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Quantum Symmetries of the $\mathcal{A}_4$ Graphs of the Di Francesco-Zuber System

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Abstract

This work is devoted to the study of quantum symmetries of the $\mathcal{A}_\triangle$ graphs of the $su(3)$ system in an algebraic point of view.
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I. GENERAL DATA

We Consider the higher Coxeter graphs[6,11,7] associated to the $su(3)$ system in 2d RCFT. The $\mathcal{A}_k$ graphs are the truncated Weyl alcooves $\mathcal{P}_+^k$ of $SU(3)_q$ at level k :

$$\mathcal{P}_+^k = \{\lambda = (\lambda_1, \lambda_2) \,|\, \lambda_1, \lambda_2 \in Z\,,\, \lambda_1, \lambda_2 \geq 0\,,\, \lambda_1 + \lambda_2 \leq k\}\,, \tag{1.1}$$

The trivial representation is $\lambda = 1 = (0,0)$ and the two fundamental irreps are $2 = (1,0)$ and $3 = (0,1)$. The altitude (or generalized Coxeter number) κ is defined by $\kappa = k+3$.
Conjugation C acts as: $C(\lambda_1,\lambda_2) = (\lambda_2,\lambda_1)$. For a fixed level k, and the Z_3 automorphism A acts as: $A(\lambda_1,\lambda_2) = (k-\lambda_1-\lambda_2,\lambda_1)$. The triality $t(\lambda)$ is defined by: $t(\lambda_1,\lambda_2) = \lambda_1 - \lambda_2 \mod 3$.
Normalized characters χ_λ^k in an irrep of $\widehat{su}(3)$ possess a simple transformation property with respect to the modular group $PSL(2,Z)$, generated by the two transformations S and T[9].
The Gannon twist operator ϖ is defined by its action: $\varpi(\lambda) = A^{kt(\lambda)}(\lambda)$, and $\varpi^2 = I$.
The fusion of representations of chiral algebras in RCFT is written as: $\lambda \otimes \mu = \sum_{\nu\in\mathcal{P}_+^k} \mathcal{N}_{\lambda\mu}^{\nu}\ \nu$, where $\mathcal{N}_{\lambda\mu}^{\nu}$ are positive integers called the fusion coefficients and satisfy the Verlinde formula: $(\mathcal{N}_\lambda)_{\mu\nu} = \sum_{\beta\in\mathcal{P}_+^k} \mathcal{S}_{\lambda\beta}\mathcal{S}_{\mu\beta}\mathcal{S}^*_{\nu\beta}/\mathcal{S}_{1\beta}$. From the definition $\mathcal{N}_{\lambda\mu\nu} \doteq \mathcal{N}_{\lambda\mu}^{\nu}$, it follows that $N_1 = I_{p\times p}$, and N_2 is the adjacency matrix of the $\mathcal{A}_k$ graph. The N_λ matrices form a faithful representation of the fusion graph

algebra $\mathcal{A}_k$ with non-negative integral structure coefficients $\mathcal{N}_{\lambda\mu}^{\nu}$, suc that:

$$\begin{array}{c} N_{(\lambda,\mu)} = N_{(\lambda-1,\mu)}N_{(1,0)} - N_{(\lambda-1,\mu-1)} - N_{(\lambda-2,\mu+1)} \ , \quad N_{(\lambda,0)} = N_{(\lambda-1,0)}N_{(1,0)} - N_{(\lambda-2,1)} \\ N_{(0,\lambda)} = \left(N_{(\lambda,0)}\right)^{Tr} \qquad , \qquad N_{(\lambda,\mu)} = 0 \text{ if } \lambda < 0 \text{ or } \mu < 0 \ . \end{array} \tag{1.2}$$

Consider a graph $\mathcal{G}_k$ of type I (i.e. with self-fusion) with r vertices a it also possesses a graph algebra represented by $r \times r$ matrices G_a. The fusion graph algebra $\mathcal{A}_k$ possesses also another representation, which is provided by the fused matrices F_λ fulfilling the previous recurrence formula: $F_\lambda F_\mu = \sum_{\nu\in\mathcal{P}_+^k} N_{\lambda\mu}^{\nu} F_\nu$, where $F_1 = I$ and the matrix F_f corresponding to the fundamental irrep $\lambda = f$ of $\mathcal{A}_k$ is the adjacency matrix of the graph $\mathcal{G}_k$ with the same Coxeter number as $\mathcal{A}_k$. The vector space $\mathcal{V}(\mathcal{G}_k)$ spanned by the vertices of the $\mathcal{G}_k$ graph is a module under the action of $\mathcal{A}_k$: $\mathcal{A}_k \times \mathcal{V}(\mathcal{G}_k) \hookrightarrow \mathcal{V}(\mathcal{G}_k)$ such that $\lambda \times a \hookrightarrow \sum_b (F_\lambda)_{ab}\, b$.In the $\mathcal{A}_k$ cases, the fused matrices F_λ coincide with the fusion algebra matrices N_λ.

II. THE OCNEANU ALGEBRA OF QUANTUM SYMMETRIES

A. The Bialgebra $\mathcal{BG}$

Consider the graded algebra of endomorphisms on essential paths[10,2]:

$$\mathcal{BG} = \mathcal{E}nd\,(EssPaths) = \underset{\lambda\in\mathcal{P}_+^k}{\oplus} \mathcal{E}nd\left(EssPaths^{\lambda}\right) = \underset{\lambda\in\mathcal{P}_+^k}{\oplus} \mathcal{E}nd\left(H^{\lambda}\right) \tag{2.1}$$

Elements of H^λ are denoted ξ and $dim(H^\lambda) = d_\lambda = \sum_{a,b\in G} (F_\lambda)_{a,b}$. Consider the dual vector subspace $\widehat{H}^\lambda$ of H^λ, then $\mathcal{E}nd\left(H^\lambda\right) = H^\lambda \otimes \widehat{H}^\lambda$ and $\mathcal{BG} = \underset{\lambda\in\mathcal{P}_+^k}{\oplus} \mathcal{H}^\lambda \otimes \widehat{\mathcal{H}}^\lambda$ with elements $e_{\xi\eta}(\lambda) = \xi \otimes \bar{\eta}$.

The vector space $\mathcal{BG}$ endowed with two semi-simple algebra structures. The composition " $\circ$ " of endomorphisms, called also vertical product and the other one denoted "★", called horizontal product or convolution. The latter comes from the fact that the dual algebra $\widehat{\mathcal{B}}$, which elements are denoted $E_{\xi\eta}(x)$, can be also endowed with a multiplication structure "$\hat{\circ}$". The $\mathcal{BG}$ is also endowed with two coalgebra structures compatibles with each of the above multiplications. However the bialgebra $(\mathcal{B}, \circ, \Delta)$ is a "weak" Hopf algebra (WHA), satisfying the axioms of[1], also called Ocneanu quantum groupoïd or "double triangle algebra" (DTA)[4]. In Fig1 are depicted elements $e_{\xi\eta}(\lambda)$ of $\mathcal{B}$ and $E_{\xi\eta}(x)$ of $\widehat{\mathcal{B}}$ which justify the name DTA for the algebra $\mathcal{B}$ (where $a = s(\xi)$ and $b = r(\xi)$ are respectively the source and the range for the path ξ, idem $c = s(\eta)$ and $d = r(\eta)$).

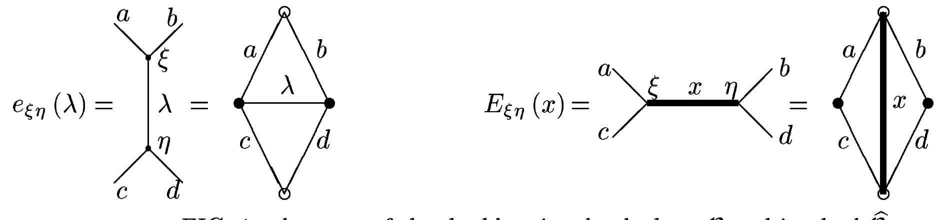


FIG. 1. elements of the double triangle algebras $\mathcal{B}$ and its dual $\widehat{\mathcal{B}}$

The algebra (WHA) $\mathcal{B}$ can be diagonalized in two different ways. On the one side the diagonalization related to the first multiplicative law $\circ$ makes $\mathcal{B}$ as a sum of simple blocks indexed by the Young diagrams λ of the $\mathcal{A}_k$ diagram, the dimension of each block is $(d_\lambda)^2$, and the whole dimension of the algebra is $\sum_\lambda (d_\lambda)^2$. On the other side the diagonalization of $\mathcal{B}$ with respect to its second multiplicative law ★ makes $\mathcal{B}$ isomorphic to a sum of another type of blocks indexed by the called "irreducible quantum symmetries" denoted x, these blocks are of sizes $(d_x)^2$ and the dimension of the algebra $\mathcal{B}$ is $\sum_x (d_x)^2$. In the sequel we will check the quadratic sum rule $\sum_\lambda (d_\lambda)^2 = \sum_x (d_x)^2$ for the $\mathcal{A}_4$ graph and its related graphs $\mathcal{A}_4^c$, $\mathcal{A}_4^{\varpi}$ and $\mathcal{A}_4^{\varpi c}$.

Let's return to the second block decomposition of $\mathcal{B}$. The minimal central projectors, using the first multiplicative law $\circ$, related to these blocks span a new (associative) algebra $\mathcal{O}c\,(\mathcal{G})$ called "Ocneanu algebra of quantum symmetries"[3,8,12]. We can associate to this algebra an Ocneanu graph denoted $\mathcal{H}_{\mathcal{O}c(\mathcal{G})}$ for which the vertices label the Ocneanu generators and are indexed by x. This graph encodes the multiplication by the Ocneanu generators $x\,y = \sum_{\mathbf{z}} \mathcal{O}_{xy}^z\, z$, where $\mathcal{O}_{xy}^z$ are the algebra structure coefficients (non-negative integers). The matrices $\mathcal{O}_x$ defined by $(\mathcal{O}_x)_{yz} = \mathcal{O}_{xy}^z$ gives an anti-representation of this algebra: $\mathcal{O}_x\,\mathcal{O}_y = \sum_z \mathcal{O}_{yx}^z\,\mathcal{O}_z$.

To every fundamental irrep f of the $\mathcal{A}_k$ algebra, we have two corresponding generators for the $\mathcal{O}c(\mathcal{G})$ algebra, denoted $\mathcal{O}_{\mathbf{f_L}}$ and $\mathcal{O}_{\mathbf{f_R}}$. The Ocneanu algebra can be realized as a kind of some tensor product $\dot{\otimes}$ for graphs of type I and its generators can be written as $x = a \dot{\otimes} b$ or a linear combination of such monoms. To each point x is associated a matrix $S_x = G_a G_b$ which characterizes the blocks of the second multiplicative law $\bigstar$. The dimension of such blocks are given by: $d_x = \sum_{a,b} (S_x)_{a,b}$.

B. "Double fusion" algebra matrices $\mathcal{V}_{\lambda\mu}$ and toric matrices $\mathcal{W}_{xy}$:

The vector space of vertices of $\mathcal{G}$ is a bimodule over the action of $\mathcal{A}_k$, and also is a module under the action of $\mathcal{O}c\,(\mathcal{G}_k)$. Each one of the two algebras $\mathcal{A}_k$ and $\mathcal{O}c\,(\mathcal{G})$ is a bimodule over the action of the other. In particular, the bimodule structure of $\mathcal{O}c\,(\mathcal{G})$ over $\mathcal{A}_k$ reads:

$$\begin{aligned} \mathcal{A}_k \times \mathcal{O}c\,(\mathcal{G}) \times \mathcal{A}_k &\hookrightarrow \mathcal{O}c\,(\mathcal{G}) \\ \lambda \times x \times \mu &\hookrightarrow \sum_y \mathcal{V}_{\lambda\mu\,,xy}\; y, \end{aligned} \tag{2.2}$$

where the $\mathcal{V}_{\lambda\mu}$ matrices form a representation of the square fusion algebra. The associativity property $(\lambda\,(\lambda'\,\mathbf{x}\,\mu)\,\mu' = (\lambda\,\lambda')\,\mathbf{x}\,(\mu\,\mu'))$ of the bimodule structure implies that these coefficients satisfy the so called modular splitting equation :

$$\mathcal{V}_{\lambda\mu}\,\mathcal{V}_{\lambda'\mu'} = \sum_{\lambda'',\mu''\in\mathcal{P}_+^k} \mathcal{N}_{\lambda\lambda'}^{\lambda''}\,\mathcal{N}_{\mu\mu'}^{\mu''}\,\mathcal{V}_{\lambda''\mu''}. \tag{2.3}$$

the $\mathcal{V}_{\lambda\mu}$ are related to the toric matrices by: $(\mathcal{V}_{\lambda\mu})_{xy} = (\mathcal{W}_{xy})_{\lambda\mu}$. Moreover,the "fundamental" left and right chiral generators of the $\mathcal{O}c\,(\mathcal{G})$ algebra (there are four of them in the $su\,(3)$ case) are given by: $\mathcal{V}_{f0} = \mathcal{O}_{f_L}$ and $\mathcal{V}_{0f} = \mathcal{O}_{f_R}$.

C. Quantum symmetries of $\mathcal{A}_4$ graphs:

The irreps of the $\mathcal{A}_4$ graph are indexed by their corresponding Young diagrams. It is displayed in Fig2 together with its conjugated $\mathcal{A}_4^c$, twisted $\mathcal{A}_4^{\varpi}$, and twisted conjugate $\mathcal{A}_4^{\varpi c}$ graphs (see[6]).

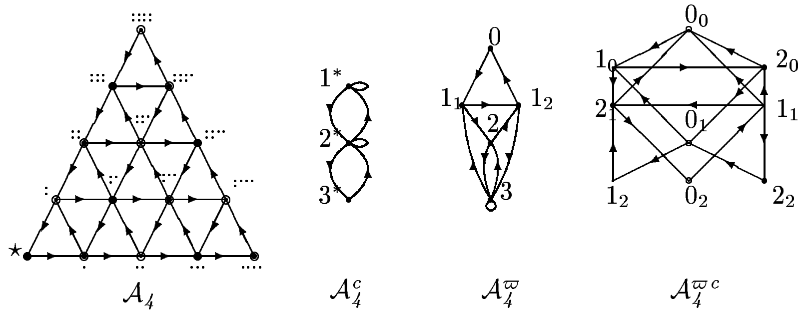


FIG. 2. The $\mathcal{A}_4$ graphs

1. Ocneanu algebra $\mathcal{O}c(\mathcal{A}_4)$ and its generators:

The Ocneanu algebras of quantum symmetries associated to $su(3)$ graphs are a kind of square tensor product[2,3,5,8] of the graph algebra $\mathcal{G}$ over the ambichiral subalgebra $\mathcal{J}$:

$$\mathcal{O}c(\mathcal{G}) = \mathcal{G} \dot{\otimes} \mathcal{G} = \mathcal{G} \otimes_{\mathcal{J}} \mathcal{G} \tag{2.4}$$

In the $\mathcal{A}_4$ case, the ambichiral algebra $\mathcal{J}$ is isomorphic to $\mathcal{A}_4$, so $\mathcal{O}c(\mathcal{A}_4) = \mathcal{A}_4 \dot{\otimes} \mathcal{A}_4 = \mathcal{A}_4 \otimes_{\mathcal{J}} \mathcal{A}_4$, where: $au \dot{\otimes} b = \mathbf{a} \dot{\otimes} \mathbf{u^*b}$, for all $a, b, u \in \mathcal{A}_4$ and u^* is the conjugate of u.

The toric matrices $\mathcal{W}_{xy}$ and the square algebra matrices $\mathcal{V}_{\lambda\mu}$ are given by: $(\mathcal{W}_{xy})_{\lambda,\mu} = (\mathcal{V}_{\lambda\mu})_{x,y} = \left(N_\lambda N_\mu^*\right)_{x,y}$, (we note here that the indices x and y are of type λ and μ). The toric matrix associated to the unit generator of $\mathcal{O}c(\mathcal{A}_4)$ is: $\mathcal{M} = \mathcal{W}_{11} = \mathbb{I}_{15}$,which then commutes with the $\mathcal{S}$ and $\mathcal{T}$ matrices, hence is modular invariant. This implies that the Ocneanu algebra $\mathcal{O}c(\mathcal{A}_4)$ is commutative because there is no entries in $\mathcal{M}$ which are equal or more than 2, and the number of points in the Ocneanu graph $\mathcal{H}_{\mathcal{O}c(\mathcal{A}_4)}$ is equal to the number of entries 1 in $\mathcal{M}$, then there are $Trace\left[\mathcal{M}\mathcal{M}^{\mathcal{T}}\right] = 15$ vertices in $\mathcal{H}_{\mathcal{O}c(\mathcal{A}_4)}$. The four fundamental generators of the Ocneanu algebra $\mathcal{O}c(\mathcal{A}_4)$ are

$$\begin{array}{lcl} \mathcal{O}_{2L} = \mathcal{V}_{2,1} = N_2 N_{1^*} = N_2 & ; & \mathcal{O}_{2R} = \mathcal{V}_{1,2} = N_1 N_{2^*} = N_3 \\ \mathcal{O}_{3L} = \mathcal{V}_{3,1} = N_3 N_{1^*} = N_3 & ; & \mathcal{O}_{3R} = \mathcal{V}_{1,3} = N_1 N_{3^*} = N_2 \end{array} \tag{2.5}$$

The corresponding Ocneanu graph $\mathcal{H}_{\mathcal{O}c(\mathcal{A}_4)}$, related to $\mathcal{O}_{2L}$ and $\mathcal{O}_{2R}$, is the superposition of tow copies of the $\mathcal{A}_4$ graph, corresponding to its two fundamental generators $2 = (1,0)$ and $3 = (0,1)$. This graph is displayed in Fig3.

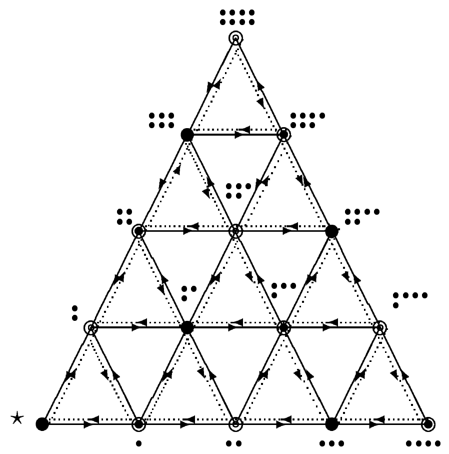


FIG. 3. The Ocneanu graph $\mathcal{H}_{\mathcal{O}c(\mathcal{A}_4)}$,

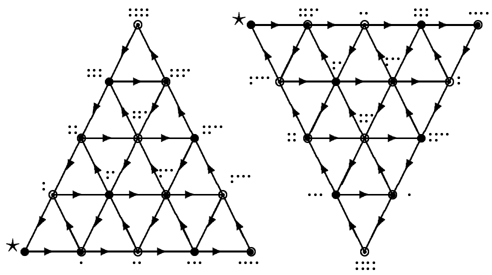
A possible realization of the points of the $\mathcal{H}_{\mathcal{O}c(\mathcal{A}_4)}$ graph is given by the 15 basis elements $\underline{\lambda} = \lambda \dot{\otimes} 1 = 1 \dot{\otimes} \lambda^*$ for $\lambda \in \mathcal{P}_+^4$. The generators of the Ocneanu algebra $\mathcal{O}c(\mathcal{A}_4)$ are realized as: $\mathcal{O}_\lambda = \mathcal{O}_{\lambda^*}^T = N_\lambda$ for $\lambda \in \mathcal{P}_+^4$. The toric matrices with one defect line $\mathcal{W}_{\lambda 1}$ are given by: $\mathcal{W}_{11} = \mathcal{M} = N_1$ and $\mathcal{W}_{\lambda 1} = \mathcal{W}_{1\lambda} = N_\lambda$. The dimensions of blocks corresponding to the first multiplicative law of the bialgebra $\mathcal{BA}_4$ are given by: $d_\lambda = \sum_{\mu,\nu\in\mathcal{P}_+^4} (F_\lambda)_{\mu,\nu}$. The linear and quadratic sums of these dimensions are respectively: $\sum_{\lambda\in\mathcal{P}_+^4} d_\lambda = 486$ and $\sum_{\lambda\in\mathcal{P}_+^4} (d_\lambda)^2 = 17766$.

To determine $d_x = \sum_{\mu,\nu\in\mathcal{P}_+^4} (S_x)_{\mu,\nu}$ dimensions of the second type of blocks that are related to the convolution law of $\mathcal{BA}_4$, let's first determine the S matrices. These matrices are given by $S_x = G_\lambda G_\mu$ when $x = \lambda \dot{\otimes} \mu$ then $S_x = G_\lambda = F_\lambda$ for $\lambda \in \mathcal{P}_+^4$ since $\mu = 1$ and $G_1 = I_{15}$. So the dimensions d_x are equal to d_λ and the linear and quadratic sum rules for the bialgebra $\mathcal{BA}_4$: $\sum_\lambda d_\lambda = \sum_x d_x = 486$ and $\sum_\lambda (d_\lambda)^2 = \sum_x (d_x)^2 = 17766$ are trivially satisfied.

2. The Ocneanu algebra $\mathcal{O}c\left(\mathcal{A}_4^\varpi\right)$:

The Ocneanu algebra related to the $\mathcal{A}_4^\varpi$ graph is $\mathcal{O}c(\mathcal{A}_4^\varpi) = \mathcal{A}_4 \dot{\otimes}_\varpi \mathcal{A}_4 = \mathcal{A}_4 \otimes_{\mathcal{A}_4^{\varpi cj}} \mathcal{A}_4$ where: $au\dot{\otimes}_\varpi b = a \dot{\otimes}_\varpi \varpi(u^*)\, b$ for all $a, b, u \in A_4$ and u^* is the conjugate of u. The toric matrices $\mathcal{W}_{x,y}^\varpi$ and the square algebra matrices $\mathcal{V}_{\lambda,\mu}^\varpi$ are given in terms of the N_λ matrices by: $(\mathcal{W}_{x,y}^\varpi)_{\lambda,\mu} = \left(\mathcal{V}_{\lambda,\mu}^\varpi\right)_{x,y} = (N_\lambda N_{\varpi(\mu^*)})_{x,y}$. The modular invariant toric matrix associated to $\mathcal{O}c\left(\mathcal{A}_4^\varpi\right)$ is: $\mathcal{M}^\varpi = \delta_{\lambda,\varpi(\mu)}$ for all $\lambda, \mu \in \mathcal{P}_+^4$.A possible realization of the points of the $\mathcal{H}_{\mathcal{O}c(\mathcal{A}_4^\varpi)}$ graph is given by the 15 basis elements: $\underline{\lambda}^\varpi = \lambda\dot{\otimes}_\varpi \mathbf{1} = \mathbf{1}\dot{\otimes}_\varpi \varpi(\lambda^*)$ for $\lambda\in\mathcal{P}_+^4$. The generators of the Ocneanu algebra $\mathcal{O}c(\mathcal{A}_4^\varpi)$ are realized as: $\mathcal{O}_\lambda^\varpi = N_\lambda$ for $\lambda\in\mathcal{P}_+^4$.

The Ocneanu graph $\mathcal{H}_{\mathcal{O}c\left(\mathcal{A}_4^\varpi\right)}$ related to the fundamental generators $\mathcal{O}_{2L}^\varpi$ and $\mathcal{O}_{2R}^\varpi$ is the graph encoded by multiplication by these two generators, it is the superposition of the two copies of the $\mathcal{A}_4$ graph corresponding to its fundamental irrep $2 = (1, 0)$ and the irrep $14 = (1, 3)$. These two graphs are drown separately in Fig4.



The $\mathcal{A}_4$ graph related to (1,0) The $\mathcal{A}_4$ graph related to (1,3)

FIG. 4. The two components of the Ocneanu graph $\mathcal{H}_{\mathcal{O}c\left(\mathcal{A}_4^\varpi\right)}$

The dimensions of blocks corresponding to the first multiplicative law of the bialgebra $\mathcal{BA}_4^\varpi$ are: $d_\lambda^\varpi = \sum_{a,b\in\mathcal{A}_4^\varpi} (F_\lambda^\varpi)_{a,b}$, then we get: $\sum_{\lambda\in\mathcal{P}_+^4} d_\lambda^\varpi = 162$ and $\sum_{\lambda\in\mathcal{P}_+^4} (d_\lambda^\varpi)^2 = 1974$. To determine the dimensions $d_x^\varpi = \sum_{a,b\in\mathcal{A}_4^\varpi} (S_\lambda^\varpi)_{a,b}$ let's first remark that the S_x encode the property of $\mathcal{A}_4^\varpi$ as a module over the Ocneanu algebra $\mathcal{O}c\left(\mathcal{A}_4^\varpi\right)$, with the $\mathcal{O}_\lambda^\varpi$ matrices as generators, in the same way as the fused matrices F_λ^ϖ encode the property of $\mathcal{A}_4^\varpi$ as a module over the fusion graph algebra A_4, with the nimreps matrices N_λ as generators. Since $\mathcal{O}_\lambda^\varpi = N_\lambda$, it is obvious that the S_x coincide with the fused matrices F_λ^ϖ, then the dimensions d^{ϖ}_{λ} are equal to d^{ϖ}_{x} and the linear and quadratic sum rules are fulfilled: $\sum_{\lambda\in\mathcal{P}^4_+} d^{\varpi}_{\lambda} = \sum_{x\in\mathcal{H}_{\mathcal{O}c(\mathcal{A}^{\varpi}_4)}} d^{\varpi}_{x} = 162$ and $\sum_{\lambda\in\mathcal{P}^4_+} (d^{\varpi}_{\lambda})^2 = \sum_{x\in\mathcal{H}_{\mathcal{O}c(\mathcal{A}^{\varpi}_4)}} (d^{\varpi}_{x})^2 = 1974$.

III. RESULTS FOR $\mathcal{A}^C_4$ AND $\mathcal{A}^{\varpi C}_4$ CASES:

The Ocneanu algebra $\mathcal{O}c\left(\mathcal{A}^c_4\right)$ is: $\mathcal{O}c(\mathcal{A}^c_4) = \mathcal{A}_4 \dot{\otimes}_c \mathcal{A}_4 = \mathcal{A}_4 \otimes_{\mathcal{A}_4} \mathcal{A}_4$, where: $au \dot{\otimes}_c b = a \dot{\otimes}_c ub$, for all $a, b, u \in \mathcal{A}_4$. The toric matrices $\mathcal{W}^c_{x,y}$ and the $\mathcal{V}^c_{\lambda,\mu}$ matrices : $\left(\mathcal{W}^c_{x,y}\right)_{\lambda,\mu} = \left(\mathcal{V}^c_{\lambda,\mu}\right)_{x,y} = (N_\lambda N_\mu)_{x,y}$. The modular invariant of $\mathcal{O}c\left(\mathcal{A}^c_4\right)$ is: $\mathcal{M}^c = \delta_{\lambda,\mu^*}$ for all $\lambda, \mu \in \mathcal{P}^4_+$. We can check that $\sum_{\lambda\in\mathcal{A}^c_4} d^c_{\lambda} = \sum_{x\in\mathcal{H}_{\mathcal{O}c(\mathcal{A}^c_4)}} d^c_{x} = 102$ and $\sum_{\lambda\in\mathcal{A}^c_4} (d^c_{\lambda})^2 = \sum_{x\in\mathcal{H}_{\mathcal{O}c(\mathcal{A}^c_4)}} (d^c_{x})^2 = 792$.

The Ocneanu algebra $\mathcal{O}c\left(\mathcal{A}^{\varpi c}_4\right)$ is: $\mathcal{O}c(\mathcal{A}^{\varpi c}_4) = \mathcal{A}_4 \dot{\otimes}_{\varpi c} \mathcal{A}_4 = \mathcal{A}_4 \otimes_{\mathcal{A}^{\varpi}_4} \mathcal{A}_4$, where: $au \dot{\otimes}_{\varpi c} b = a \dot{\otimes}_{\varpi c} \varpi(u)\, b$ for all $a, b, u \in \mathcal{A}_4$. The toric matrices $\mathcal{W}^{\varpi c}_{x,y}$ and the $\mathcal{V}^{\varpi c}_{\lambda,\mu}$ matrices are: $\left(\mathcal{W}^{\varpi c}_{x,y}\right)_{\lambda,\mu} = \left(\mathcal{V}^{\varpi c}_{\lambda,\mu}\right)_{x,y} = \left(N_\lambda N_{\varpi(\mu)}\right)_{x,y}$ and the toric matrix associated to $\mathcal{O}c\left(\mathcal{A}^{\varpi c}_4\right)$ is $\mathcal{M}^{\varpi c} = \delta_{\lambda,\varpi(\mu^*)}$ for all $\lambda, \mu \in \mathcal{P}^4_+$. We can check that: $\sum_\lambda d^{\varpi c}_\lambda = \sum_x d^{\varpi c}_x = 306$ and $\sum_\lambda (d^{\varpi c}_\lambda)^2 = \sum_x (d^{\varpi c}_x)^2 = 7182$. So the linear and quadratic sum rules are fulfilled.

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IV. REFERENCES

[1] G. Böhm, F. Nill, K. Szlachányi, *Weak Hopf Algebras I. Integral theory and $C^\star$ structure*, J. Algebra **221** (1999) 385–438.

[2] R.Coquereaux, G.Schieber, *Twisted partition functions for ADE boundary conformal field theories and Ocneanu algebra of quantum symmetries*,J. of Geom. and Phys.**781**(2002)1–43.

[3] R. Coquereaux, G. Schieber, *Determination of quantum symmetries for higher ADE systems from the modular T matrix*, J. of Math. Physics **44** (2003) 3809–3837.

[4] R. Coquereaux, R. Trinchero, *On quantum symmetries of ADE graphs*, Advances in Theor. and Math. Phys., volume **8** issue 1 (2004).

[5] R. Coquereaux, D. Hammaoui, G. Schieber, E.H. Tahri, *Comments about quantum symmetries of SU(3) graphs*, math-ph/0508002 (submitted to the J. of Geom. and Phys.)

[6] P. Di Francesco, J.-B. Zuber, *SU(N) Lattice integrable models associated with graphs*, Nucl. Phys **B338** (1990) 602–646.

[7] D. Hammaoui, G. Schieber, E.H. Tahri, *Higher Coxeter graphs associated to affine $su(3)$ modular invariants*, hep-th/0412102, J. Phys. A: Math. Gen. 38 (2005) 8256–8286.

[8] D. Hammaoui, G. Schieber, E.H. Tahri, *Quantum symmetries of higher Coxeter graphs of $su(3)$ type*, in preparation.

[9] V. G. Kǎc, D. H. Peterson, *Infinite-dimensional Lie algebras, theta functions and modular forms*, Adv. Math. **53** (1984) 125–264.

[10] A. Ocneanu, *Paths on Coxeter diagrams: from Platonic solids and singularities to minimal models and subfactors*, Notes taken by S. Goto, AMS Fields Institute Monographs **13** (1999), Rajarama Bhat et al eds.

[11] A. Ocneanu, *The Classification of subgroups of quantum SU(N)*, Lectures at Bariloche Summer School 2000, Argentina, AMS Contemp. Math. **294**, R. Coquereaux, A. García and R. Trinchero eds.

[12] V.B.Petkova, J.B.Zuber,*The many faces of Ocneanu cells*, Nucl.Phys. **B603**(2001)449–496.