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Generalized Monodromy Matrix of Two Dimensional String Effective Action

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Abstract

We study the $O(d,d)/O(d)\times O(d)$ coset reformulation of two-dimensional string effective action. We construct the generalized Monodromy Matrix $\hat{\mathcal{M}}(\omega)$ for $O(d,d)$ string effective action by using general integrability conditions and T-duality group properties.

Keywords: *T-duality Symmetries, Integrable models, Monodromy Matrix.*

I. INTRODUCTION

The unexpected appearance of noncompact global symmetries was one of the most intriguing discoveries to emerge from the study of supergravity theories in the 1970's. More recently, non compact groups have been found to play a significant role in string theory.

In the context of Einstein gravity and supergravity theories, when a four dimensional action is dimensionally reduced to two space-time dimensions[1–4], the resulting action, in general, describes a σ-model on a coset[5], G/H , where G turns out to be a non compact group in many cases and H is its maximal compact subgroup.

In order to study integrability properties of such models, one of the approaches involves constructing the monodromy matrix associated with the system under consideration[6–9]. In the case of σ-models in flat space, it is customary to introduce a constant spectral parameter and then construct a convenient current which is curvature free. This single condition fulfills the requirement of integrability of the field equations. On the other hand, when we consider a σ-model in a curved background, the spectral parameter assumes space-time dependence to satisfy consistency requirements in order that the flat curvaturelessness of the appropriate currents are maintained.

The purpose of this paper is to construct the $O(d,d)/O(d)\times O(d)$ coset and the monodromy matrix, $\hat{\mathcal{M}}(\omega)$, for a two-dimensional string effective action, obtained from a D-dimensional effective action, which is compactified on T^d and thus enjoys isometries along $d = D - 2$ spatial directions. We provide the prescription for the construction of the general monodromy matrix and its transformation property under $O(d,d)$ transformations. In this procedure, we are able to synthesize the classical integrability properties and the T-duality symmetry of string theory in a novel way. Furthermore, we demonstrate explicitly the transformation of $\hat{\mathcal{M}}(\omega)$ under T-duality.

II. $O(D,D)$ TWO DIMENSIONAL STRING EFFECTIVE ACTION

A global symmetry of a supergravity theory is generally associated with a non compact Lie group G. The scalar fields $\{\phi^i\}$ in the theory parameterize the coset G/H, where H represents the maximal compact subgroup of G. One of these principals models for T-duality is the $O(d,d)/O(d)\times O(d)$ coset. Then, the key questions are, how to reformulate this latter by using non compact symmetries and how to exploit this $O(d,d)$ symmetry in the construction process of generalized monodromy matrix.

Our starting point is the $O(d,d)/O(d)\times O(d)$ reformulations, where the d^2 moduli fields that arise in the toroidal compactification of the NS-NS string effective action[10,11] parameterize the $O(d,d)/O(d)\times O(d)$ coset.

The $O(d,d)$ group has $2d$ dimensions and its representation is given by

$$\Omega = \begin{pmatrix} P & Q \\ R & S \end{pmatrix} \in O(d,d) \tag{2.1}$$

where $\{P,Q,R,S\}$ are $d\times d$ matrices such that Ω preserves the bilinear form η

$$\Omega^T\eta\Omega = \eta \quad , \quad \eta \equiv \begin{pmatrix} 0 & I_d \\ I_d & 0 \end{pmatrix} \tag{2.2}$$

with I_d is the $d\times d$ identity matrix. The group properties are written as

$$P^TR + R^TP = 0 \quad , \quad Q^TS + S^TQ = 0 \quad , \quad P^TS + R^TQ = I_d \tag{2.3}$$

and the inverse of Ω is given by

$$\Omega^{-1} = \eta\Omega^T\eta \tag{2.4}$$

The maximal compact subgroup of $O(d,d)$ is $O(d)\times O(d)$, this latter has d^2-d dimension and it represent a symmetry of the space of solutions which appears in the frame work of string field theory[12]. The dimension of $O(d,d)/O(d)\times O(d)$ is d^2.

To construct the $O(d,d)/O(d)\times O(d)$ theory we follow the procedure used in various supergravity theories[10,11,13]. Recall that the purpose of this process is to derive the general monodromy matrix $\hat{\mathcal{M}}(\omega)$, for a 2-dimensional string effective action, obtained from a D-dimensional effective action, which is compactified on T^d and thus enjoys isometries along $d = D-2$ spatial directions. Indeed, in this process, we are able to synthesize the classical integrability properties and the duality symmetry of string theory in a new way.

The way to do this is to consider as starting point the D-dimensional tree level string effective action[13]

$$S = \int dx^D\sqrt{-G}(R + (\partial\phi)^2 - \frac{1}{12}H^2) \tag{2.5}$$

where $G_{\mu\nu}$ is the metric in D-dimensions in the string frame, G its determinant, R the corresponding scalar curvature, ϕ is the dilaton and $H = dB$, where B represents the moduli coming from the reduction of the B-field in D space-time dimensions.

The reduced action takes the following form[14,15]

$$S = \int dx^0dx^1\sqrt{-g}\exp(-\bar{\phi})\left(R + (\partial\bar{\phi})^2 + \frac{1}{8}Tr(\partial_\alpha M^{-1}\partial^\alpha M)\right) \tag{2.6}$$

where $g_{\alpha\beta}$ $(\alpha,\beta = 0,1)$ is the two dimensional space-time metric, and

$$\bar{\phi} = \phi - \frac{1}{2}\log\det G \tag{2.7}$$

the parametrization of the $O(d,d)/O(d) \times O(d)$ coset is determined by introducing a $2d \times 2d$ matrix V which plays the role of a "vielbein" for the $2d \times 2d$ symmetric matrix M belonging to $O(d,d)$, in the sense that

$$M = \begin{pmatrix} G^{-1} & -G^{-1}B \\ BG^{-1} & G - BG^{-1}B \end{pmatrix} \tag{2.8}$$

and the lower triangular matrix V is given by

$$V = \begin{pmatrix} E^{-1} & 0 \\ BE^{-1} & E^{T} \end{pmatrix} \in O(d,d) \quad , \quad V^{T}\eta V = \eta \tag{2.9}$$

where E is a $d \times d$ vielbein satisfying $E^{T}E = G$.It is easy to check that

$$M = V^{T}V \quad , \quad M^{T}\eta M = \eta \quad , \quad M^{-1} = \eta M \eta \tag{2.10}$$

the action (2.6) is invariant under the global $O(d,d)$ transformations,

$$g_{\alpha\beta} \rightarrow g_{\alpha\beta} \quad , \quad \bar{\phi} \rightarrow \bar{\phi} \quad , \quad M \rightarrow \Omega^{T}M\Omega \tag{2.11}$$

this transformation acts on G and B in a rather complicated non linear way. Then, with this M properties, we can write the line element of $O(d,d)/O(d) \times O(d)$ coset space as

$$dS^{2}_{t\arg et} = -\frac{1}{8}Tr(\eta dM\eta dM) = -\frac{1}{4}Tr(dGdG^{-1} + G^{-1}dBG^{-1}dB) \tag{2.12}$$

It should be noted that the matrix V transforms as

$$V \rightarrow \Omega^{T}V \tag{2.13}$$

In fact, V belongs to the coset $O(d,d)/O(d) \times O(d)$.

III. GENERALIZED INTEGRABILITY CONDITIONS

Let us consider a general Sigma model in flat space-time, defined on the $O(d,d)/O(d) \times O(d)$ coset. With $V \in O(d,d)/O(d) \times O(d)$ and $M = V^{T}V$, we can decompose the current $V^{-1}\partial_{\alpha}V$ which belong to the Lie algebra of G as

$$V^{-1}\partial_{\alpha}V = P_{\alpha} + Q_{\alpha} \tag{3.1}$$

where $Q_{\alpha} \in O(d) \times O(d)$ and P_{α} belongs to the complement and are satisfy

$$P^{T}_{\alpha} = P_{\alpha} \quad , \quad Q^{T}_{\alpha} = -Q_{\alpha} \tag{3.2}$$

Furthermore, the current in (3.1) is invariant under a local $O(d) \times O(d)$ transformation

$$V \rightarrow Vh(x) \tag{3.3}$$

where P_{α} and Q_{α} are transformed as

$$P_{\alpha} \rightarrow h^{-1}(x)P_{\alpha}h(x) \quad , \quad Q_{\alpha} \rightarrow h^{-1}(x)Q_{\alpha}h(x) \ + h^{-1}(x)\partial_{\alpha}h(x) \tag{3.4}$$

Let us next describe how to construct the generalized monodromy matrix for such systems from the general properties of integrability such as the general conserved quantities with general functions depending on spectral parameter (without special form of these functions, in fact are the general ones). So, the general current decomposition become

$$\hat{V}^{-1}\partial_{\alpha}\hat{V} = Q_{\alpha} + f(t)P_{\alpha} + g(t)\epsilon_{\alpha\beta}P^{\beta} \tag{3.5}$$

and from the zero curvature condition with a potential which depends on a constant spectral parameter as follows

$$\partial_\alpha(\hat{V}^{-1}\partial_\beta\hat{V})+\partial_\beta(\hat{V}^{-1}\partial_\alpha\hat{V})+\Big[(\hat{V}^{-1}\partial_\alpha\hat{V}),(\hat{V}^{-1}\partial_\beta\hat{V})\Big]=0 \tag{3.6}$$

To start with, setting the anti-symmetric tensor field to zero $B=0$, in this case, we can write

$$M^{(B=0)}=\begin{pmatrix} G^{-1} & 0 \\ 0 & G \end{pmatrix} \quad , \quad V^{(B=0)}=\begin{pmatrix} E^{-1} & 0 \\ 0 & E \end{pmatrix} \tag{3.7}$$

Furthermore, assume that E and G matrices are diagonal as follows

$$E=diag(\exp(\frac{1}{2}(\lambda+\psi_1),\exp(\frac{1}{2}(\lambda+\psi_2),...,\exp(\frac{1}{2}(\lambda+\psi_d)) \tag{3.8}$$

$$G=diag(\exp(\lambda+\psi_1),\exp(\lambda+\psi_2),...,\exp(\lambda+\psi_d)) \tag{3.9}$$

with $\sum\limits_i \psi_i=0$, so that $\lambda=\frac{1}{d}\log\det G$, as adopted in[15]. Then

$$P_\alpha=\frac{1}{2}\Big((\hat{V}^{B=0})^{-1}\partial_\alpha\hat{V}+((\hat{V}^{B=0})^{-1}\partial_\alpha\hat{V})^T\Big)=\begin{pmatrix} -E^{-1}\partial_\alpha E & 0 \\ 0 & E^{-1}\partial_\alpha E \end{pmatrix} \tag{3.10}$$

$$Q_\alpha=\frac{1}{2}((\hat{V}^{B=0})^{-1}\partial_\alpha\hat{V}-((\hat{V}^{B=0})^{-1}\partial_\alpha\hat{V})^T) \tag{3.11}$$

Indeed, from general expressions of (3.5) and (3.6), we obtain as results

$$\hat{V}^{-1}\partial_+\hat{V}=Q_++(f(t)+g(t))P_+ \tag{3.12}$$

$$\hat{V}^{-1}\partial_-\hat{V}=Q_-+(f(t)+g(t))P_- \tag{3.13}$$

and after some computation, we find that

$$\begin{gathered}\partial_\alpha(\hat{V}^{-1}\partial_\beta\hat{V})+\partial_\beta(\hat{V}^{-1}\partial_\alpha\hat{V})+\Big[(\hat{V}^{-1}\partial_\alpha\hat{V}),(\hat{V}^{-1}\partial_\beta\hat{V})\Big]= \\ (f'+g')(\partial_\alpha tP_\beta-\partial_\beta tP_\alpha)+(f+g)(\partial_\beta P_\alpha-\partial_\alpha P_\beta+2\left[P_\alpha,P_\beta\right]) \\ +(4fg-1)\left[P_\alpha,P_\beta\right]+\left[Q_\alpha,Q_\beta\right]=0\end{gathered}$$

This latter leads to spectral parameters forms as follows

$$\partial_+t=\frac{P_+\partial_0t+H(t)}{P_0} \tag{3.14}$$

$$\partial_-t=\frac{P_-\partial_0t-H(t)}{P_0} \tag{3.15}$$

with $H(t)$ is a general function which depends on $f(t)$ and $g(t)$ functions (more details for this $H(t)$ function and its expression are given in our next work)[16]. Since $Q_\pm=0$, we can write P_+ and P_- as

$$P_+=\begin{pmatrix} -E^{-1}\partial_+E & 0 \\ 0 & E^{-1}\partial_+E \end{pmatrix} \quad , \quad P_-=\begin{pmatrix} -E^{-1}\partial_-E & 0 \\ 0 & E^{-1}\partial_-E \end{pmatrix} \tag{3.16}$$

and the general currents as[16]

$$\hat{V}^{-1}\partial_+\hat{V}=(f(t)+g(t))\begin{pmatrix} -E^{-1} & 0 \\ 0 & E^{-1} \end{pmatrix}\partial_+E \tag{3.17}$$

$$\hat{V}^{-1}\partial_-\hat{V} = (f(t)+g(t))\begin{pmatrix} -E^{-1} & 0 \\ 0 & E^{-1} \end{pmatrix}\partial_- E \tag{3.18}$$

Thus, the spectral parameters which play the basic role in the integrable models and the generalized monodromy matrix particularly become

$$\partial_+ t = t(f(t)+g(t))\phi^{-1}\partial_+\phi \tag{3.19}$$

$$\partial_- t = t(f(t)+g(t))\phi^{-1}\partial_-\phi \tag{3.20}$$

where ϕ is the dilaton field.

Therefore, from the motion equations in[14] and these above results (with other details which will appear in our next work under study[16]), we determine that the monodromy matrix has the form

$$\hat{\mathcal{M}}^{(B=0)} = \hat{V}^{(B=0)}(x,t)((\hat{V}^{(B=0)})^T(x,\frac{1}{t})) = \begin{pmatrix} \hat{\mathcal{M}}(\omega) & 0 \\ 0 & \hat{\mathcal{M}}^{-1}(\omega) \end{pmatrix} \tag{3.21}$$

with

$$\eta^{\infty}(\hat{V}(x,t)) = \eta(\hat{V}(x,\frac{1}{t})) = (\hat{V}^{-1}(x,\frac{1}{t}))^T \tag{3.22}$$

and

$$\hat{\mathcal{M}} = \hat{V}(x,t)\hat{V}^T(x,\frac{1}{t}) \tag{3.23}$$

where η^{∞} is the symmetric space automorphism[17] and ω is the constant of integration for the spectral parameter, ω depend in this case on $f(t)$ and $g(t)$ functions and the dilaton field (its expression will be given in[16]). We deduce finally, that the monodromy matrix for $B = 0$ case can be written as

$$\hat{\mathcal{M}}_i(\omega) = \frac{\omega_i - \omega}{\omega_i + \omega} \tag{3.24}$$

and we can deduce the monodromy matrix for $B \neq 0$ from $B = 0$ case or directly, but we can see that the two ways to follow give the same result and are similar.

IV. CONCLUSION

In this work, we have explored the non compact $O(d,d)$ group that appears in toroidal compactification of oriented closed bosonic strings. Using the dimensional reduction methods, it is shown that this non compact group is an exact symmetry of the classical low-energy effective field theory. Then, we have reviewed approach to construct the $O(d,d)/O(d) \times O(d)$ coset using a generalized "vielbein" formalism and a triangular $2d \times 2d$ matrix V giving rise to the generalized monodromy matrix[16]. We have also shown that there exists an interesting connection between the integrability conditions of the two dimensional string effective action and T-duality properties.The symmetries of the string effective theory are encoded in the general monodromy matrix $\hat{\mathcal{M}}(\omega)$ only sensitive to the global $O(d,d)$ rotations.

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V. REFERENCES

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