African Journal Of Mathematical Physics Volume **4***(2007)11-18*

Stability of Some Solitonic Systems with Real Scalar Fields

J. Sadeghi [a,b,*] and A. Mohammadi [b,†]

a. Institute for Studies in Theoretical Physics and Mathematics (IPM),
P .O .Box 19395-5531, Tehran, Iran
b. Sciences Faculty, Department of Physics, Mazandaran University,
P .O .Box 47415-416, Babolsar, Iran
* pouriya@ipm.ir, † a.mohammadi@umz.ac.ir

Abstract

In this article we consider the stability of the solitons in certain systems of scalar fields in 1+1 dimensions. We obtain the soliton solution by solving sets of first order differential equations. We also study the stability of these solutions with the help of Jacobi Polynomials.
Keywords: Solitons, Linear Stability, Jacobi Polynomial.

I. INTRODUCTION

Classical finite - energy solutions of field theory are generally called solitons [1-2]. As it turns out, the interactions among the various fields involved are nonlinear. Therefore in Lagrangian formulation of the problem one is encountered with the task of solving sets of second order nonlinear differential equations of motion.

To simplify things we consider the case of real scalar fields in 1+1 dimensions. We also utilize the method of Refs. [3-7] where they consider a more restricted class of systems of real scalar fields. For these systems one has to solve sets of first order nonlinear differential equations to obtain the soliton solutions.

In order to discuss the stability of systems, one usually has to consider second order operators. In Ref [7] a method is introduced that one uses first order operators instead of second order operators. This will make calculations much easier.

The plan of this article is as follows. In section 2 we discuss a general system of real scalar fields. For a specific class of systems, sets of first order equations are obtained. In section 3, first we introduce the first order operators that will make the treatment of linear stability much easier. We also find the soliton solutions for some models in field theories. As we know, the associated Jacobi equation in form of first order is well known . So with the help of first order operators and Jacobi polynomials we study the stability of these solutions. Finally in section 4 we present our conclusions.

II. SYSTEMS OF REAL SCALAR FIELDS

In order to discuss the systems of scalar fields in 1+1 dimensions, we consider the following Lagrangian density

$$l = \frac{1}{2}\partial_\alpha \Phi \partial^\alpha \Phi + \frac{1}{2}\partial_\alpha \chi \partial^\alpha \chi - U(\Phi, \chi) \tag{2.1}$$

where $U(\Phi, \chi)$ is non- linear function of two fields Φ and χ, the unit system $c = \hbar = 1$ is chosen and $x_\alpha = (t, -x)$, $x^\alpha = (t, x)$. Here we consider some classes of potentials first introduced in Refs. [1-4]. These potentials have the following form,

$$U(\Phi, \chi) = \frac{1}{2}W_\Phi^2 + \frac{1}{2}W_\chi^2, \tag{2.2}$$

where $W = W(\Phi, \chi)$ is a smooth arbitrary function of fields Φ and χ. In here we can write the form of W_Φ and W_χ

$$W_\Phi = \frac{\partial W}{\partial \Phi} \qquad W_\chi = \frac{\partial W}{\partial \chi}.$$

Also from Euler - Lagrange equations in 1+1 dimension we have,

$$\frac{\partial^2 \Phi}{\partial t^2} - \frac{\partial^2 \Phi}{\partial x^2} + W_\Phi W_{\Phi\Phi} + W_\chi W_{\Phi\chi} = 0, \tag{2.3}$$

and

$$\frac{\partial^2 \chi}{\partial t^2} - \frac{\partial^2 \chi}{\partial x^2} + W_\Phi W_{\chi\Phi} + W_\chi W_{\chi\chi} = 0. \tag{2.4}$$

For the static case we have,

$$\frac{d^2\Phi}{dx^2} = W_\Phi W_{\Phi\Phi} + W_\chi W_{\Phi\chi}, \tag{2.5}$$

and

$$\frac{d^2\chi}{dx^2} = W_\Phi W_{\chi\Phi} + W_\chi W_{\chi\chi}. \tag{2.6}$$

The energy of static field configurations is ,

$$E = \frac{1}{2}\int_{-\infty}^{\infty} dx \left[(\frac{d\Phi}{dx})^2 + (\frac{d\chi}{dx})^2 + 2U(\Phi, \chi) \right]. \tag{2.7}$$

In general we can write the structure of static energy as $E = E_M + E'$, where

$$E' = \frac{1}{2}\int_{-\infty}^{\infty} \left[(\frac{d\Phi}{dx} - W_\Phi)^2 + (\frac{d\chi}{dx} - W_\chi)^2 \right] dx \tag{2.8}$$

and E_M becomes

$$E_M = W\left(\Phi(\infty), \chi(\infty)\right) - W\left(\Phi(-\infty), \chi(-\infty)\right) \tag{2.9}$$

If we consider the following expressions,

$$\frac{d\Phi}{dx} = W_\Phi, \qquad \frac{d\chi}{dx} = W_\chi \tag{2.10}$$

we will have $E' = 0$ and $E_M = E$, in this case the energy is a lower bound. This solution is known as BPS solution. We notice that expressions (2.10) are equivalent to equations (2.5) and (2.6). Therefore in our analysis, we utilize these first order equations instead of equations (2.5) and (2.6).

III. LINEAR STABILITY

In order to discuss the classical linear stability we consider the following equations,

$$\Phi(x,t) = \bar{\phi}(x) + \eta_n(x)\cos\omega_n t, \tag{3.1}$$

and

$$\chi(x,t) = \bar{\chi}(x) + \xi_n(x)\cos\omega_n t, \tag{3.2}$$

where $\bar{\phi}(x)$ and $\bar{\chi}(x)$ are static solutions of first order equations. Inserting equation (3.1) and (3.2) in equations (2.3), (2.4) and (2.10), we obtain,

$$\begin{pmatrix} -\frac{d^2}{dx^2} + V_{11} & V_{12} \\ V_{21} & -\frac{d^2}{dx^2} + V_{22} \end{pmatrix} \begin{pmatrix} \eta_n \\ \xi_n \end{pmatrix} = \omega_n^2 \begin{pmatrix} \eta_n \\ \xi_n \end{pmatrix}. \tag{3.3}$$

Equation (3.3) can be written as,

$$S_2 \begin{pmatrix} \eta_n \\ \xi_n \end{pmatrix} = \omega_n^2 \begin{pmatrix} \eta_n \\ \xi_n \end{pmatrix}, \qquad S_2 = -\frac{d^2}{dx^2} + V, \tag{3.4}$$

where V is,

$$V = \begin{pmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{pmatrix}. \tag{3.5}$$

And the matrix elements are,

$$\begin{aligned} V_{11} &= \bar{W}_{\Phi\Phi}^2 + \bar{W}_{\chi\Phi}^2 + \bar{W}_{\Phi}\bar{W}_{\Phi\Phi\Phi} + \bar{W}_{\chi}\bar{W}_{\Phi\Phi\chi} \\ V_{12} = V_{21} &= \bar{W}_{\Phi\Phi}\bar{W}_{\Phi\chi} + \bar{W}_{\chi\chi}\bar{W}_{\Phi\chi} + \bar{W}_{\Phi}\bar{W}_{\Phi\Phi\chi} + \bar{W}_{\chi}\bar{W}_{\Phi\chi\chi} \\ V_{22} &= \bar{W}_{\chi\chi}^2 + \bar{W}_{\Phi\chi}^2 + \bar{W}_{\Phi}\bar{W}_{\Phi\chi\chi} + \bar{W}_{\chi}\bar{W}_{\chi\chi\chi} \end{aligned} \tag{3.6}$$

With the help of eigenvalue equation we are able to diagonalize the second-order Schrödinger operator. This calculation to be done by finding the normal mode fluctuation. After that one can simplify the calculation by introducing first order operators S_1^+ and S_1^-, defined by

$$S_1^{\pm} = \begin{pmatrix} \pm\frac{d}{dx} + v_+ & 0 \\ 0 & \pm\frac{d}{dx} + v_- \end{pmatrix}, \tag{3.7}$$

where

$$v_{\pm} = \frac{1}{2}\left(\bar{W}_{\Phi\Phi} + \bar{W}_{\chi\chi}\right) \pm (R)^{\frac{1}{2}}, \tag{3.8}$$

with

$$R = \left[\frac{1}{4}(\bar{W}_{\Phi\Phi} - \bar{W}_{\chi\chi})^2 + \bar{W}_{\Phi\chi}^2\right]. \tag{3.9}$$

Due to equation (3.7) we note that S_1^+ and S_1^- are adjoint of each other. Now the second order operators S_2 can be decomposed as $S_2 = S_1^+S_1^-$. Moreover by using the $V_{\pm}$ we rewrite the S_2 operator as,

$$S_2^{\pm} = -\frac{d^2}{dx^2} + V_{\pm}, \qquad V_{\pm} = v_{\pm}^2 + \frac{dv_{\pm}}{dx}, \tag{3.10}$$

Now if ω be a real number, the perturbation terms in equations (3.1) and (3.2) will be oscillatory. Therefore our static solution is a stable one. On the other hand if ω be imaginary,the perturbation terms will increase exponentially and the solution will be unstable. Now we will consider some specific systems and with the help of these operators and Jacobi polynomials we study the stability of these systems. At first we consider no-interaction cases with $W_{\phi\chi} = 0$.

As a first example we consider the following system which is known as Φ^6 theory,

$$U(\Phi) = \frac{1}{2}\lambda^2\Phi^2(\Phi^2 - a^2)^2. \tag{3.11}$$

For a review of the classical solutions with orbit method and quantization of this model see Ref. [1]. However we give a new treatment of the stability. There are two topological solutions

$$\Phi^2(x) = \frac{1}{2}a^2\left[1 \pm \tanh(\lambda a^2 x)\right] \tag{3.12}$$

where positive and negative sign are kink and anti-kink, respectively. By using the equations (18) , (19) and (20) one can obtain the following expression,

$$S_2^+ = -\frac{d^2}{dx^2} + \frac{5}{2}\lambda^2 a^4 + \frac{3}{2}\lambda^2 a^4 \tanh(\lambda a^2 x) - \frac{3}{4}\lambda^2 a^4 \sec h^2(\lambda a^2 x), \tag{3.13}$$

$$S_2^+ = -\frac{d^2}{dx^2} + \frac{5}{2}\lambda^2 a^4 - \frac{3}{2}\lambda^2 a^4 \tanh(\lambda a^2 x) - \frac{15}{4}\lambda^2 a^4 \sec h^2(\lambda a^2 x), \tag{3.14}$$

where the first one is related to kink and the second one is related to anti-kink.

The second example is defined by,

$$U(\Phi,\chi) = \frac{1}{2}\left[\lambda\Phi(\Phi^2 - a^2) + \mu\Phi\chi^2\right]^2 + \frac{1}{2}\mu^2\Phi^4\chi^2. \tag{3.15}$$

where first pair of solutions of this system is given by,

$$\Phi^2(x) = \frac{1}{2}a^2\left[1 \pm \tanh(\lambda a^2 x)\right], \qquad \chi(x) = 0. \tag{3.16}$$

Now we are going to compute the operators S_2^+ and S_2^-. The results for kink solutions are,

$$S_2^+ = -\frac{d^2}{dx^2} + \frac{5}{2}\lambda^2 a^4 + \frac{3}{2}\lambda^2 a^4 \tanh(\lambda a^2 x) - \frac{3}{4}\lambda^2 a^4 \sec h^2(\lambda a^2 x), \tag{3.17}$$

and

$$S_2^- = -\frac{d^2}{dx^2} + \frac{1}{2}\mu^2 a^4 + \frac{1}{2}\mu^2 a^4 \tanh(\lambda a^2 x) - \frac{\mu}{2}(\lambda - \frac{\mu}{2})a^4 \sec h^2(\lambda a^2 x). \tag{3.18}$$

Also for anti-kink solutions we have,

$$S_2^+ = -\frac{d^2}{dx^2} + \frac{5}{2}\lambda^2 a^4 - \frac{3}{2}\lambda^2 a^4 \tanh(\lambda a^2 x) - \frac{15}{4}\lambda^2 a^4 \sec h^2(\lambda a^2 x) \tag{3.19}$$

$$S_2^- = -\frac{d^2}{dx^2} + \frac{1}{2}\mu^2 a^4 - \frac{1}{2}\mu^2 a^4 \tanh(\lambda a^2 x) - \frac{\mu}{2}(\lambda + \frac{\mu}{2})a^4 \sec h^2(\lambda a^2 x). \tag{3.20}$$

In order to discuss the stability of above systems, we use the equation (3.4) as $S_2\eta(x) = \omega^2\eta(x)$. Here also the condition $\omega^2 > 0$ ensures the stability of systems. First we consider the S_2^+ from equations (23) and (27). If we define parameter $\lambda a^2 = M$ then we get,

$$\left[-\frac{d^2}{dx^2} + \frac{5}{2}M^2 + \frac{3}{2}M^2\tanh(Mx) - \frac{3}{4}M^2\sec h^2(Mx)\right]\eta_n(x) = \omega_n^2\eta_n(x), \tag{3.21}$$

Now we introduce the variable y where $y = \tanh(Mx)$, then we write the fluctuation $\eta_{n,m}(y) = U(y)P_{n,m}(y)$. Then the equation (27) becomes,

$$(1-y^2)P''_{n,m}(y)+\left[2(1-y^2)\frac{U'}{U}-2y\right]P'_{n,m}(y)+$$
$$\left[(1-y^2)\frac{U''}{U}-2y\frac{U'}{U}-\frac{5}{2}\frac{1}{(1-y^2)}-\frac{3}{2}\frac{y}{(1-y^2)}+\frac{3}{4}+\frac{\omega_n^2}{M^2}\frac{1}{(1-y^2)}\right]P_{n,m}(y)=0. \tag{3.22}$$

From Refs. [8-11] the differential equation for the associated Jacobi function is given by,

$$(1-y^2)P''_{n,m}(y)-[\alpha-\beta+(\alpha+\beta+2)y]\,P'_{n,m}(y)+$$
$$\left[n(\alpha+\beta+n+1)-\frac{m(\alpha+\beta+m+(\alpha-\beta)y)}{1-y^2}\right]P_{n,m}(y)=0 \tag{3.23}$$

Upon comparing the coefficients of P' in equation (32) and (33) we obtain $U(y)$, which is given by the expression,

$$U(y)=(1-y)^{\frac{\alpha}{2}}(1+y)^{\frac{\beta}{2}}, \tag{3.24}$$

where

$$\eta_{n,m}(y)=(1-y)^{\frac{\alpha}{2}}(1+y)^{\frac{\beta}{2}}P_{n,m}(y). \tag{3.25}$$

We notice that $\eta_{n,m}(y)\cos\omega_n t$ is the fluctuation, and the condition for the stability of our solutions is $\omega^2\geq 0$. Now if we compare the coefficients of $P_{n,m}$ in equation (32) and (33) and after some algebra calculation we obtain,

$$\omega^2=-\lambda^2a^4((m+\alpha)^2-4), \tag{3.26}$$

from the stability condition we have,

$$-(2+\alpha)\leq m\leq(2-\alpha).$$

In order to study the stability of the second model, we have to consider the operator S_2^- in equation (28). Using the same method we find the ω^2 for this operator, which is given by,

$$\omega^2=-\frac{4}{9}\lambda^2a^4((m+\alpha)^2-\frac{\mu^2}{\lambda^2}), \tag{3.27}$$

also from stability condition we have,

$$-(\frac{\mu}{\lambda}+\alpha)\leq m\leq(\frac{\mu}{\lambda}-\alpha).$$

For anti - kink solutions which are defined by the equations (24) and (29), the ω^2 is obtained by,

$$\omega^2=-\lambda^2a^4((m+\alpha)^2-1), \tag{3.28}$$

also the stability condition leads us to have,

$$-(1+\alpha)\leq m\leq(1-\alpha).$$

And from equation (30) we get

$$\omega^2=-\lambda^2a^4((m+\alpha)^2), \tag{3.29}$$

where in this case we have only zero mode.

In order to account interactions between two fields with nontrivial systems, we must have $W_{\phi\chi}$ nonzero. From Ref.[7] we use trial orbit method for calculation second pair of solutions,

$$\Phi^2(x) = \frac{1}{2}a^2\left[1-\tanh(\mu a^2 x)\right],$$
$$\chi^2(x) = \frac{1}{2}a^2(\frac{\lambda}{\mu}-1)\left[1+\tanh(\mu a^2 x)\right], \tag{3.30}$$

here parameters λ and μ are restricted to satisfy $\frac{\lambda}{\mu}\geq 1$. Note that the limit $\frac{\lambda}{\mu}\to 1$ transforms the second pair of solutions into the first one. Also the second pair of solutions field configurations obey,

$$\Phi^2(x) + (\frac{\lambda}{\mu}-1)^{-1}\chi^2(x) = a^2. \tag{3.31}$$

In this case the fluctuations are coupled, and R is given by eq.(19)

$$R = \frac{1}{4}(9\lambda^2-\mu^2+6\mu\lambda)\phi^4 + \frac{1}{4}\mu^2\chi^4 + \frac{1}{4}(6\mu\lambda+14\mu^2)\phi^2\chi^2$$
$$+\frac{1}{4}(-6\lambda^2+2\mu\lambda)a^2\phi^2 - \frac{1}{2}\mu\lambda a^2\chi^2 + \frac{1}{4}a^4\lambda^2 \tag{3.32}$$

By choosing $\lambda = \frac{3}{2}\mu$ and eq.(40) we can rewrite eq.(41)

$$R = \frac{1}{4}(\mu\phi^2+\mu a^2)^2. \tag{3.33}$$

In this case, the Schrödinger operators corresponding to fluctuations about the normal modes can be written as,

$$S_2^+ = -\frac{d^2}{dx^2} + \frac{9}{2}\lambda^2a^4 - \frac{9}{2}\lambda^2a^4\tanh(\frac{2}{3}\lambda a^2x) - \frac{15}{9}\lambda^2a^4\sec h^2(\frac{2}{3}\lambda a^2x)$$
$$S_2^- = -\frac{d^2}{dx^2} + \frac{4}{9}\lambda^2a^4 - \frac{8}{9}\lambda^2a^4\sec h^2(\frac{2}{3}\lambda a^2x) \tag{3.34}$$

Now we are going to consider S_2^+ and using Jacobi polynomials, we will arrive at

$$\omega^2 = -\frac{4}{9}\lambda^2a^4((m+\alpha)^2), \tag{3.35}$$

from the stability condition we have just zero mode. And also for S_2^- we obtain,

$$\omega^2 = -\frac{4}{9}\lambda^2a^4((m+\alpha)^2-1), \tag{3.36}$$

this restricts us to take,

$$-(1+\alpha)\leq m\leq(1-\alpha).$$

As a third example we consider a system of coupled scalar field which is defined by the following potential,

$$U(\Phi,\chi) = \frac{1}{2}\lambda^2\Phi^2(\chi^2-a^2)^2 + \frac{1}{2}\lambda^2\chi^2(\Phi^2-a^2)^2. \tag{3.37}$$

If we impose the condition $\Phi^2=\chi^2$, then the solutions for this theory are,

$$\Phi^2 = \chi^2 = \frac{1}{2}a^2\left[1\pm\tanh(\sqrt{2}\lambda a^2x)\right]. \tag{3.38}$$

Now we are going to compute S_2^+ and S_2^-. The kink solutions are,

$$S_2^+ = -\frac{d^2}{dx^2} + \frac{5}{2}\lambda^2a^4 + \frac{3}{2}\lambda^2a^4\tanh(\sqrt{2}\lambda a^2x) - (\frac{9}{4}-\frac{3\sqrt{2}}{2})\lambda^2a^4\sec h^2(\sqrt{2}\lambda a^2x), \tag{3.39}$$

$$\omega^2 = -2\lambda^2 a^4((m+\alpha)^2 - 2), \tag{3.40}$$

where

$$-(\sqrt{2}+\alpha) \le m \le (\sqrt{2}-\alpha).$$

And also have,

$$S_2^- = -\frac{d^2}{dx^2} + \frac{5}{2}\lambda^2 a^4 + \frac{3}{2}\lambda^2 a^4 \tanh(\sqrt{2}\lambda a^2 x) - (\frac{1}{4} + \frac{\sqrt{2}}{2})\lambda^2 a^4 \sec h^2(\sqrt{2}\lambda a^2 x), \tag{3.41}$$

$$\omega^2 = -2\lambda^2 a^4((m+\alpha)^2 - 2), \tag{3.42}$$

where

$$-(\sqrt{2}+\alpha) \le m \le (\sqrt{2}-\alpha).$$

Also for anti-kink solutions we have,

$$S_2^+ = -\frac{d^2}{dx^2} + \frac{5}{2}\lambda^2 a^4 - \frac{3}{2}\lambda^2 a^4 \tanh(\sqrt{2}\lambda a^2 x) - (\frac{9}{4} + \frac{3\sqrt{2}}{2})\lambda^2 a^4 \sec h^2(\sqrt{2}\lambda a^2 x), \tag{3.43}$$

$$\omega^2 = -2\lambda^2 a^4((m+\alpha)^2 - \frac{1}{2}), \tag{3.44}$$

where $-(\frac{\sqrt{2}}{2}+\alpha) \le m \le (\frac{\sqrt{2}}{2}+\alpha)$. And also

$$S_2^- = -\frac{d^2}{dx^2} + \frac{9}{4}\lambda^2 a^4 + \lambda^2 a^4 \tanh(\sqrt{2}\lambda a^2 x) - (3-\sqrt{2})\lambda^2 a^4 \sec h^2(\sqrt{2}\lambda a^2 x), \tag{3.45}$$

$$\omega^2 = -2\lambda^2 a^4((m+\alpha)^2 - \frac{13}{8}), \tag{3.46}$$

where

$$-(\sqrt{\frac{13}{8}}+\alpha) \le m \le (\sqrt{\frac{13}{8}}-\alpha).$$

IV. CONCLUSION

In this articles we have considered field theories with polynomial interaction between the fields. In some of these models we have $W_{\phi\chi} \neq 0$. For the Φ^6 model we obtain the solutions and we also made a stability analysis. For the second model we choosed a class of solutions and we obtained some bound states. For the third model the fluctuations for specific pair of solutions are given by the condition $\Phi^2 = \chi^2$. It would be interesting to use the associated Jacobi polynomial to investigate the soliton solutions for other field theories.

V. REFERENCES

[1] R. Jackiw, Rev. Mod. Phys **49**, 681 (1977)
[2] R. Rajaraman, Solitons and Instantons (Amsterdam: North - Holland) (1982)
[3] D. Bazeia, M. M. Santos and R. F. Ribeiro, Phys. Lett. A **208**, 84 (1995)
[4] D. Bazeia, M. M. Santos , Phys. Lett. A **217**, 28 (1996)
[5] D. Bazeia, D. Ribeiro and M. M. Santos , Phys. Rev. D **54**, 1852 (1996)
[6] D. Bazeia, D. Ribeiro and M. M. Santos , Phys. Rev. E **54**, 2943 (1996)
[7] D. Bazia, J. R. S. Nascimento, R. F. Riberiro, and D. Toledo, J. Phys. A **30**(1997)8157-8166
[8] M.A. Jafarizadeh and H. Fakhri, Phys. Lett. A **230**, 164 (1997).
[9] M.A. Jafarizadeh and H. Fakhri, Ann. Phys. (N.Y.) **262**, 260 (1998).
[10] A.F. Nikiforov and V.B. Uvarov, *"Special Functions of Mathematical Physics: A Unified Introduction with Applications", (Basel.Boston: Birkhäuser, (1988)).*
[11] H. Fakhri and J. Sadeghi Mod. Phys. Lett. A **19**, 615-625 (2004).