(C^\infty(M)) \simeq M$. Formula (2.1) rewritten as[5]

$$d(\omega_1, \omega_2) = \sup_{a \in \mathcal{A}} \{|\omega_1(a) - \omega_2(a)| \,/\, \|[D, a]\| \leq 1\} \tag{2.2}$$

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[2] The noncompact case has been studied in [6].
[3] To get familiar with this formula, one can study the example $N = \mathbb{R}$, $D = \frac{d}{dx}$, $\mathcal{H} = L_2(\mathbb{R})$. Then $\|[D, f]\| = \|f'\| = \sup_{z \in \mathbb{R}}\{|f'(z)|\}$ so that (2.1) $\leq |x - y| = d_{geo}(x, y)$. This upper bound is reached by the function $z \mapsto z$.
[4] State: linear positive application τ from $\mathcal{A}$ to $\mathbb{C}$. Pure state ω: state that does not decompose as a convex combination of other states, $\omega \neq \lambda\tau_1 + (1 - \lambda)\tau_2$.

defines a distance d between states which
- makes sense whether $\mathcal{A}$ is commutative or not,
- is coherent with the classical case, $d = d_{geo}$, when $\mathcal{A} = C^\infty(M)$,
- does not involve some notions ill-defined in a quantum context such as paths between points.

In [7] we computed d in a n-point space ($\mathcal{A} = \mathbb{C}^n$) as well as for other finite dimensional algebras, like $\mathcal{A} = M_2(\mathbb{C})$ which yields a metric on the two sphere[5]. Finite dimensional spectral triples are particularly interesting in products of geometries. Namely given a spin manifold M and a spectral triple $(\mathcal{A}_I, \mathcal{H}_I, D_I)$, one defines

$$\mathcal{A} = C^\infty(M) \otimes \mathcal{A}_I, \mathcal{H} = L_2(M, S) \otimes \mathcal{H}_I, D = -i\gamma^\mu\partial_\mu \otimes \mathbb{I}_\mathbb{I} + \gamma^{\not{5}} \otimes \mathbb{D}_\mathbb{I} \tag{2.3}$$

which again is a spectral triple. $\mathcal{P}(\mathcal{A})$ is the set of couples $(\omega_x, \omega \in \mathcal{P}(\mathcal{A}_I))$ so that for finite dimensional $\mathcal{A}_I$ (2.3) describes a geometry which is product of the discrete by the continuum. For instance $\mathcal{A}_I = \mathbb{C}^2$ yields a two sheet model, two copies of M indexed by the pure states of $\mathbb{C}^2$ and which are at finite distance from one another, although there is no "path" between them.

III. FLUCTUATION OF THE METRIC

Inspired by the commutative case, $Diff(M) = Aut(C^\infty(M))$, one describes the symmetries of a noncommutative geometry in terms of automorphisms of $\mathcal{A}$. The group $Aut(\mathcal{A})$ naturally splits into inner automorphisms,

$$In(\mathcal{A}) : \alpha_u(a) = uau^*,$$

given by unitary elements ($uu^* = \mathbb{I}$) of $\mathcal{A}$ and outer automorphisms,

$$Out(\mathcal{A}) = Aut(\mathcal{A})/In(\mathcal{A}).$$

The latest have a nice interpretation in quantum field theory as *flow of time*[4]. The formers are characteristic of the noncommutative case (otherwise $In(\mathcal{A})$ is trivial). Remarkably, the action of $In(\mathcal{A})$ on a geometry $(\mathcal{A}, \mathcal{H}, D)$ via the replacement of the representation π by $\pi \circ \alpha_u$ is equivalent to substituting D with

$$D_A \doteq D + A + JAJ^{-1} \tag{3.1}$$

where $A \doteq u[D, u^*]$. This appears a particular instance of the so called *fluctuations of the metric*, defined in a more general manner by taking

$$A = \sum_i a_i[D, b_i], \quad a_i, b_i \in \mathcal{A}. \tag{3.2}$$

In case of the product geometry (2.3) explicit computations[8] yield $A = H - i\gamma^\mu A_\mu$ where H is a scalar field on M with value in $\mathcal{A}_I$ (the Higgs field in the standard model[1]) and A_μ is a 1-form field with value in $Lie(U(\mathcal{A}_I))$, that is to say a gauge field. Therefore via the fluctuations of the metric both the Higgs field and gauge fields acquire a metric interpretation. In [10] we focused on the Higgs field only, assuming $A_\mu = 0$. In [3] Connes considers the example $H = 0$ for the internal geometry $\mathcal{A}_I = M_n(\mathbb{C})$. The vanishing of H is obtained by taking $D_I = 0$ so that the fluctuated Dirac operator

$$D_A = -i\gamma^\mu(\partial_\mu + A_\mu)$$

is nothing but[6] the usual covariant Dirac operator on the $U(n)$ trivial bundle $P = M \times \mathbb{C}P^{n-1}$ with connection 1-form A_μ. The latest defines both a Carnot-Carathéodory distance d_H via (1.1) and a noncommutative distance d via (2.2) (with D_A instead of D). It is not difficult to show that whatever M and n, $d \leq d_H$. Also $d = d_H$ as soon as the holonomy of the connection is trivial. However when the holonomy is not trivial d does not necessarily equals d_H.

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[5] $\mathcal{P}(M_2(\mathbb{C})) = \mathbb{C}P^{n-1}$ is in one to one correspondence with S^2, see eq.(4.1) below.

[6] Note the slight abuse of notation: we canceled out the JAJ^{-1} term in (3.1) since it commutes with the representation and does not appear in the commutator.

IV. COUNTER-EXAMPLE: THE CARDIO-TORUS

Take $M = S^1$ and $\mathcal{A}_I = M_2(\mathbb{C})$. Then $\mathcal{P}(\mathcal{A})$ is a trivial bundle on the circle with fiber $\mathbb{C}P^1$, maps to the 2-sphere via the Hopf map

$$\xi = \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} \in \mathbb{C}P^1 \mapsto \begin{pmatrix} x_\xi = 2\mathrm{Re}(\xi_1 \bar{\xi_2}) \\ y_\xi = 2\mathrm{Im}(\xi_1 \bar{\xi_2}) \\ z_\xi = |\xi_1|^2 - |\xi_2|^2 \end{pmatrix} \in S^2. \tag{4.1}$$

Let us fix a trivialization on P and write ξ_x the point in the fiber over x corresponding to $\xi \in \mathbb{C}P^1$. Take

$$A_\mu = \begin{pmatrix} 0 & 0 \\ 0 & \theta \end{pmatrix}$$

with $\theta \in]0,1[$. Then for any $\xi_x, \zeta_y \in P$

$$d(\xi_x, \zeta_y) = +\infty$$

if and only if $z_\xi \neq z_\zeta$. Hence the set of points at finite noncommutative distance from ξ_x is a two torus $T_\xi = M \times S^1$. Let us parameterize the S^1 fiber by $\phi \in [0, 2\pi[$ such that $0 = \xi_x = \xi_x^0$ and define $\xi_x^k \doteq 2k\theta\pi$ as the end point of the horizontal lift starting at ξ_x^{k-1} of the basis S^1 (see figure 2).

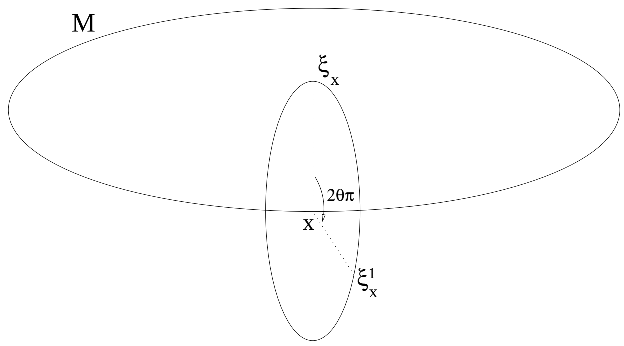

FIG. 2. The torus T_ξ.

Assuming the basis has radius 1,

$$d_H(\xi_x, \xi_x^k) = 2k\pi.$$

In case θ is irrational, any neighborhood of 0 contains some $\phi_k \doteq 2k\theta\pi$ mod $[2\pi]$ with k arbitrarily large. Hence, as plotted in figure IV, d_H "destroys" the S^1 structure of the fiber. On the contrary a rather long calculation fully detailed in [9] shows that

$$d(0, \phi) = C \sin\frac{\phi}{2}$$

for any $\phi \in [0, 2\pi[$, with $C = \frac{4\pi|\bar{\xi_1}\xi_2|}{|\sin\theta\pi|}$ a constant. It is interesting to compare d to the euclidean distance on the circle $d_E(0, \phi) = \min\,(\phi, 2\pi - \phi)$. Both distances keep in mind the S^1 structure of the fiber but while d_E is not smooth at the cut-locus $\phi = \pi$, d is smooth. Hence from the metric point of view the fiber over x equipped with the noncommutative distance d is *smoother than a circle.*

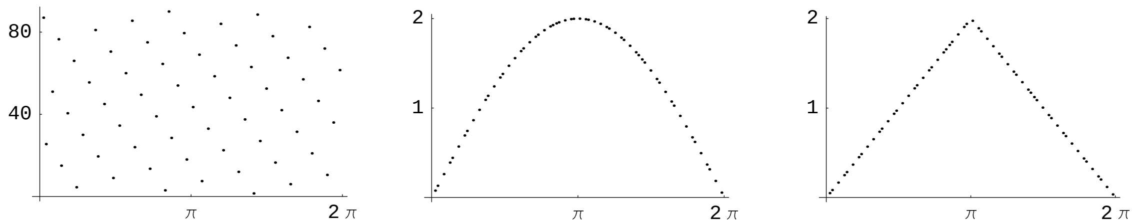

FIG. 3. From left to right: d_H, d, d_E.

In polar coordinates, $d(0,\phi)$ is the euclidean length on the cardioid $\frac{C}{4}(1+\cos\phi)$. However one has to be careful with this interpretation since the identification of ξ_x to $\phi = 0$ is arbitrary. Identifying 0 to ζ_x with $\zeta \neq \xi$ and $z_\xi = z_\zeta$, one would have similarly find $d(0,\phi) = C \sin\frac{\phi}{2}$. Hence the noncommutative distance d is invariant by translation on the fiber. On the contrary the euclidean distance on the cardioid is not invariant by translation. Therefore assuming that two observers O_1, O_2 respectively located at ξ_x and ζ_x are measuring d, both will agree that the fiber they are lying on is a cardioid but both will pretend to be localized at this particular point opposite to the "sharp" of the cardioid (see figure 4).

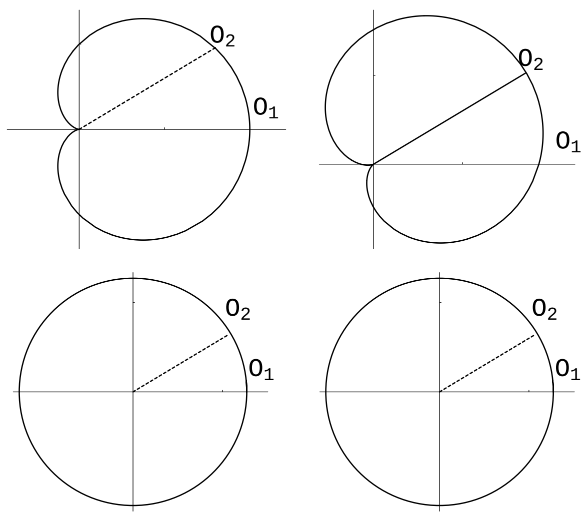

FIG. 4. On the left, the world according to 0_1; on the right, the world according to O_2. At top the world is a cardioid, O_1's and O_2's visions are not compatible with an embedding of the S^1-fiber into a 2-dimensional riemannian space. At bottom the world is an euclidean circle, from the intrinsic points of view of the O_i's as well as for an outside observer embedded into the euclidean plane.

Both are equally right. The point is that their measurements are not compatible with an "outside" riemannian point of view, as this would be the case if they were measuring d_E. Say it briefly, the fiber equipped with d is *not* a riemannian manifold, but a geometry in which every one can, on the same footing, pretend to be the center of the world[7]. Whether such a statement might have cosmological consequences is left to further investigation...

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[7]Thanks to Paulo Almeida for this nice formulation.

V. CONCLUSION

As a conclusion let us mention few questions,

- what is the physical meaning of this "smoothness from an intrinsic metric point of view" ?

- how to deal with other basis-manifold than S^1 ? (a related question for sub-riemannian geometers: given a minimal horizontal curve, is it possible to deform it keeping its length fixed and reducing the number of times its projection on M selfintersects ?)

- what is the metric when both H and A_μ are different from 0 ?

and also underline that [9] is intended to be a preliminary step towards the study of the metric aspect of the noncommutative torus (the situation should be quite similar, except that the pure state space is then a twisted bundle).

VI. REFERENCES

[1] A. H. Chamseddine, A. Connes, *Spectral Action Principle*, C. Math. Phys. **186** (1996) 737-750.

[2] A. Connes, *Noncommutative geometry*, Academic Press (1994).

[3] A. Connes, *Gravity Coupled with Matter and the Foundation of Noncommutative Geometry*, C. Math. Phys. **182** (1996) 155–176.

[4] A. Connes, C. Rovelli, *Von Neumann algebra automorphisms and time-thermodynamics relation in generally covariant quantum theories*, Class. Quantum Grav **11** 2899-2917 (1994). P. Martinetti, C.Rovelli, *Diamond's temperature: Unruh effect for bounded trajectories and the thermal time hypothesis*, Class. Quant. Grav **20** (2003) 4919-4932.

[5] A. Connes, J. Lott, *The metric aspect of noncommutative geometry*, proceedings of 1991 Cargèse summer conference, ed. by J. Fröhlich et al. (Plenum, New York 1992).

[6] V. Gayral, J. M Gracia Bondia, B. Iochum, Y. Schucker, J.C. Varilly, *Moyal planes are spectral triples*, hep-th/0307241.

[7] B. Iochum, T. Krajewski, P. Martinetti, *Distance in finite spaces from non commutative geometry*, Journ. Geom. Phys. **37** (2001) 100–125.

[8] D. Kastler, D. Testard, *Quantum forms of tensor products*, C. Math. Phys. **155** (1993) 135-142. F. J. Vanhecke, *On the product of real spectral triples*, Lett. Math. Phys. 50 (1999), no. 2.

[9] P. Martinetti, *Carnot-Carathodory metric and gauge fluctuation in noncommutative geometry*, hep-th/0506147.

[10] P. Martinetti, R. Wulkenhaar, *Discrete Kaluza Klein from scalar fluctuations in non commutative geometry*, J.Math.Phys. **43** (2002) 182-204.

[11] R. Montgomery, *A tour of subriemannian geometries, their geodesics and applications*, AMS (2002).