African Journal Of Mathematical Physics Volume **4***(2007)33-43*

Vectorial Polarized Manifolds

A. Awane [1], A.Chkiriba[1], M.Goze[2], E. Azizi[1] and M. Bent Bah[1]

1. Equipe de Geometrie Differentielle et Applications. Faculte des Sciences Ben Msik. B.P. 7955. Boulevard Driss Harti. Casablanca. Maroc.,
2. Universite de Haute Alsace, Faculte des Sciences et Techniques, 4, rue des fr'eres Lumi'ere. 68093 Mulhouse cedex France.
Corresponding author: a.awane@yahoo.fr

Abstract

We introduce and develop the notion of vectorial polarized manifolds, some properties of this last structure and various vectorial Hamiltonian mappings in the case of polarized Poisson manifolds and a variety of vectorial Hamiltonian mappings associated to diverse vectorial polarized systems in dimension ≤ 4 are given.
Keywords : Hamiltonian systems,Poisson manifolds, symplectic structures. Generalized Hamiltonian Dynamics of Nambu. **MSC 2000 :** 53D10, 53D17, 70Hxx, 70Hxx.

I. INTRODUCTION

The $k-$symplectic structure was introduced by A. Awane in his dissertation in 1984 (see for examples[1] and[20], developed by Puta Mircea in 1988[20]; analog properties were developed also by M. De Léon, Mc Lean, K. Norris, M.Salgado see[14], and generalized by L.K.Norris[19], etc...

A polarized structure on an even dimensional smooth manifold M is a pair (θ, E) constituted by a closed differential $2-$form θ of maximum rank and by an $n-$codimensional integrable subbundle E of TM which is Lagrangian with respect to the $2-$form θ. Locally, there exists coordinate system $(x^i, y^i)_{1\leq i\leq n}$ (the Darboux coordinate system) such that :

$$\theta = \sum_{i=1}^{n} dx^i \wedge dy^i$$

and the subbundle E is defined by $dy^1 = \ldots = dy^n = 0$.

The notion of a polarized manifold plays an important role in theory of the geometric quantization of Kostant-Souriau, see for example P.Molino[16] and N.Woodhouse[21]. Interesting properties were putting in evidence by A. Weinstein, P. Dazord, J.M. Morvan, P. Molino, P.Libermann, etc...

Let us recall back that one of the main motivations which led to introduce the notion of the $k-$symplectic structure, as extension of the geometry of polarization[3], is to propose a geometric support of the equations of Nambu[18], in analogy with the well known symplectic geometry and classic Hamiltonian formalism. Some properties of the Poisson structure subordinate to a $k-$symplectic manifold have led us to put in evidence the notion of vectorial polarized Poisson structure; for a fixed finite dimensional real vector space V, this last structure is defined on a foliated manifold $(M, \mathfrak{F})$ by a pair $(\mathfrak{H}(M, \mathfrak{F}), P)$, such that $\mathfrak{H}(M, \mathfrak{F})$ is a submodule of the space $\mathcal{C}^{\infty}(\mathcal{M}, \mathcal{V})$ of V-valued smooth functions on M, over the ring of basic functions for the foliation $\mathfrak{F}$, and P is $\mathcal{C}^{\infty}(\mathcal{M})-$bilinear antisymmetric mapping

$$P : \bigwedge_1 (M, V) \times \bigwedge_1 (M, V) \longrightarrow \mathcal{C}^{\infty}(\mathcal{M}, \mathcal{V})$$

allowing to find usual case for $V = \mathbb{R}$.

A notable feature of the Hamiltonian description of classical dynamics is Liouville's theorem, which states that the volume of phase space occupied by an ensemble of systems is conserved. The theorem plays, amongst other things, a fundamental role in statistical mechanics. On the other hand, Hamiltonian dynamics is not the only formalism that makes a statistical mechanics possible. Any set of equations which lead to a Liouville theorem in a suitably defined phase space will do (provided, of course, that ergodicity may be assumed). Nambu proposes a possible generalization of the Hamiltonian dynamics for a 3-dimensional space.

In this optic, the $k-$symplectic geometry propose a geometrical structure in which cohabit differential $2-$forms $\theta^1, \ldots, \theta^k$, such that the Hamiltonian mappings are $\mathbb{R}^{\urcorner}-$valued H whose components H^p are related to Hamiltonian systems X_H by relationship :

$$i(X_H)\theta^p = -dH^p,$$

in order to find Nambu-Hamilton equations conserving the specific features of the classical symplectic geometry.

In this perspective, a $k-$symplectic structure is a $(k+1)-$tuple $(\theta^1, \ldots, \theta^k; E)$ such that $\theta^1, \ldots, \theta^k$ constitute a non degenerate system vanishing on the tangent vector fields to leaves.

The generalized Darboux theorem shows that there exists about each point x_0 of M a local coordinate system $(x_i^p, y^i)_{1 \leq p \leq k, 1 \leq i \leq n}$ defined on a neighborhood such that :

$$\theta^p_{|U} = dx_i^p \wedge dy^i$$

and the subbundle E is defined by $dy^1 = \ldots = dy^n = 0$.

The study of an exterior system shows that in high dimension, an infinity of systems not algebraically equivalent can be put in obviousness. $k-$symplectic systems are defined directly by conditions of regularity; they can be interpreted as models of exterior systems of maximum rank vanishing on the tangent vector fields of the generalized Lagrangian foliation.

In this work, we introduce the notion of the vectorial polarized structure on manifold M. This structure is given by (θ, E), when θ is a closed vectorial valued $2-$form on M vanishing on the section of a subbundle E. The polarized manifolds and the $k-$symplectic manifolds are particular vectorial polarized manifolds.

Also, we give in this work, some properties of this last structure and various vectorial Hamiltonian mappings in the case of polarized Poisson manifolds and also a variety of vectorial Hamiltonian mappings associated to diverse vectorial polarized systems in dimension ≤ 4.

II. VECTORIAL POLARIZED MANIFOLDS

A. Definition

Let $(M, \mathfrak{F})$ be a foliated manifold, M is an $m-$dimensional manifold endowed with a $p-$dimensional foliation; and let V be an $\mathbb{R}-$ vector space of dimension k.

Let us fix a basis $(e_r)_{1\leq r\leq k}$ of V, and let $\bigwedge_2(M,V) = \bigwedge_2(M)\otimes V$, be a $V-$valued differential two forms; that is, the space of

$$\theta = \theta^{\alpha}\otimes e_{\alpha} = \theta^1\otimes e_1 + \ldots + \theta^k \otimes e_k$$

where $\theta^1,\ldots,\theta^k \in \bigwedge_2(M)$.

We say that $(\theta\ ,E)$ is a vectorial polarized structure on M, if the following conditions are satisfied :

1. the $V-$valued $2-$form θ is non degenerate, that is :

$$\forall x\in M,\ \forall X\in T_xM,\ i(X)\theta = 0 \Longrightarrow X=0$$

2. each leaf of $\mathfrak{F}$ is maximal totally isotropic with respect to θ.

B. Hamiltonian systems

Suppose that M is equipped with a vectorial polarized structure $(\theta\ ;E)$ and let

$$j:\mathfrak{X}(\mathfrak{M})\longrightarrow \Lambda_1(\mathfrak{M})\otimes\mathfrak{V}$$

defined by

$$j(X) = i(X)\theta,\ \forall X\in\mathfrak{X}(\mathfrak{M}).$$

A vector field X on M is called a vectorial polarized Hamiltonian system if it is an infinitesimal automorphism for the vectorial polarized structure $(\theta\ ;E)$, that is, if the following conditions are satisfied :

1. X is foliate;
2. $i(X)\theta$ is closed.

We denote by $\mathfrak{L}(\mathfrak{M},\mathfrak{F})$ the $\mathcal{C}^{\infty}(\mathcal{M})-$module of infinitesimal automorphisms of the pair $(\theta\ ,E)$. Let X be a vectorial polarized Hamiltonian system. By Poincaré lemma, every $x\in M$, there exists an open neighborhood U of x and a (smooth) mapping $H:U\longrightarrow V$ satisfying the relationship :

$$i(X)\theta_{|U} = -dH_{|U}.$$

Conversely, if a smooth mapping $H:M\longrightarrow V$, satisfies

$$dH = dH^{\alpha}\otimes e_{\alpha}\in j(\mathfrak{L}(\mathfrak{M},\mathfrak{F})),$$

there exists a unique vector field on M, denoted X_H and called the vectorial polarized Hamiltonian system associated with H, such that :

$$i(X_H)\theta = -\,dH$$

on M; the vector field X_H is called *a (strongly) Hamiltonian system.*

A smooth mapping $H:M\longrightarrow V$ satisfying $dH\in j(\mathfrak{L}(\mathfrak{M},\mathfrak{F}))$ is called a vectorial polarized Hamiltonian mapping of the vectorial polarized structure $(\theta\ ;E)$.

C. Poisson bracket of a vectorial polarized structure

Let H and K be two vectorial polarized Hamiltonian mappings and X_H, X_K the associated polarized Hamiltonian systems. The Lie bracket $[X_H,Y_K]$ is a polarized Hamiltonian system. More precisely, the mapping denoted $\{H,K\}$ of M into V defined by

$$\{H,K\} = \theta(X_H,X_K) = \theta^{\alpha}(X_H,X_K)e_{\alpha}$$

satisfies

$$[X_H,X_K] = X_{\{H,K\}}.$$

The mapping $\{H,K\}$ is called the (vectorial) *Poisson bracket* of the (vectorial) Hamiltonian mappings H and K.

We denote by $\mathfrak{H}(M,\mathfrak{F},\mathfrak{V})$ the space of all vectorial polarized Hamiltonian mappings.

Two important classes of the vectorial polarized structure.

III. POLARIZED MANIFOLDS

A real polarization on M, is a vectorial polarization (θ, E) such that $m = 2p$ and $V = \mathbb{R}$.

Theorem III.1 *(Darboux theorem). Every point of M has an open neighborhood U with local coordinates system $\left(x^1, \ldots, x^n, y^1, \ldots, y^n\right)$, such that*

$$\theta = dx^1 \wedge dy^1 + \ldots + dx^n \wedge dy^n$$

and $\mathfrak{F}$ is defined by equations $dy^1 = \ldots = dy^n = 0$.

And with respect to an adapted coordinates system $\left(x^1, \ldots, x^n, y^1, \ldots, y^n\right)$, the polarized Hamiltonian mapping H takes the form :

$$H = \sum_{i=1}^{n} a_i(y^1, \ldots, y^n)x^i + b(y^1, \ldots, y^n) \tag{3.1}$$

where $a_1, \ldots, a_n, b$ are basic functions for $\mathfrak{F}$.

Recall that, by the symplectic duality $\zeta : X \longmapsto i(X)\theta$, between the tangent bundle TM and the cotangent bundle T^*M, we associate to θ a non degenerate bivector P (the Poisson tensor) defined by :

$$P(\alpha, \beta) = \theta\left(\zeta^{-1}(\alpha), \zeta^{-1}(\beta)\right) \text{ for all } \alpha, \beta \in \bigwedge\nolimits_1 (M),$$

and, we have an antisymmetric linear mapping $\underline{P} : \bigwedge_1 (M) \longrightarrow \mathfrak{X}(\mathfrak{M})$, given by

$$\langle \beta, \underline{P}(\alpha) \rangle = P(\alpha, \beta),$$

IV. K-SYMPLECTIC MANIFOLDS

A k-symplectic structure on M, is a vectorial polarization (θ, E) such that $m = n(k+1)$ and $p = nk$.

A. Canonical k-symplectic structure on $\mathbb{R}^{n(1+k)}$

Consider $\mathbb{R}^{n(1+k)}$ equipped with its Cartesian coordinates $(x_i^\alpha, y^i)_{1 \leq \alpha \leq k, 1 \leq i \leq n}$. Let E be the sub-bundle of $T\mathbb{R}^{n(1+k)}$ defined by the equations

$$dy^1 = 0, \ldots, dy^n = 0$$

and let

$$\theta = \theta^\alpha \otimes e_\alpha = \left(dx_i^\alpha \wedge dy^i\right) \otimes e_\alpha$$

The pair (θ, E) defines a k-symplectic structure on $\mathbb{R}^{n(1+k)}$ called *the canonical k-symplectic structure.* This structure induces a natural k-symplectic structure on the torus $\mathbb{T}^{n(1+k)}$.

B. The generalized Darboux theorem (see for example[1] and[3].

Let M be an $n(n+1)-$dimensional manifold. If the $(k+1)$-tuple $(\theta = \theta^\alpha \otimes e_\alpha, E)$ is a k-symplectic structure on M then for every point p of M there exists an open neighborhood U of M containing p equipped with local coordinates $(x_i^\alpha, y^i)_{1 \leq \alpha \leq k, 1 \leq i \leq n}$ called an adapted coordinate system, such that the $V-$valued differential form θ *is* represented on U by :

$$\theta = \theta^\alpha \otimes e_\alpha = \left(dx_i^\alpha \wedge dy^i\right) \otimes e_\alpha,$$

and E is defined by equation : $dy^1 = 0, \ldots, dy^n = 0$.

Proposition IV.1 *Let $H = (H^\alpha)_{1\leq\alpha\leq k}$ be a vectorial polarized Hamiltonian mapping and let X_H be the associated polarized Hamiltonian system. With respect to an adapted coordinates system $(x_i^\alpha, y^i)_{1\leq\alpha\leq k,1\leq i\leq n}$, the components H^p of H and X_H can be written:*

$$H^\alpha = x_j^\alpha f^j(y^1,\dots,y^n) + g^\alpha(x^1,\dots,x^n)$$

and

$$X_H = -\left(x_j^\alpha \frac{\partial f^j}{\partial y^s}(y^1,\dots,y^n) + \frac{\partial g^\alpha}{\partial y^s}(y^1,\dots,y^n)\right)\frac{\partial}{\partial x_s^\alpha} + f^s(y^1,\dots,y^n)\frac{\partial}{\partial y^s},$$

where f^j and g^α are smooth basic functions on U.

Remark IV.1 *Assume that $k \geq 2$. It follows from the proof of the previous proposition that, if the Pfaffian forms $i(X)\theta^1,\dots,i(X)\theta^k$ are closed (or equivalently $L_X\theta^1 = \dots = L_X\theta^k = 0$), then the vector field X is necessarily an infinitesimal automorphism of $\mathfrak{F}$.*

With respect to an adapted coordinate system $(x_i^\alpha, y^i)_{1\leq\alpha\leq k,1\leq i\leq n}$ the components $\{H,K\}^\alpha$ of $\{H,K\}$ are given by:

$$\{H,K\}^\alpha = \sum_{s=1}^n \left(\frac{\partial H^\alpha}{\partial x_s^\alpha}\frac{\partial K^\alpha}{\partial y^s} - \frac{\partial H^\alpha}{\partial y^s}\frac{\partial K^\alpha}{\partial x_s^\alpha}\right),$$

Let $\mathfrak{H}(M)$ be the set of Hamiltonian mappings of the k-symplectic structure $(\theta^1,\dots,\theta^k;E)$. The association $(H,K)\longmapsto\{H,K\}$, of $\mathfrak{H}(M)\times\mathfrak{H}(M)$ into $\mathfrak{H}(M)$, is a skew-symmetric $\mathbb{R}$-bilinear mapping satisfying the Jacobi identity.

Proposition IV.2 $(\mathfrak{H}(M),\{,\})$ *is an infinite-dimensional Lie algebra.*

V. NAMBU'S STATISTICAL MECHANICS

Let (x,y,z) be a triplet of dynamical variables (a canonical triplet) which span a 3-dimensional phase space M. This is a formal generalization of conventional phase space spanned by a canonical pair (p,q). Next, we will introduce two functions H and G depending on (x,y,z) which serve as a pair of "Hamiltonians" to determine the motion of points in phase space. More precisely Nambu has postulated the following Hamilton equation :

$$\begin{cases} \frac{dx}{dt} = \frac{D(H,G)}{D(y,z)}, \\ \frac{dy}{dt} = \frac{D(H,G)}{D(z,x)}, \\ \frac{dz}{dt} = \frac{D(H,G)}{D(x,y)}, \end{cases}$$

where $D(H,G)/D(y,z)$ denote the Jacobian

$$\frac{D(H,G)}{D(y,z)} = \frac{\partial H}{\partial y}\frac{\partial G}{\partial z} - \frac{\partial H}{\partial z}\frac{\partial G}{\partial y}.$$

The above equations are called *Nambu's equations of motion*, and the vector field whose integral curves are given by Nambu's equations of motion will be denoted by $X^n_{(H,G)}$ and called the *dynamical system of Nambu.*

Consider the space $M = \mathbb{R}^3$ equipped with its canonical 2-symplectic structure $(\theta^1, \theta^2; E)$ defined by

$$\begin{cases} \theta^1 &= dx \wedge dz, \\ \theta^2 &= dy \wedge dz, \\ E &= \ker dz. \end{cases}$$

The Hamiltonian mapping of the 2-symplectic structure are the mapping

$$H : M \longrightarrow \mathbb{R}^2$$

whose components are given by

$$\begin{cases} H^1 &= f(z)x + g^1(z), \\ H^2 &= f(z)y + g^2(z), \end{cases}$$

where f, g^1 and g^2 are smooth real functions depending only on the variable z. The integral curves of the Hamiltonian system X_H of the 2-symplectic structure are given by the following equations :

$$\frac{dx}{dt} = -\frac{\partial H^1}{\partial z},$$

$$\frac{dy}{dt} = -\frac{\partial H^2}{\partial z},$$

and

$$\frac{dz}{dt} = \frac{\partial H^1}{\partial x} = \frac{\partial H^2}{\partial y}.$$

Theorem V.1 *Let $H = (H^1, H^2)$ with $H^1 = f(z)x + g^1(z)$ and $H^2 = f(z)y + g^2(z))$ be Hamiltonian mappings of the 2-symplectic structure. Then the Hamiltonian system X_H and the dynamical system of Nambu X_H^n are related by*

$$X_H^n = f(z)X_H.$$

Corollary V.1 *The mapping*

$$(f(z))^{-1}H = (x + h^1(z), y + h^2(z))$$

is a solution of Nambu's equations of motions on a domain where $f(z)$ is a non-vanishing function and

$$h^1(z) = (f(z))^{-1}g^1(z) \quad , \quad h^2(z) = (f(z))^{-1}g^2(z) .$$

VI. VECTORIAL POLARIZED POISSON MANIFOLDS

A. Definition

Let $(M, \mathfrak{F})$ be a foliated manifold, M is an $n-$dimensional manifold endowed with a $p-$dimensional foliation; and let V be an $\mathbb{R}-$ vector space of dimension k.

Let us fix a basis $(e_r)_{1\leq r\leq k}$ of V with dual basis $(\omega^r)_{1\leq r\leq k}$, and let $\bigwedge_1(M, V) = \bigwedge_1(M) \otimes V$ be the space of $V-$valued differential forms of degree 1, that is, the space of

$$\alpha = \alpha^1 \otimes e_1 + \ldots + \alpha^k \otimes e_k$$

where $\alpha^1, \ldots, \alpha^k \in \bigwedge_1(M)$. Locally, on an open neighborhood U endowed with local coordinates system $(x^1, \ldots, x^n)$, each element $\alpha \in \bigwedge_1(M, V)$ has the form :

$$\alpha_{|U} = \sum_{r=1}^{k}\sum_{i=1}^{n} \alpha_i^r dx^i \otimes e_r$$

where $\alpha_i^r : U \longrightarrow \mathbb{R}$ are smooth mappings.

We denote by E_V^o the annihilator of the subbundle E in $\bigwedge_1(M, V)$, it is the space of $V-$valued $1-$forms on M vanishing on the cross sections of E.

Definition VI.1 *Let $(M, \mathfrak{F})$ be a foliated manifold, let $\mathfrak{H}(M, \mathfrak{F})$ be a submodule of $\mathcal{C}^\infty(\mathcal{M}, \mathcal{V})$ over the ring $\mathfrak{B}(\mathfrak{M}, \mathfrak{F})$, and let*

$$P : \bigwedge_1(M, V) \times \bigwedge_1(M, V) \longrightarrow \mathcal{C}^\infty(\mathcal{M}, \mathcal{V})$$

be an antisymmetric $\mathcal{C}^\infty(\mathcal{M})-$bilinear. We say that $(\mathfrak{H}(M, \mathfrak{F}), P)$ is a vector polarized Poisson structure on M, if the following properties are satisfied :

1. *$P(\alpha, \beta) = 0$ for all $\alpha, \beta \in E_V^o$,*
2. *for all $H, K \in \mathfrak{H}(M, \mathfrak{F})$, $P(dH, dK) \in \mathfrak{H}(M, \mathfrak{F})$,*
3. *the correspondence $(H, K) \longmapsto \{H, K\} = P(dH, dK)$, from $\mathfrak{H}(M, \mathfrak{F}) \times \mathfrak{H}(M, \mathfrak{F})$ with values in $\mathfrak{H}(M, \mathfrak{F})$, confers to $\mathfrak{H}(M, \mathfrak{F})$ a law of Lie algebra,*
4. *each $H \in \mathfrak{H}(M, \mathfrak{F})$ corresponds to a vector field X_H such that :*

$$\langle dK, X_H \rangle = \{H, K\},$$

for all $K \in \mathfrak{H}(M, \mathfrak{F})$.

P will be called a vector polarized Poisson tensor.

Let us consider an open U of M endowed with an adapted local coordinates system $(x^1, \ldots, x^p, y^1, \ldots y^q)$. Since P is zero on the annihilator E_V^o of the subbundle E in $\bigwedge_1(M, V)$, then the tensor P has the form :

$$P = A_{pq}^{ijr}\left((\frac{\partial}{\partial x^i} \otimes \omega^p) \wedge (\frac{\partial}{\partial x^j} \otimes \omega^q)\right) \otimes e_r + B_{pq}^{ijr}\left((\frac{\partial}{\partial x^i} \otimes \omega^p) \wedge (\frac{\partial}{\partial y^j} \otimes \omega^q)\right) \otimes e_r \qquad (6.1)$$

where $A_{pq}^{ijr}, B_{pq}^{ijr} : U \longrightarrow \mathbb{R}$ are differential mappings. The Jacobi identity implies that we have :

$$\begin{array}{l}
\frac{\partial A_{uv}^{abr}}{\partial x^l} A_{rw}^{lcv} - \frac{\partial A_{uv}^{abr}}{\partial y^m} B_{wr}^{cmv} + \frac{\partial A_{vw}^{bcr}}{\partial x^l} A_{ru}^{lav} - \frac{\partial A_{vw}^{bcr}}{\partial y^m} B_{ur}^{amv} + \frac{\partial A_{wu}^{car}}{\partial x^l} A_{rv}^{lbv} - \frac{\partial A_{wu}^{car}}{\partial y^m} B_{vr}^{bmv} = 0, \\
\frac{\partial A_{uv}^{abr}}{\partial x^l} B_{rw}^{lcv} + \frac{\partial B_{vw}^{bcr}}{\partial x^l} A_{ru}^{lav} - \frac{\partial B_{vw}^{bcr}}{\partial y^m} B_{ur}^{amv} - \frac{\partial B_{uw}^{acr}}{\partial x^l} A_{rv}^{lbv} + \frac{\partial B_{uw}^{acr}}{\partial y^m} B_{vr}^{bmv} = 0, \\
\frac{\partial B_{uv}^{abv}}{\partial x^l} A_{rv}^{lbv} - \frac{\partial B_{uw}^{acr}}{\partial x^l} B_{rv}^{lbv} = 0.
\end{array} \qquad (6.2)$$

For each element $\alpha \in \bigwedge_1(M, V)$ we can associate a $\mathcal{C}^\infty(\mathcal{M})-$linear mapping

$$P(\alpha, .) : \bigwedge_1(M, V) \longrightarrow \mathcal{C}^\infty(\mathcal{M}, \mathcal{V})$$

such that $P(\alpha, .)(\beta) = P(\alpha, \beta)$ for each $\beta \in \bigwedge_1(M, V)$, the linear mapping $P(\alpha, .)$ coincides with the vector field $\underline{P}(\alpha)$ for $k = 1$.

VII. MODEL VECTOR POLARIZED POISSON MANIFOLDS

Let us consider the model space $\mathbb{R}^n = \mathbb{R}^p \times \mathbb{R}^q$ endowed with the $p-$dimensional model foliation $\mathfrak{F}$ defined by the equations $dy^1 = \ldots dy^q = 0$, where (x^i, y^j), with $i = 1, \ldots, p$ and $j = 1, \ldots, q$, are the Cartesian coordinates system and let $V = \mathbb{R}^{\urcorner}$ be the real space in which one fixes the canonical basis $(e_r)_{1 \leq r \leq k}$ with dual basis $(\omega^r)_{1 \leq r \leq k}$.

Let $(\mathfrak{H}(M, \mathfrak{F}), P)$ be a vector polarized Poisson structure on $\mathbb{R}^n$. The vector polarized Poisson bivector P takes the form (6.1) and satisfies the Jacobi identity (6.2), and $\mathfrak{H}(M, \mathfrak{F})$ is a submodule of $\mathcal{C}^\infty(\mathbb{R}^\backslash, \mathcal{V})$ over the ring $\mathfrak{B}(\mathbb{R}^n, \mathfrak{F})$ of basic functions. We give now some examples of polarized Poisson structures widening the space of vector polarized Hamiltonian mappings subordinate to the $k-$symplectic manifolds.

1. Let $\mathfrak{H}(\mathbb{R}^n, \mathfrak{F})$ be the $\mathfrak{B}(\mathfrak{M}, \mathfrak{F})-$submodule of $\mathcal{C}^\infty(\mathbb{R}^n, V)$ spanned by $e_1 \ldots, e_k$. thus $\mathfrak{H}(\mathbb{R}^n, \mathfrak{F}) = \mathfrak{B}(\mathbb{R}^n, \mathfrak{F}) \times \ldots \times \mathfrak{B}(\mathbb{R}^n, \mathfrak{F})$ (k times) and for all vector polarized Poisson bivector P, the associated Lie algebra $(\mathfrak{B}(\mathfrak{M}, \mathfrak{F}), \{,\})$ is abelian.

2. For $V = \mathbb{R}^2$ we consider the $\mathfrak{B}(M, \mathfrak{F})-$submodule $\mathfrak{H}(\mathbb{R}^n, \mathfrak{F})$ of $\mathcal{C}^\infty(\mathbb{R}^n, V)$ spanned by the mappings :

$$X^I : (x, y) \longmapsto x^I e_1 \ \ (i = 1, \ldots, p).$$

$$X^{ij} : (x, y) \longmapsto x^i x^j e_2 \ \ (i, j = 1, \ldots, p)$$

and by the vectors e_1, e_2. The components H^1 and H^2 of each element H of $\mathfrak{B}(\mathfrak{M}, \mathfrak{F})$ take the form:

$$\begin{aligned} H^1 &= \sum_{i=1}^{p} f_i(y^1, \ldots, y^q)x^i + g^1(y^1, \ldots, y^q) \quad (i = 1, \ldots, p) \\ H^2 &= \sum_{i,j=1}^{p} f_{ij}(y^1, \ldots, y^q)x^i x^j + g^2(y^1, \ldots, y^q) \quad (i, j = 1, \ldots, p) \end{aligned} .$$

where $f_i, f_{ij}, g^1, g^2 \in \mathfrak{B}(\mathfrak{M}, \mathfrak{F})$.

3. Suppose that $p = mk$. We denote by $(x, y) = (x^{ra}, y^1, \ldots, y^q)_{1 \leq a \leq m,\, 1 \leq r \leq k}$ the Cartesian coordinates system of $\mathbb{R}^n$ and let $\mathfrak{H}(\mathbb{R}^n, \mathfrak{F})$ be the $\mathfrak{B}(\mathfrak{M}, \mathfrak{F})-$submodule of $\mathcal{C}^\infty(\mathbb{R}^n, V)$ spanned by the mappings

$$X^a : (x, y) \longmapsto \sum_{r=1}^{k} x^{ra} e_r \ \ (a = 1, \ldots, m)$$

and by the vectors $e_1, \ldots, e_k$. The component H^r of each element H of $\mathfrak{H}(M, \mathfrak{F})$ takes the form :

$$H^r = \sum_{a=1}^{m} f_a(y^1, \ldots, y^n)x^{ra} + g^r(y^1, \ldots, y^n) \quad (r = 1, \ldots, k).$$

where $f_a, g^r \in \mathfrak{B}(\mathfrak{M}, \mathfrak{F})$.

4. In the previous notations, we consider the $\mathfrak{B}(M, \mathfrak{F})-$submodule $\mathfrak{H}(\mathbb{R}^n, \mathfrak{F})$ of $\mathcal{C}^\infty(\mathbb{R}^n, V)$ spanned by the mappings

$$X^{ab} : (x, y) \longmapsto \sum_{r=1}^{k} x^{ra} x^{rb} e_r \ \ (a, b = 1, \ldots, m)$$

and by the vectors $e_1, \ldots, e_k$. The component H^r of each element H of $\mathfrak{B}(\mathfrak{M}, \mathfrak{F})$ has the form :

$$H^r = \sum_{a,b=1}^{m} f_{ab}(y^1, \ldots, y^n)x^{ra}x^{rb} + b^r(y^1, \ldots, y^n) \quad (r = 1, \ldots, k).$$

where $f_{ab}, g^r \in \mathfrak{B}(\mathfrak{M}, \mathfrak{F})$.

5. In the previous notations, we consider the $\mathfrak{B}(M, \mathfrak{F})$ −submodule $\mathfrak{H}(\mathbb{R}^n, \mathfrak{F})$ of $\mathcal{C}^\infty(\mathbb{R}^n, V)$ spanned by the mappings :

$$X^a : (x, y) \longmapsto \sum_{r=1}^{k} x^{ra} e_r \ \ (a = 1, \dots, m)$$

$$X^{ab} : (x, y) \longmapsto \sum_{r=1}^{k} x^{ra} x^{rb} e_r \ \ (a, b = 1, \dots, m)$$

and by the vectors $e_1, \dots, e_k$. The components H^r of each element H of $\mathfrak{B}(\mathfrak{M}, \mathfrak{F})$ take the form :

$$H^r = \sum_{a=1}^{m} f_a(y^1, \dots, y^n) x^{ra} + \sum_{a,b=1}^{m} f_{ab}(y^1, \dots, y^n) x^{ra} x^{rb} + b^r(y^1, \dots, y^n),$$

where $(r = 1, \dots, k)$ and $f_a, f_{ab}, g^r \in \mathfrak{B}(\mathfrak{M}, \mathfrak{F})$.

VIII. LOCAL MODELS OF VECTORIAL POLARIZED SYSTEMS

For the polarized manifolds and the k−symplectic manifolds, there is a unique model : the Darboux model and its generalization. In the case of the vectorial polarized systems, there is not a unique model.

We give here, some local models of the vectorial polarized systems, when the dimension of the space is ≤ 4, and the vectorial polarized Hamiltonian systems and the vectorial polarized Hamiltonian mappings having singles expressions.

A. For $k = 2$ and $m = 3$

Every $\mathbb{R}^2$-valued form $\theta = \theta^1 \otimes e_1 + \theta^2 \otimes e_2$ in $\mathbb{R}^3$ admits a maximal solution of dimension 2 then is 2−symplectic system and can be written under the following local form :

$$\begin{cases} \theta^1 = dx^1 \wedge dx^3 \\ \theta^{32} = dx^2 \wedge dx^3 \end{cases}$$

B. For $k = 3$, $m = 3$

Consider the $\mathbb{R}^3$-valued form in $\mathbb{R}^3$ given by

$$\theta = \theta^1 \otimes e_1 + \theta^2 \otimes e_2 + \theta^3 \otimes e_3$$

with rank 3, such that the system $\{\theta^1, \theta^2, \theta^3\}$ is not is algebraically equivalent to 2−system. Then, locally, this system is algebraically equivalent to the following model

$$\begin{cases} \theta^1 = dx^2 \wedge dx^3 \\ \theta^2 = dx^3 \wedge dx^1 \\ \theta^3 = dx^1 \wedge dx^2 \end{cases}$$

This system has a 1−dimensional maximal solution. The vectorial polarized Hamiltonian mappings take the following expressions :

$$\begin{cases} H^1 = ax^2x^3 + b_3x^2 - b_2x^3 + \alpha_1 \\ H^2 = -ax^1x^3 + b_1x^3 - b_3x^1 + \alpha_2 \\ H^3 = -ax^1x^2 + b_2x^1 - b_1x^2 + \alpha_3 \end{cases}$$

where $a, b_1, b_2, b_3, \alpha_1, \alpha_2, \alpha_3$ are real numbers..

C. For $k = 2$ and $m = 4$

Let $\theta = \theta^1 \otimes e_1 + \theta^2 \otimes e_2$ a $\mathbb{R}^2$-valued form in $\mathbb{R}^4$ with maximum rank and with maximal solution of dimension 2 . then locally we

$$\theta^1 = dx^1 \wedge dy^1$$
$$\theta^2 = dx^2 \wedge dy^2.$$

This system has a 2−dimensional maximal solution (the foliation defined by $dy^1 = dy^2 = 0$). The vectorial polarized Hamiltonians mappings take the following expressions :

$$\begin{cases} H^1 = f^1\left(y^1, y^2\right)\ x^1 + g^1\left(y^1, y^2\right) \\ H^2 = f^2\left(y^1, y^2\right)\ x^2 + g^2\left(y^1, y^2\right) \end{cases}$$

where f^1, f^2, g^1, g^2 are basic functions for the foliation defined by $dy^1 = dy^2 = 0$

D. For $k = 3$ and $m = 4$

We consider only

1. The 3−symplectic system

$$\left(S^1\right) \quad \begin{cases} \theta^1 = dx^1 \wedge dy \\ \theta^2 = dx^2 \wedge dy \\ \theta^3 = dx^3 \wedge dy \end{cases}$$

2. The system $\left(S^2\right)$

$$\left(S^2\right) \quad \begin{cases} \theta^1 = dx^1 \wedge dy^2 + dx^2 \wedge dy^1 \\ \theta^2 = dx^1 \wedge dy^1 \\ \theta^3 = dx^2 \wedge dy^2 \end{cases}$$

This system has a 2−dimensional maximal solution (the foliation defined by $dy^1 = dy^2 = 0$).

The vectorial polarized Hamiltonian mappings take the following expressions :

$$\begin{cases} H^1 = f^1\left(y^1, y^2\right)\ x^1 + g^1\left(y^1, y^2\right) \\ H^2 = f^2\left(y^1, y^2\right)\ x^2 + g^2\left(y^1, y^2\right) \\ H^3 = f^2\left(y^1, y^2\right)\ x^1 - f^1\left(y^1, y^2\right)\ x^2 + g^3\left(y^1, y^2\right) \end{cases}$$

where $f^1, f^2, f^3, g^1, g^2, g^3$ are basic functions for the foliation defined by $dy^1 = dy^2 = 0$.

3. The system $\left(S^3\right)$

$$\left(S^3\right) \quad \begin{cases} \theta^1 = dx^1 \wedge dx^2 + dx^3 \wedge dx^4 \\ \theta^2 = dx^1 \wedge dx^3 - dx^2 \wedge dx^4 \\ \theta^3 = dx^1 \wedge dx^4 + dx^2 \wedge dx^3 \end{cases}$$

Locally, this system has a 1−dimensional maximal solution.

The vectorial polarized Hamiltonian mappings take the following expressions :

$$H^1\left(x_1, x_2, x_3, x_4\right) = ax_1 + bx_2 + cx_3 + dx_4 + \alpha_1$$
$$H^2\left(x_1, x_2, x_3, x_4\right) = -dx_1 - cx_2 + bx_3 + ax_4 + \alpha_2$$
$$H^3\left(x_1, x_2, x_3, x_4\right) = cx_1 - dx_2 - ax_3 + bx_4 + \alpha_3$$

where $a, b, c, d, \alpha_1, \alpha_2, \alpha_3$ are real numbers.

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