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On a Graded q-Differential Algebra

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Abstract

We construct the structure of a graded q-differential algebra on a $\mathbf{Z}_N$-graded algebra by means of a graded q-commutator. We apply this construction to a reduced quantum plane and study the first order differential calculus on a reduced quantum plane induced by the N-differential of the graded q-differential algebra.

I. GRADED Q-DIFFERENTIAL ALGEBRA

In this section given a $\mathbf{Z}_N$-graded algebra we construct the graded q-differential algebra. Let us remind the definition of a graded q-differential algebra ([1]). A unital associative algebra is said to be a graded q-differential algebra ($q \in \mathbf{C}, q \neq 1$)if it is a $\mathbf{Z}_N$-graded (or $\mathbf{Z}$-graded)algebra endowed with the linear mapping d of degree $+1$ satisfying the graded q-Leibniz rule and $d^N = 0$ in the case when q is a primitive N-th root of unity. The linear mapping d is called an N-differential of a graded q-differential algebra.

Let $\mathcal{A}$ be an associative unital $\mathbf{Z}$ (or $\mathbf{Z}_N$)-graded algebra over the complex numbers $\mathbf{C}$ and $\mathcal{A}^k \subset \mathcal{A}$ be the subspace of homogeneous elements of a grading k. The grading of a homogeneous element w will be denoted by $|w|$, which means that if $w \in \mathcal{A}^k$ then $|w| = k$. Let q be a complex number such that $q \neq 1$. The q-commutator of two homogeneous elements $w, w' \in \mathcal{A}$ is defined by the formula

$$[w, w']_q = ww' - q^{|w||w'|} w'w. \tag{1.1}$$

Using the associativity of an algebra $\mathcal{A}$ and the property $|ww'| = |w| + |w'|$ of its graded structure it is easy to show that for any homogeneous elements $w, w', w'' \in \mathcal{A}$ the q-commutator has the property

$$[w, w'w'']_q = [w, w']_q w'' + q^{|w||w'|} w'[w, w'']_q. \tag{1.2}$$

Given an element $v \in \mathcal{A}^1$ one can define the mapping $d_v : \mathcal{A}^k \to \mathcal{A}^{k+1}$ by the following formula

$$d_v w = [v, w]_q, \qquad w \in \mathcal{A}^k.$$

It follows from the property of q-commutator (1.2) that d_v is a graded q-differential on an algebra $\mathcal{A}$, i.e. it is a homogeneous linear mapping of degree 1 satisfying the graded q-Leibniz rule

$$d_v(ww') = d_v(w)w' + q^{|w|}wd_v(w'), \tag{1.3}$$

where w, w' are the homogeneous elements of $\mathcal{A}$.

Lemma 1.1 *For any integer $k \geq 2$ the k-th power of the q-differential d_v can be written as follows*

$$d_v^k w = \sum_{i=0}^{k} p_i^{(k)} v^{k-i} w v^i, \tag{1.4}$$

where w is a homogeneous element of $\mathcal{A}$ and

$$p_i^{(k)} = (-1)^i q^{|w|_i} \frac{[k]_q!}{[i]_q![k-i]_q!} = (-1)^i q^{|w|_i} \begin{bmatrix} k \\ i \end{bmatrix}_q,$$

$$|w|_i = i|w| + \frac{i(i-1)}{2}.$$

This lemma can be proved by means of the mathematical induction and the following identities

$$p_0^{(k)} = p_0^{(k+1)} = 1, \quad p_{k+1}^{(k+1)} = -q^{|w|+k} p_k^{(k)},$$

$$p_i^{(k+1)} = p_i^{(k)} - q^{|w|+k} p_{i-1}^{(k)}, \quad 1 \leq i \leq k.$$

Theorem 1.1 *If N is an integer such that $N \geq 2$, $\mathcal{A}$ is a $\mathbf{Z}_N$-graded algebra, q is a primitive N-th root of unity and $v^N = \alpha e$, where $\alpha \in \mathbf{C}$ and e is the unity element of an algebra $\mathcal{A}$, then $d_v^N w = 0$ for any $w \in \mathcal{A}$.*

It follows from the Lemma 1.1 that if q is a primitive N-th root of unity then for any integer $l = 1, 2, \ldots, N-1$ the coefficient $p_l^{(N)}$ contains the factor $[N]_q$ which is equal to zero in the case of q being a primitive N-th root of unity and this implies $p_l^{(N)} = 0$. Thus $d_v^N(w) = v^N w + (-1)^N q^{|w|_N} w v^N$. Taking into account that $v^N = \alpha e$ we obtain $d_v^N(w) = (1 + (-1)^N q^{|w|_N})\alpha w$. The first factor in the right-hand side of the above formula equals to zero. Indeed if N is an odd number then $1 - (q^N)^{\frac{N-1}{2}} = 0$. In the case of an even integer N we have $1 + (q^{\frac{N}{2}})^{N-1} = 1 + (-1)^{N-1} = 0$, and this ends the proof.

Let $\mathcal{A}$ be an associative unital $\mathbf{Z}_N$-graded algebra over the complex numbers $\mathbf{C}$ with unit element denoted by e. Then from the property (1.3) and the Theorem 1.1 it follows

Corollary. *If there exists an element $v \in \mathcal{A}$ of grading 1 such that $v^N = \alpha e, \alpha \in \mathbf{C}$ then an algebra $\mathcal{A}$ endowed with the homogeneous linear mapping $d_v : \mathcal{A}^k \to \mathcal{A}^{k+1}$ of degree +1, defined by $d_v w = [v, w]_q$, where $w \in \mathcal{A}$, and q is a primitive N-th root of unity, is a $\mathbf{Z}_N$-graded q-differential algebra and d_v is its N-differential.*

Let us remind that a first order differential calculus over an associative unital algebra $\mathcal{B}$ is a pair $(\mathcal{M}, d)$, where $\mathcal{M}$ is a $(\mathcal{B}, \mathcal{B})$-bimodule and d is a linear mapping $d : \mathcal{B} \to \mathcal{M}$ which satisfies the Leibniz rule $d(ww') = d(w)w' + wd(w')$, where $w, w' \in \mathcal{B}$. The subspace $\mathcal{A}^0$ of elements of grading zero of a $\mathbf{Z}_N$-graded algebra $\mathcal{A}$ is a subalgebra, and N-differential d_v restricted to this subalgebra induces a first order differential calculus $(\mathcal{A}^1, d_v)$ where the space $\mathcal{A}^1$ of elements of grading 1 has a $(\mathcal{A}^0, \mathcal{A}^0)$-bimodule structure. Indeed it follows from the associativity of the algebra $\mathcal{A}$ and its $\mathbf{Z}_N$-graded structure that for each k the mappings $\mathcal{A}^0 \times \mathcal{A}^k \to \mathcal{A}^k$ and $\mathcal{A}^k \times \mathcal{A}^0 \to \mathcal{A}^k$ determined by the algebra multiplication $(r, w) \to rw$, $(w, s) \to ws$, where $w \in \mathcal{A}^k$ and $r, s \in \mathcal{A}^0$, induce a $(\mathcal{A}^0, \mathcal{A}^0)$-bimodule structure on $\mathcal{A}^k$. In the next section we consider a reduced quantum plane from a point of view of graded q-differential algebra and study the first order differential calculus induced by the N-differential.

II. REDUCED QUANTUM PLANE AS A Q-DIFFERENTIAL ALGEBRA

An exterior calculus with exterior differential d satisfying $d^N = 0$ has been studied in ([2].[3],[4]). In this section we construct and study this kind of exterior calculus on a reduced quantum plane with the help of the construction described in the previous section. Let us remind that the unital associative algebra $\mathbf{C}_{rq}$ generated, over the complex numbers $\mathbf{C}$, by the two variables x and y satisfying the relations $x\,y = q\,y\,x,\ x^N = y^N = \mathbf{1}$, where q is a primitive N-th root of unity and $\mathbf{1}$ is the unity element of $\mathbf{C}_{rq}$, can be considered as an algebra of polynomials over a reduced quantum plane. Let us mention that this algebra has a representation by $N \times N$ complex matrices.

The set of monomials $B = \{\mathbf{1}, y, x, x^2, yx, y^2, \ldots, y^k x^l, \ldots, y^{N-1}x^{N-1}\}$ can be taken as the basis of the vector space of the algebra $\mathbf{C}_{rq}$. Having chosen the basis B we can endow this vector space with a $\mathbf{Z}_N$-graded structure as follows: if a polynomial $w \in \mathbf{C}_{rq}$ written in terms of the monomials in the basis B has the form

$$w = \sum_{l=0}^{N-1} \beta_l y^k x^l, \quad \beta_l \in \mathbf{C}, \quad k \in \mathbf{N}, \tag{2.1}$$

then we shall refer to it as the homogeneous polynomial of grading k, where $k \in \mathbf{Z}_N$. Let us denote the grading of a homogeneous polynomial w by $|w|$ and the subspace of the homogeneous polynomials of grading k by $\mathbf{C}^k_{rq}$. It is obvious that

$$\mathbf{C}_{rq} = \mathbf{C}^0_{rq} \oplus \mathbf{C}^1_{rq} \oplus \ldots \oplus \mathbf{C}^{N-1}_{rq}. \tag{2.2}$$

In particular a polynomial r of grading zero has the form

$$r = \sum_{l=0}^{N-1} \beta_l x^l, \qquad \beta_l \in \mathbf{C}, r \in \mathbf{C}^0_{rq}. \tag{2.3}$$

It is easy to show that $\mathbf{Z}_N$-graded structure defined by (2.2)on a vector space $\mathbf{C}_{rq}$ is consistent with the algebra structure of $\mathbf{C}_{rq}$, i.e. for any two homogeneous polynomials we have $|ww'| = |w| + |w'|$. Consequently $\mathbf{C}_{rq}$ is a Z_N-graded algebra with respect to (2.2), and there exists an element v of grading one of this algebra satisfying $v^N = \alpha \cdot \mathbf{1}$, where $\alpha \in \mathbf{C}$. Indeed one can take for instance $v = y$ which satisfies all mentioned above conditions. According to the first section we can endow a reduced quantum plane with the structure of a graded q-differential algebra defining the N-differential by the formula $d_v w = [v, w]_q$, where q is a primitive N-th root of unity and $w \in \mathbf{C}_{rq}$.

As it was mentioned previously the subspace $\mathbf{C}^0_{rq}$ of elements of grading zero is a subalgebra of the algebra $\mathbf{C}_{rq}$. From a point of view of differential geometry we can interpret the generator x as a coordinate of a one-dimensional space, the subalgebra $\mathbf{C}^0_{rq}$ as an algebra of (polynomial) functions or differential forms of degree 0 on this one-dimensional space. Let us remind that the subalgebra $\mathbf{C}^0_{rq}$ is a commutative algebra generated by x satisfying the single relation $x^N = \mathbf{1}$.

The subspace $\mathbf{C}^k_{rq}$ of polynomials of grading k is a bimodule over the algebra of functions $\mathbf{C}^0_{rq}$. If we put a homogeneous polynomial w of grading k to the form

$$w = y^k \sum_{l=0}^{N-1} \beta_l x^l = y^k\, r, \quad r = \sum_{l=0}^{N-1} \beta_l x^l \in \mathbf{C}^0_{rq}, \tag{2.4}$$

and take into account that the polynomial $r = (y^k)^{-1}w = y^{N-k}w$ is uniquely determined then we can conclude that $\mathbf{C}^k_{rq}$ is a free right module over $\mathbf{C}^0_{rq}$ generated by y^k. Thus extending our differential-geometric interpretation to the whole algebra $\mathbf{C}_{rq}$ we can interpret the bimodule $\mathbf{C}^k_{rq}$ as a module of differential forms of degree k over the algebra of functions $\mathbf{C}^0_{rq}$ and d_v as an exterior differential.

It is well known that a bimodule structure on a free right module is uniquely determined by the homomorphism of the corresponding algebra, and in our case this means that there exists a homomorphism $A_k : \mathbf{C}^0_{rq} \to \mathbf{C}^0_{rq}$ such that

$$r\, y^k = y^k\, A_k(r), \qquad r \in \mathbf{C}^0_{rq}. \tag{2.5}$$

It is easy to find that $A_k(x) = q^k\,x$ and $A_k = A_1^k, A_0 = I$ where $I : \mathbf{C}^0_{rq} \to \mathbf{C}^0_{rq}$ is the identity mapping. Thus we have the set $\{A_k\}_{k=0}^{N-1}$ of homomorphisms of the algebra $\mathbf{C}^0_{rq}$.

Since $\mathbf{C}^1_{rq}$ is a free right module over $\mathbf{C}^0_{rq}$ there exists an invertible element $u \in \mathbf{C}^0_{rq}$ such that $v = y \cdot u$. We can take v for a generator of the free right module $\mathbf{C}^1_{rq}$. Using the relation (2.5) and the commutativity of the algebra $\mathbf{C}^0_{rq}$ we find that the relation determining the bimodule structure in terms of the new generator v has the same form (**??**) as in the case of the generator y. Now we can write the polynomial $d_v w$ in the form of an element of the right module generated by v as follows

$$\begin{aligned} d_v w &= vw - wv = vw - vA_1(w) \\ &= v(w - A_1(w)) = v\,\Delta_0(w). \end{aligned}$$

where $\Delta_0 = I - A_1 : \mathbf{C}^0_{rq} \to \mathbf{C}^0_{rq}$. It is easy to check that given any polynomials $w, w' \in \mathbf{C}^0_{rq}$ the mapping Δ_0 satisfies the following properties

$$\Delta_0(ww') = \Delta_0(w)w' + A_1(w)\Delta_0(w'), \tag{2.6}$$

$$\Delta_0(x^k) = (1-q)[k]_q\,x^k. \tag{2.7}$$

This formula shows that $d_v x$ can be taken as a generator of the right module $\mathbf{C}^1_{rq}$.

It is well known[5] that a differential on a unital associative algebra induces the right partial derivatives which satisfy a generalized Leibniz rule. In our case we have only one derivative $\partial : \mathbf{C}^0_{rq} \to \mathbf{C}^0_{rq}$ which is defined by the formula $d_v w = d_v x\ \partial w,\ \forall w \in \mathbf{C}^0_{rq}$. The explicit formula for the partial derivative has the form $\partial w = (1-q)^{-1} x^{N-1} \Delta_0(w)$. It follows immediately from (2.6),(2.7) that this derivative satisfies the generalized Leibniz rule $\partial(ww') = \partial(w) \cdot w' + A_1(w) \cdot \partial(w')$ and $\partial x^k = [k]_q\,x^{k-1}$.

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III. REFERENCES

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