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On the Critical Space-Time Dimensions of Parasuperstrings

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Abstract

We investigate the extension of the superstring theory in the formalism of the paraquantization. Unlike the quantum superstrings which are defined at space-time dimension D=10, in the case of the parasuperstrings, the four space-time dimensions D=3,4,6,10, (in which the classical superstrings can be formulated) survive. The consequences of this study on the heterotic parastrings is investigated, we found that in addition to the ordinary case (26,10) of the space-time dimensions in which the heterotic string can be formulated, in the case of the heterotic parastrings, only survives the case (14,6) which is based on the only possible group E8.

I. INTRODUCTION

A first study of the paraquantum Poincaré algebra was done by F.Ardalan and F.Mansouri [1]. This study is based on the particular manner in which the center of mass variables of the string are to be handled. Indeed, these authors impose on the center of mass coordinates and the total energy momentum operators of the string x^{μ}, p^{μ} to satisfy ordinary commutation relations. This is done by the choice of a specific direction in the paraspace of the Green components , characterized by the anzatz $x^{\mu(\beta)} = x^{\mu}\delta_{\beta 1}$ and $p^{\mu(\beta)} = p^{\mu}\delta_{\beta 1}$, where $x^{\mu(\beta)}$ (resp $p^{\mu(\beta)}$) are the Green components of x^{μ} (resp p^{μ}). This requires relative paracommutation relations between the center of mass coordinates and the excitation modes of the string which are exclusively anomalous bilinear commutation relations in terms of the Green components. Because of the separation of $\beta = 1$ and $\beta \neq 1$ in the precedent anzatz, these bilinear commutations relations can not be rewritten in trilinear commutation relations form which are the basis of the paraquantization. In this hypothesis, they find that the resulting parabosonic (resp paraspinning) string theories are Poincaré invariant if the dimension D of the space-time and the order Q of the paraquantization are related by the expressions $D = 2 + \frac{24}{Q}$ (resp $D = 2 + \frac{8}{Q}$).

A second study of the paraquantum Poincaré algebra is done by N.Belaloui (N.B) and H.Bennacer (H.B) [2], [3] the paraquantization is done without the Ardalan and Mansouri hypothesis on the center of mass variables. Indeed the paraquantum system is so that the operators $X^{\mu}(\sigma,\tau)$, $\mathcal{P}^{\nu}(\sigma',\tau)$, and $\psi^{\rho}(\sigma'',\tau)$ satisfy the paraquantum trilinear commutation relations. Unlike the Ardalan and Mansouri work, this is done by requiring that both the center of mass variables and the excitation modes of the string verify paraquantum commutation relations. To satisfy this, one must, among other things, have $\left[X^{\mu(\alpha)}(\sigma,\tau), P^{\nu(\alpha)}(\sigma',\tau)\right] = \imath g^{\mu\nu}\delta(\sigma-\sigma')$, but the Ardalan and Mansouri hypothesis, characterized by the anzatz $x^{\mu(\beta)} = x^{\mu}\delta_{\beta 1}$ and $p^{\mu(\beta)} = p^{\mu}\delta_{\beta 1}$, leads to the result $\left[X^{\mu(\alpha)}(\sigma,\tau), P^{\nu(\alpha)}(\sigma',\tau)\right] =$

$\imath g^{\mu\nu}\left[\delta(\sigma-\sigma')-(1-\delta_{\alpha 1})\right]$, which is not compatible with the paraquantization of the variables $X^{\mu}(\sigma,\tau)$, $P^{\mu}(\sigma,\tau)$ and $\psi_a^{\mu}(\sigma,\tau)$. With the only use of the trilinear relations , one can prove that :

$$\begin{aligned}
\left[p^{\mu},M^{\nu\rho}\right] &= -\imath g^{\mu\nu}p^{\rho}+\imath g^{\mu\rho}p^{\nu} \\
\left[M^{\mu\nu},M^{\rho\sigma}\right] &= \imath g^{\nu\rho}M^{\sigma\mu}-\imath g^{\mu\sigma}M^{\nu\rho}-\imath g^{\nu\sigma}M^{\rho\mu}+\imath g^{\mu\rho}M^{\nu\sigma}
\end{aligned}$$

Now, for the first commutator $[p^{\mu},p^{\nu}]$ of the algebra, one can only write $\left[p^{\mu},[p^{\nu},p^{\sigma}]_{+}\right]=0$, in the same way for the coordinates x^{μ} where one can only write $\left[x^{\mu},[x^{\nu},x^{\sigma}]_{+}\right]=0$. These relations are equivalent to $[p^{\mu},p^{\nu}]=\Theta^{\mu\nu}$ and $[x^{\mu},x^{\nu}]=\Lambda^{\mu\nu}$ where, in terms of the Green components, $\Theta^{\mu\nu}=2\sum_{\alpha\neq\beta}p^{\mu(\alpha)}p^{\nu(\beta)}\neq 0$! and $\Lambda^{\mu\nu}=2\sum_{\alpha\neq\beta}x^{\mu(\alpha)}x^{\nu(\beta)}\neq 0$! for $Q\neq 1$, these are noncommuting coordinates and momentum coordinates so that , for $Q\neq 1$, we are working in a noncommutative space-time. In particular, one can have paraspinning strings with critical dimensions $D=10,6,4,3$ (respectively in orders $Q=1,2,4,8$). This coincide with the dimensions in which the classical superstrings can be formulated.

In this work, we investigate the existence possibilities of the $D=3,4,6$ parasuperstrings with the N.B and H.B definition of the paraquantum system, we begin by developping some points given in [4] where now, we derive the susy generators algebras in the cases $D=3,4$ and 6 with respect to their susy groups for both a parabose-parafermi and a bose-parafermi systems. We construct the general form of the partition function and derive these later in the cases $D=3,4,6$. We determine the spectrum in these cases, we found a common chord between the number of states and the one given by the partition functions. In the second point, we investigate the consequences of this study on the heterotic parastring, we found that, in addition to the ordinary case (26,10), the heterotic parastring can only survives in the case (14,6) which is constructed from the only possible group E_8.

II. PARAQUANTUM FORMALISM OF THE SUPERSTRING

A. Introduction

Let us first introduce the notations we will use in this work through a brief summary of some familiar results in superstring theory . In the transverse gauge, the action is postulated as :

$$S=-\frac{1}{2\pi}\int d\sigma d\tau\left(\partial_a X^i\partial^a X_i-i\overline{S}^{Aa}\rho^B\partial_B S^{Aa}\right) \tag{2.1}$$

where $A,B=1,2$ and a is the components index of the spinor S^A with

$$S=\begin{pmatrix}S^1\\ S^2\end{pmatrix}\qquad;\qquad \overline{S}=S^T\rho^1$$

and

$$\rho^1=\begin{pmatrix}0&-\imath\\ \imath&0\end{pmatrix}\qquad;\qquad \rho^2=\begin{pmatrix}0&\imath\\ \imath&0\end{pmatrix}\qquad;\qquad \left\{\rho^A,\rho^B\right\}=-2\eta^{AB}$$

The solutions are :

$$X^i(\sigma,\tau)=x^i+p^i\tau+\sum_{n\neq 0}\frac{1}{n}\alpha_n^i(0)\exp(-\imath n\tau)\cos n\sigma \tag{2.2}$$

$$\begin{cases}S^{1a}(\sigma,\tau)=\dfrac{1}{\sqrt{2}}\displaystyle\sum_{n\in Z}S_n^{1a}\exp\left[-\imath n(\tau-\sigma)\right]\\ S^{2a}(\sigma,\tau)=\dfrac{1}{\sqrt{2}}\displaystyle\sum_{n\in Z}S_n^{2a}\exp\left[-\imath n(\tau+\sigma)\right]\end{cases} \tag{2.3}$$

with $i,a=\overline{1,D-2}$

B. Paraquantization

In this gauge, the paraquantum operators x^-, p^+, x^i, p^i ,α_n^i and s_n^a verify the trilinear relations [4]:

$$\left[s_n^a\ ,\left[s_m^b\ ,s_l^c\ \right]_-\right] = 2\left(\delta^{ab}\delta_{n+m}s_l^c\ -\delta^{ik}\delta_{n+l}s_m^b\ \right) \tag{2.4}$$

$$\left[\alpha_n^i,\left[\alpha_m^j,\alpha_l^k\right]_+\right] = 2(\delta^{ij}n\delta_{n+m,0}\alpha_l^k+n\delta^{ik}\delta_{n+l,0}\alpha_m^j) \tag{2.5}$$

$$\left[x^i,\left[p^j,pk\right]_+\right] = 2i\left(\delta^{ij}pk+\delta^{ik}p^j\right) \tag{2.6}$$

$$\left[\alpha_n^i,\left[\alpha_m^j,A\right]_+\right] = 2\delta^{ij}n\delta_{n+m}A \tag{2.7}$$

$$\left[s_n^a\ ,\left[s_m^b\ ,B\right]_+\right] = 2\delta^{ab}\delta_{n+m}B \tag{2.8}$$

$$\left[x^i,\left[p^j,C\right]_+\right] = 2\imath\delta^{ij}C \tag{2.9}$$

$$\left[x^-,\left[p^+,D\right]_+\right] = 2\imath D \tag{2.10}$$

and all the others are null. Here D, E, F, and G represent the following operators: A= x^-, p^+, x^k, p^k or s_l^a $B=x^-$, p^+, x^i, p^i or α_n^i, $C=x^-$, p^+, x^k, α_n^k or s_n^a, $D=x^-$, x^i, p^i α_n^i or s_n^a.

C. Green decomposition

Applying the Green decomposition

$$x^i=\sum_{\alpha=1}^{Q}x^{i(\alpha)}\quad ;p^i=\sum_{\alpha=1}^{Q}p^{i(\alpha)}\quad ;\quad \alpha_n^i=\sum_{\beta=1}^{Q}\alpha_n^{i(\beta)}$$

$$x^-=\sum_{\alpha=1}^{Q}x^{-(\alpha)}\quad ;\quad p^+=\sum_{\alpha=1}^{Q}p^{+(\alpha)};\quad s_n^a=\sum_{\beta=1}^{Q}s_n^{a(\alpha)} \tag{2.11}$$

the set of the precedent trilinear commutation relations is equivalent to these anomalous bilinear relations

$$\left[x^{i(\alpha)},p^{j(\alpha)}\right]=\imath\delta^{ij}\quad ;\quad \left[x^{i(\alpha)},p^{j(\beta)}\right]_+=0\quad \alpha\neq\beta \tag{2.12}$$

$$\left[x^{-(\alpha)},p^{+(\alpha)}\right]=\imath\quad ;\quad \left[x^{-(\alpha)},p^{+(\beta)}\right]_+=0\quad \alpha\neq\beta \tag{2.13}$$

$$\left[\alpha_n^{i(\alpha)},\alpha_m^{j(\alpha)}\right]=n\delta^{ij}\delta_{n+m,0}\quad ;\quad \left[\alpha_n^{i(\alpha)},\alpha_m^{j(\beta)}\right]_+=0\quad \alpha\neq\beta \tag{2.14}$$

$$\left[s_n^{a(\alpha)},s_m^{b(\alpha)}\right]_+=\delta^{ab}\delta_{n+m,0}\quad ;\quad \left[s_n^{a(\alpha)},s_m^{b(\beta)}\right]=0\quad \alpha\neq\beta \tag{2.15}$$

$$\left[\alpha_n^{i(\alpha)},s_m^{a(\alpha)}\right]=0\quad ;\quad \left[\alpha_n^{i(\alpha)},s_m^{a(\beta)}\right]_+=0\quad \alpha\neq\beta \tag{2.16}$$

and all the other commutators (and anticommutators) of the type $\left[A^{(\alpha)},B^{(\alpha)}\right]=0$ (and $\left[A^{(\alpha)},B^{(\beta)}\right]_+=0$, for $\alpha\neq\beta$).

III. PARASUSY SYSTEM

To have a susy system, the counting of the number of the independent degrees of freedom within the spinors is important when we fix the gauge. The classical superstrings can be formulated in $D=3$ and the the fermions are Majorana, in $D=4$ and the fermions are Majorana or Weyl, in $D=6$ and the fermions are Weyl, in $D=10$ and the fermions are Majorana and Weyl. In D dimensions, the two Dirac spinors S^1, S^2 have $(2+2)$ complex components in three dimensions, $(4+4)$ complex components in four dimensions, $(8+8)$ complex components in six dimensions, and $(32+32)$ complex components in ten dimensions. Restricting the spinors to being Majorana or Weyl reduces them down by half to $(2+2)$ real components in $D=3$ and so on. Choosing the light cone gauge will further reduces them down by half once again to $(1+1)$ real components in $D=3$ etc... Lastly, when we go on-shell, the number of components goes down by half again to 1 real component in $D=3$ and so on. Finally, we have the same numbers of the bosonic and the fermionic independent degrees of freedom. This is recapitulated in the following table:

Conditions	$D=3$	$D=4$	$D=6$	$D=10$
Dirac	$(2+2)\ C.C$	$(4+4)\ C.C$	$(8+8)\ C.C$	$(32+32)\ C.C$
Spinors types	$Majorana$ $(2+2)\,R.C$	$Maj\ or\ Weyl$ $(2+2)\,C.C$	$Weyl$ $(4+4)\,C.C$	$Majorana-Weyl$ $(16+16)\,R.C$
Light cone	$2\ R.C$	$(2+2)\ R.C$	$(4+4)\ R.C$	$(8+8)\ R.C$
On-Shell	$1\ R.C$	$2\ R.C$	$4\ R.C$	$8\ R.C$
Transverses $D-2$	$1\ R.C$	$2\ R.C$	$4\ R.C$	$8\ R.C$
Susy Groups	$O(1)$	$U(1)$	$Sp(1)=Spin(3)$	$SO(8)=Spin(8)$

Before developing these two parasusy models, let us recall some remarks. In the ordinary case ($D=10$), the 16 charges are the components of a Majorana-Weyl spinors which satisfies this susy algebra $\left([Q,Q]_{+}\sim(1\pm\Gamma)\,\Gamma.p\right)$. In $spin(8)$ notation which contains two spinorial representations, this equation splits up two 3 pieces. It is the same thing for $D=6$

$$D=10\longrightarrow\left\{\begin{array}{c}8_v\\ 8_s\rightarrow Q^a\\ 8_c\rightarrow Q^{\dot a}\end{array}\right.\Longrightarrow\left\{\begin{array}{l}\left[Q^a,Q^b\right]_+=2p^+\delta^{ab}\\ \left[Q^{\dot a},Q^{\dot b}\right]_+=2\delta^{\dot a\dot b}H\\ \left[Q^a,Q^{\dot a}\right]_+=\sqrt{2}\gamma^i_{a\dot a}p^i\end{array}\right.$$

$$D=6\rightarrow\left\{\begin{array}{c}4_v\\ 4_s\longrightarrow Q^a\\ 4_c\longrightarrow Q^{\dot a}\end{array}\right.$$

$$D=3,\ 4\longrightarrow\left\{\begin{array}{c}(D-2)_v\\ (D-2)_s\longrightarrow Q^a\end{array}\right.$$

Let us now construct the susygenerators algebra:
For $D=3$ and 4, and in the case of paraboson-parafermion susy system, because of these trilinear commutation relations

$$\left[p^+,\left[\alpha^i_n,\alpha^j_m\right]_+\right]=0 \tag{3.1}$$

$$\left[p^+,\left[s^a_n,s^b_m\right]_-\right]=0 \tag{3.2}$$

$$\left[p^+,\left[p_i,p_i\right]_+\right]=0 \tag{3.3}$$

the susy generators are given by this expression

$$Q^a=\frac{i}{2}\left[\left(p^+\right)^{\frac{1}{2}},\left(\gamma^+s_0\right)^a\right]_++\frac{i}{2}\left(p^+\right)^{\frac{-1}{2}}\sum_{n=-\infty}^{\infty}\left[\left(\gamma_is_{-n}\right)^a,\alpha^i_n\right]_+ \tag{3.4}$$

one can then prove this relation for $D=4$

$$\left[Q^a,\overline{Q}^b\right]_+=-2\left(h\gamma_\mu p^\mu\right)^{ab} \tag{3.5}$$

and this one for $D=3$

$$\left[Q,\overline{Q}\right]_+=-2\gamma_\mu p^\mu \tag{3.6}$$

which correspond to the algebra of the supersymmetric quantum mechanic (SSQM).
Now, in the boson parafermion case, described by these combined relations

$$\left[s_n^a\ ,\left[s_m^b\ ,s_l^c\ \right]_-\right] = 2\left(\delta^{ab}\delta_{n+m}s_l^c\ -\delta^{ac}\delta_{n+l}s_m^b\ \right) \tag{3.7}$$

$$\left[s_n^a\ ,\left[s_m^b\ ,A\right]_+\right]_+ = 2\delta^{ab}\delta_{n+m}A\ (A\neq s_l^c) \tag{3.8}$$

$$\left[\alpha_n^i,\alpha_m^j\right] = n\delta^{ij}\delta_{n+m,0} \tag{3.9}$$

$$\left[x^i,p^j\right] = i\delta^{ij} \tag{3.10}$$

$$\left[x^-,p^+\right] = \imath \tag{3.11}$$

the supercharges are given by this expression

$$Q^a = i\left(p^+\right)^{\frac{1}{2}}\left(\gamma^+ s_0\right)^a\ +i\left(p^+\right)^{\frac{-1}{2}}\sum_{n=-\infty}^{\infty}\left(\gamma_i s_{-n}\right)^a\alpha_n^i \tag{3.12}$$

so that we can prove this trilinear relation between the susygenerators

$$\left[Q^a\ ,\left[\overline{Q}^b\ ,\overline{Q}^c\right]_-\right] = 2\left[-2\left(h\gamma_\mu p^\mu\right)^{ab}\overline{Q}^c+2\left(h\gamma_\mu p^\mu\right)^{ac}\overline{Q}^b\right] \tag{3.13}$$

which correspond to the algebra of the paraSSQM in the sense of Becker and Debergh.

In the same way, for $D=6$, and in the paraboson-parafermion case, the two supercharges are given by these relations

$$Q^a = \frac{1}{2}\left[\left(2p^+\right)^{\frac{1}{2}},s_0^a\right]_+ \tag{3.14}$$

$$Q^{\dot a} = \left(p^+\right)^{\frac{-1}{2}}\gamma_{\dot a a}^i\sum_{n=-\infty}^{\infty}\frac{1}{2}\left[s_{-n}^a,\alpha_n^i\right]_+ \tag{3.15}$$

and one can demonstrate that they satisfy exactly the same algebra as in the ordinary case ($D=10$).

$$\left[Q^a,Q^b\right]_+ = 2p^+\delta^{ab} \tag{3.16}$$

$$\left[Q^{\dot a},Q^{\dot b}\right]_+ = 2\delta^{\dot a\dot b}H \tag{3.17}$$

$$\left[Q^a,Q^{\dot a}\right]_+ = \sqrt{2}\gamma_{a\dot a}^i p^i \tag{3.18}$$

where the Hamiltonian is given by

$$H = \frac{1}{p^+}\left\{\frac{1}{2}\sum_{n=1}^{\infty}\left(\left[\alpha_{-n}^i,\alpha_n^i\right]_+ + n\left[s_{-n}^a\ ,s_n^a\ \right]_-\right)+\frac{1}{2}p_i^2\right\} \tag{3.19}$$

However, in the boson-parafermion case, the supercharges Q^a and $Q^{\dot a}$ defined by

$$Q^a = \left(2p^+\right)^{\frac{1}{2}}s_0^a \tag{3.20}$$

$$Q^{\dot a} = \left(p^+\right)^{\frac{-1}{2}}\gamma_{\dot a a}^i\sum_{n=-\infty}^{\infty}s_{-n}^a\alpha_n^i \tag{3.21}$$

satisfy a parasupersymmetric algebra described by these six trilinear commutation relations

$$\left[Q^a\ ,\left[Q^b\ ,Q^c\right]_-\right] = 2\left(2p^+\right)\left[\delta^{ab}Q^c\ -\delta^{ac}Q^b\right] \tag{3.22}$$

$$\left[Q^a\ ,\left[Q^b\ ,Q^{\dot a}\right]_-\right] = 2\left(2p^+\delta^{ab}Q^{\dot a}-\sqrt{2}\gamma_{a\dot a}^i p^i Q^b\right) \tag{3.23}$$

$$\left[Q^{\dot a}\ ,\left[Q^a\ ,Q^b\right]_-\right] = 2\left(\sqrt{2}\gamma_{a\dot a}^i p^i Q^b-\sqrt{2}\gamma_{b\dot a}^j p^j Q^a\right) \tag{3.24}$$

$$\left[Q^{\dot a}\ ,\left[Q^{\dot b}\ ,Q^{a}\right]_{-}\right]=2\left(2\delta^{\dot a\dot b}HQ^{a}-\sqrt{2}\gamma^{i}_{a\dot a}p^{i}Q^{\dot b}\right)\tag{3.25}$$

$$\left[Q^{a}\ ,\left[Q^{\dot a}\ ,Q^{b}\right]_{-}\right]=2\left(\sqrt{2}\gamma^{i}_{a\dot a}p^{i}Q^{b}-\sqrt{2}\gamma^{i}_{a\dot b}p^{j}Q^{\dot a}\right)\tag{3.26}$$

$$\left[Q^{\dot a}\ ,\left[Q^{\dot b}\ ,Q^{\dot c}\right]_{-}\right]=2\left(2H\delta^{\dot a\dot b}Q^{\dot c}-2H\delta^{\dot a\dot c}Q^{\dot b}\right)\tag{3.27}$$

where H is the Hamiltonian given by

$$H=\frac{1}{2p^{+}}\left\{\sum_{n=-\infty}^{\infty}\left(\alpha^{i}_{-n}\alpha^{i}_{n}+\frac{1}{2}n\left[s^{a}_{-n}\ ,s^{a}_{n}\ \right]_{-}\right)\right\}\tag{3.28}$$

In connection with some works on the extension of the SSQM to the paraSSQM (see for example [5, 6, 7] and the references herein), one can see that these results obey to what we find in these literature.

IV. SPECTRUM

In this part, we proceed to analyse the spectrum. To do this and for the space-time dimensions D=3, 4, 6, we construct, on one hand the four first levels of physical states and on the other hand, the partition functions. First, by the use of both the trilinear relations and Green decomposition, we calculate the masses of the first levels of the physical states and verify that the mass didn't depend on the space-time dimensions, in the same time, this suggests a general form of these physical states. One can then write all the physical states in the first, second, third and fourth levels and determine their numbers. For example, the fourth level is described by these states.(applied on the fundamental states $|0\rangle$

$(c.1)$	$\frac{1}{4}\left[\alpha^{i}_{-1},\alpha^{j}_{-1}\right]_{+}\left[\alpha^{k}_{-1},\alpha^{l}_{-1}\right]_{+}$	$(c.11)$	$\frac{1}{2}s^{a}_{-2}\left[\alpha^{i}_{-1},s^{b}_{-1}\right]_{+}$
$(c.2)$	$\frac{1}{2}s^{a}_{-1}\left(\alpha^{i}_{-1}\left[\alpha^{j}_{-1},\alpha^{k}_{-1}\right]_{+}\right)$	$(c.12)$	$s^{a}_{-2}\left\langle s^{b}_{-1},s^{c}_{-1}\right\rangle$
$(c.3)$	$\frac{1}{2}\left[\alpha^{i}_{-1},\alpha^{j}_{-1}\right]_{+}\left\langle s^{a}_{-1},s^{b}_{-1}\right\rangle$	$(c.13)$	$\alpha^{i}_{-2}\left\langle s^{a}_{-1},s^{b}_{-1}\right\rangle$
$(c.4)$	$\alpha^{i}_{-1}\left\langle s^{a}_{-1},s^{b}_{-1},s^{c}_{-1}\right\rangle$	$(c.14)$	$\left\langle s^{a}_{-2},s^{b}_{-2}\right\rangle$
$(c.5)$	$\left\langle s^{a}_{-1},s^{b}_{-1},s^{c}_{-1},s^{d}_{-1}\right\rangle$	$(c.15)$	$\frac{1}{2}\left[s^{a}_{-3},s^{b}_{-1}\right]_{-}$
$(c.6)$	$\frac{1}{2}\alpha^{i}_{-2}\left[\alpha^{j}_{-1},\alpha^{k}_{-1}\right]_{+}$	$(c.16)$	$\frac{1}{2}\left[s^{a}_{-3},\alpha^{i}_{-1}\right]_{+}$
$(c.7)$	$\frac{1}{2}\left[\alpha^{i}_{-2},\alpha^{j}_{-2}\right]_{+}$	$(c.17)$	$\frac{1}{2}\left[\alpha^{i}_{-3},s^{a}_{-1}\right]_{+}$
$(c.8)$	$\frac{1}{2}\alpha^{i}_{-2}\left[\alpha^{j}_{-1},s^{a}_{-1}\right]_{+}$	$(c.18)$	$\frac{1}{2}\left[\alpha^{i}_{-3},\alpha^{j}_{-1}\right]_{+}$
$(c.9)$	$\frac{1}{2}s^{a}_{-2}\left[\alpha^{i}_{-1},\alpha^{j}_{-1}\right]_{+}$	$(c.19)$	s^{a}_{-4}
$(c.10)$	$\frac{1}{2}\left[\alpha^{i}_{-2},s^{a}_{-2}\right]_{+}$	$(c.20)$	α^{i}_{-4}

$|0\rangle$ means $|i\rangle$ or $|a\rangle$ where:

- $|i\rangle$: represent the $(D-2)$ physical transverse polarizations of a massless vector field
- $|a\rangle$: represent the $(D-2)$ components spinor partner

and where we have introduced the notation for the product of parafermionic operators:

$$\left\langle s^{a}_{-1},s^{b}_{-1}\right\rangle\rightarrow\left\{\frac{1}{2}\left[s^{a}_{-1},s^{b}_{-1}\right]_{-},\left(s^{a}_{-1}\right)^{2}\right\}$$

$$\left\langle s^{a}_{-1},s^{b}_{-1},s^{c}_{-1}\right\rangle\rightarrow\left\{\frac{1}{2}s^{a}_{-1}\left[s^{b}_{-1},s^{c}_{-1}\right]_{-},\frac{1}{2}\left[\left(s^{a}_{-1}\right)^{2},s^{b}_{-1}\right]_{-},\left(s^{a}_{-1}\right)^{3}\right\}$$

$$\left\langle s^{a}_{-1},s^{b}_{-1},s^{c}_{-1},s^{d}_{-1}\right\rangle\rightarrow\left\{\frac{1}{4}\left[s^{a}_{-1},s^{b}_{-1}\right]_{-}\left[s^{c}_{-1},s^{d}_{-1}\right]_{-},\frac{1}{2}s^{a}_{-1}\left[\left(s^{b}_{-1}\right)^{2},s^{c}_{-1}\right]_{-}\right.$$
$$\left.,\frac{1}{2}\left[\left(s^{a}_{-1}\right)^{3},s^{b}_{-1}\right]_{-},\left(s^{a}_{-1}\right)^{4}\right\}$$

then, we determine and recapitulate the number of parafermionic and parabosonic states in this table (D is the space-time dimension)

$States$	$Number$
$(c.1)$	$(D-2)+\frac{3}{2}(D-2)(D-3)+\frac{1}{2}(D-2)(D-3)(D-4)+\frac{(D-2)(D-3)(D-4)(D-5)}{4!}$
$(c.2)$	$\left[(D-2)+(D-2)(D-3)+\frac{1}{3!}(D-2)(D-3)(D-4)\right](D-2)$
$(c.3)$	$\left[(D-2)+\frac{1}{2}(D-2)(D-3)\right]\left[(D-2)+\frac{1}{2}(D-2)(D-3)\right]$
$(c.4)$	$\begin{cases} (D-2)\left[(D-2)+(D-2)(D-3)+\frac{1}{3!}(D-2)(D-3)(D-4)\right] & Q\neq 1,2 \\ 64 \quad for & Q=2 \end{cases}$
$(c.5)$	$\begin{cases} (D-2)+\frac{3}{2}(D-2)(D-3)+\frac{1}{2}(D-2)(D-3)(D-4) & \\ +\frac{1}{4!}(D-2)(D-3)(D-4)(D-5) & Q\neq 1,2 \\ 19 \quad for & Q=2 \end{cases}$
$(c.6)$	$(D-2)\left[(D-2)+\frac{1}{2}(D-2)(D-3)\right]$
$(c.7)$	$(D-2)+\frac{1}{2}(D-2)(D-3)$
$(c.8)$	$(D-2)^3$
$(c.9)$	$(D-2)\left[(D-2)+\frac{1}{2}(D-2)(D-3)\right]$
$(c.10)$	$(D-2)^2$
$(c.11)$	$(D-2)^3$
$(c.12)$	$(D-2)\left[(D-2)+\frac{1}{2}(D-2)(D-3)\right]$
$(c.13)$	$(D-2)\left[(D-2)+\frac{1}{2}(D-2)(D-3)\right]$
$(c.14)$	$(D-2)+\frac{1}{2}(D-2)(D-3)$
$(c.15)$	$(D-2)^2$
$(c.16)$	$(D-2)^2$
$(c.17)$	$(D-2)^2$
$(c.18)$	$(D-2)^2$
$(c.19)$	$(D-2)$
$(c.20)$	$(D-2)$

one can then recapitulate the degeneration of each level for the dimensions 3, 4, 6 and 10 in the next table

V. PARTITION FUNCTION

Now, and on the other hand, let us construct the partition function. To do this, we adopt the same procedure as in the ordinary case. We treat separately the parabosonic and parafermionic cases and postulate the parastrings partition function as follows

$$f(x)=Tr\,x^N=Tr\,x^{\frac{1}{2}\sum_{n=1}^{\infty}\left(\left[\alpha_{-n}^i,\alpha_n^i\right]_+ +n\left[s_{-n}^a\ ,s_n^a\ \right]_-\right)}$$

by the use of the precedent trilinear relations, one obtain the general form of the partition function where Q is the order of the paraquantization:

$$f(x)=2(D-2)\prod_{n=1}^{\infty}\left(\frac{1+x^n+x^{2n}+..........+x^{nQ}}{1-x^{2n}}\right)^{D-2}$$

Finally, in this table, one can see the coherence between the coefficients in the development of the partition function and the degeneration of the fourth first levels for different values of D

$\alpha' M^2$	0	1	2	3	4	Partition Functions
$D=3$	2	4	10	20	40	$2+4x+10x^2+20x^3+40x^4.....$
$D=4$	4	16	56	160	420	$4+16x+56x^2+160x^3+420x^4....$
$D=6$	8	64	342	1504	5552	$8+64x+352x^2+1504x^3+5552x^4...$
$D=10$	16	256	2304	15360	84224	$16+256x+2304x^2+15360x^3+84224x^4...$

VI. HETEROTIC PARASTRINGS

The superstrings theory presents some anomalies. In the ordinary case $D = 10$, these anomalies are canceled for the only groups $SO(32)$ or $E_8 \otimes E_8$. D.J.Gross and J.A.Harvey [8, 9] have introduced these two groups in the new hybrid theory of superstrings called heterotic. In the heterotic parasuperstring case, one show that, in addition to the ordinary case (26,10) , only survives the case (14,6) which is constructed from the only possible group E_8.To do this, one can first notice that, the compactification is from $D = 2 + \frac{24}{Q}$ to $D' = 2 + \frac{8}{Q}$, the lattice dimension is then $(D - D')$. The order of the paraquantization must be the same for the left movers of the parabosonic strings and the right movers for the parasuperstrings, so that the possible cases are:

$$\begin{aligned} Q = 1 &\longrightarrow (D, D') = (26, 10) \\ Q = 2 &\longrightarrow (D, D') = (14, 6) \\ Q = 4 &\longrightarrow (D, D') = (8, 4) \\ Q = 8 &\longrightarrow (D, D') = (5, 3) \end{aligned}$$

Now, like as the parasuperstrings, one can construct the formalism of the heterotic parastring. One important point here is the modular invariance of a one loop-amplitude of four particles which will determine the properties of the lattice Λ. From the one-loop amplitude in the ordinary case [10] and the one given by Ardalan and Mansouri [11] in the parabosonic strings case, the one-loop amplitude in the paraquantum formalism is given by the following expression:

$$A_{Loop} \sim \int \prod_{i=1}^{4} d^2 z_i \, |\omega|^{-2} \left[-\frac{4\pi}{\ln|\omega|} \right]^{\frac{Q(D'-2)+2}{2}} \times \prod_{1 \leq i \leq 4} \left[\chi(C_{JI}, \omega) \right]^{\frac{1}{2} k_i k_j} B(\overline{\omega}, \overline{z}_i, K_i) \tag{6.1}$$

with

$$B(\overline{\omega}, \overline{z}_i, K_i) = \overline{\omega}^{-1} f(\overline{\omega})^{-Q(D-2)} \left[\psi(\overline{C}_{JI}, \overline{\omega}) \right]^{K_i K_j} L(\overline{\omega}, \overline{z}_i, K_i) \tag{6.2}$$

and

$$\chi(z, \omega) = \exp\left[\frac{\ln^2 |z|}{2 \ln \overline{\omega}} \right] \left| \frac{1-z}{\sqrt{z}} \prod_{m=1}^{\infty} \frac{(1 - \omega^m z)\left(1 - \frac{\omega^m}{z}\right)}{(1 - \omega^m)^2} \right| \tag{6.3}$$

$$\psi(\overline{z}, \overline{\omega}) = \exp\left[\frac{\ln^2 \overline{z}}{2 \ln \overline{\omega}} \right] \frac{1-\overline{z}}{\sqrt{\overline{z}}} \prod_{m=1}^{\infty} \left(\frac{(1 - \overline{\omega}^m \overline{z})\left(1 - \frac{\overline{\omega}^m}{\overline{z}}\right)}{(1 - \overline{\omega}^m)^2} \right) \tag{6.4}$$

$$L(\overline{\omega}, \overline{z}_i, K_i) = \sum_{P \in \Lambda} \exp\left[\frac{1}{2} \ln \overline{\omega} \left(P - \sum_{i=1}^{4} \frac{\ln \overline{z}_i}{\ln \overline{\omega}} Q_i \right)^2 \right] \tag{6.5}$$

and

$$Q_i = \sum_{j=1}^{i-1} K_i \tag{6.6}$$

$$\nu_i = \sum_{j=1}^{i} \frac{\ln z_i}{2\pi i} \tag{6.7}$$

$$\tau = \frac{\ln \omega}{2\pi i} \tag{6.8}$$

$$C_{ji} = z_i z_{i+1} ....z_j \tag{6.9}$$

$$f(\omega) = \prod_{m=1}^{\infty} (1 - \omega^m) \tag{6.10}$$

This amplitude must be modular invariant, then, we consider the modular transformations T and S defined by

$$T : \tau \longrightarrow \tau + 1$$
$$S : \tau \longrightarrow -\frac{1}{\tau}$$

The periodicity of the closed string conduct to an even lattice, after a tedious calculation, the modular invariance in this case requires that the lattice must be self-dual, in addition to the space-time condition given by $D' = 2 + \frac{8}{Q}$. So:

$$Modular\ invariance \iff \begin{cases} \Lambda = \Lambda^* & self-dual\ lattice \\ D' = 2 + \dfrac{8}{Q} \end{cases} \tag{6.11}$$

The even and self-dual lattices exist only for $D = 8n$ with n integer, then the only possibilities for (D, D') are :

$$Q = 1 \longrightarrow (D, D') = (26, 10) \qquad \text{(ordinary case)}$$
$$Q = 2 \longrightarrow (D, D') = (14, 6)$$

The only group which describe the eight dimensions in an even and-self dual lattice is the exceptional group E_8

VII. GENERATORS SUSY ALGEBRA

In the same way like in the parasuperstrings theory, we define the parasupercharges in this case by :

$$Q^a = \frac{i}{2} \left[(p^+)^{\frac{1}{2}}, (\gamma^+ s_0)^a \right]_+ + i\,(p^+)^{\frac{-1}{2}} \sum_{n=-\infty}^{\infty} \left[(\gamma_i s_{-n})^a \alpha_n^i \right]_+ \tag{7.1}$$

these generators obey to an ordinary supersymmetric quantum mechanic algebra, i.e to the following ordinary anticommutation relations

$$\left[Q^a, \overline{Q}^b \right]_+ = -2\,(h\gamma_\mu p^\mu)^{ab} \tag{7.2}$$

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