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Invariant k-Symplectic Structures on Principal S^1-Bundles

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Abstract

The aim of this work is to study invariant k-symplectic structures on the total space of a principal bundle of structural group the circle S^1 in order to provide further examples of k-symplectic manifolds. We show that the existence of such structures gives rise to some constraint relations on the topology of the base space.

Keywords: k-symplectic structure, principal fibre bundle, torus-bundle, S^1-bundle.
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I. INTRODUCTION

Mathematical and physical considerations (the local study of the Pfaffian systems and Nambu's statistical mechanics) have led to introduce the k-symplectic structure . By the Heisenberg group of order k in the sense of Goze-Haraguchi, we see that the $k-$symplectic geometry is related to the $k-$contact systems in analogy with the well known relationship between symplectic and contact structures[2].

A $k-$symplectic structure on an $n\,(k+1)-$dimensional manifold is given by an $n-$codimensional foliation $\mathfrak{F}$ and a system of k differential closed $2-$forms vanishing on the subbundle of TM defined by $\mathfrak{F}$ with transversal characteristic spaces. The fundamental example is given by the natural $k-$symplectic structure on a fibre product of k Lagrangian fibrations; in particular, we have a $k-$symplectic structure on the Whitney sum $T^*B \oplus \ldots \oplus T^*B$ over an $n-$dimensional manifold B.

This paper is devoted to study invariant $k-$symplectic structure on the total space of a principal S^1-bundle with a codimension one foliation defined by a Pfaffian form. The invariance of the forms under the circle action on the total space, shows that differential forms defining the invariant $k-$symplectic structure on the total space of a principal circle bundle take a particular expressions, whose existence depends on the topology of the base space. This work is an extension of the symplectic case, studied in[3].

II. INVARIANT FORMS ON T^P-BUNDLE

Let M be a smooth differentiable manifold of dimension $n(k+1)$ provided with k closed forms $\omega^1, \omega^2, \ldots, \omega^k$ of degree 2 and with an $n-$codimensional foliation.
We denote by E the sub-bundle of TM defined by tangent vectors to the leaves of the foliation and by $\Gamma(E)$ the set of all cross-sections of the M-bundle $TM \to M$.
The set of all differential p-forms on M will be denoted by $\Lambda^p(M)$. We recall that the tuple $(\omega^1, \omega^2, \ldots, \omega^k; E)$ is called k-symplectic structure on M if the following conditions hold:

1. $C_x(\omega^1) \cap \ldots \cap C_x(\omega^k) = \{0\}$ for each $x \in M$
2. $\omega^a(X,Y) = 0$ for all X , $Y \in \Gamma(E)$ and $a \in \{1, \ldots, k\}$
   where $C_x(\omega^a) = \{X \in T_xM \mid i(X)\omega^a = 0\}$ is the characteristic space of the the differential form ω^a

Example II.1 *: Canonical k-symplectic structure on* $\mathbf{R^{n(k+1)}}$
Consider the real space $\mathbf{R^{n(k+1)}}$ *equipped with its Cartesian coordinates* $(x^{ai}, x^i)_{1 \le a \le k, 1 \le i \le n}$
Let E be the sub-bundle of TM defined by the equations

$$dx^1 = 0, ..., dx^n = 0$$

and let ω^a be the differential form on the real space given by:

$$\omega^a = \sum_{i=1}^{n} x^{ai} \wedge x^i \ \ (a = 1, \cdots, k)$$

The $(k+1)$-tuple $(\omega^1, \omega^2, ..., \omega^k; E)$ defines a k-symplectic structure on $\mathbf{R^{n(k+1)}}$ *called the canonical k-symplectic structure.*

The Darboux theorem shows that every k-symplectic manifold is locally isomorphic to $\mathbf{R^{n(k+1)}}$; that is to see, each point of M admits a neighborhood U and a local coordinate system $(x^{ai}, x^i)_{1 \le a \le k, 1 \le i \le n}$, called adapted to the k-symplectic structure $(\omega^1, \omega^2, \ldots, \omega^k; E)$ on M, such that the differential form ω^a is represented in U by:

$$\omega^a = \sum_{i=1}^{n} x^{ai} \wedge x^i$$

and the sub-bundle E is defined by the equations:

$$dx^1 = 0, \ldots, dx^n = 0$$

We assume that M is the total space of a principal bundle $\pi(M, B, T^p)$ over a differentiable smooth and closed manifold B with structural group the torus T^p; and also, we assume that the foliation is defined by n Pfaffian forms $\alpha^1, \ldots, \alpha^n$, of course we have :

$$d\alpha^i \wedge \alpha^1 \wedge \ldots \wedge \alpha^n = 0 \qquad \forall i \in \{1, \ldots, n\}$$

Condition (2) of definition (1) means that the system $\{\omega^1, \omega^2, \ldots, \omega^k\}$ is vanishing on the tangent vectors to the leaves of the foliation or equivalently that ω^a belongs to the ideal spanned by $\alpha^1, \ldots, \alpha^n$. This fact is expressed by :

$$\omega^a \wedge \alpha^1 \wedge \ldots \wedge \alpha^n = 0 \qquad \forall a \in \{1, \ldots, k\}$$

Our objective is to study the existence of k-symplectic structures on M. Such structures are considered to be invariant under the action of T^p on the total space. We have :

Definition II.1 *A $k-$symplectic structure $(\omega^1, \omega^2, ..., \omega^k; E)$ on M is called invariant under the action of T^p on the total space(or simply invariant) if both ω^a and the α^i are invariant.*

In this paragraph, we give an expression the differential forms $\omega^1, \omega^2, \ldots, \omega^k, \alpha^1, \ldots, \alpha^n$ that reflect their invariance. Let's first give some notations. $\eta = (\eta^1, \ldots, \eta^p)$ stands for a given connection 1-form on the fibre bundle, and $\Omega_M = (\Omega_M^1, \ldots, \Omega_M^p)$ for its curvature.The torus T^p being an abelian Lie group, the structural equations are given by :

$$\Omega_M^i = d\eta^i \qquad \forall i \in \{1, \ldots, p\}$$

and the Ω_M^i are basic invariant 2-forms; then, there exists a (unique) 2-form Ω on the base space B such that $\Omega_M = \pi^\star(\Omega)$ (see for example[4,6]); and hence

$$d\eta^i = \pi^\star(\Omega^i) \qquad \forall i \in \{1, \ldots, p\}$$

Proposition II.1 *A Pfaffian form α on M is invariant if and only if it can be written*[8]*:*

$$\alpha = \sum_{i=1}^{p} \pi^\star(f_i)\eta^i + \pi^\star(\psi)$$

where $f_i \in \Lambda^0(M)$ and $\psi \in \Lambda^1(M)$

Proposition II.2 *The base space B is assumed to be connected*[8] *. A closed 2-form ω on M is invariant if and only if it can be written:*

$$\omega = \sum_{i=1}^{p} \pi^\star(\varphi_i) \wedge \eta^i + \sum_{i<j} \Gamma_{ij}\eta^i \wedge \eta^j + \pi^\star(\theta)$$

where $\varphi_i \in \Lambda^1(B)$, $\theta \in \Lambda^2(B)$ and Γ_{ij} are constant real (skew-symmetric in i and j) satisfying:

$$d\varphi_i = \sum_{j=1}^{p} \Gamma_{ij}\Omega^j \quad \text{and} \quad d\theta = \sum_{i=1}^{p} \varphi_i \wedge \Omega^i$$

Γ_{ij} are real valued functions on M. The connectedness of the base space allows them to be constant real.

III. INVARIANT K-SYMPLECTIC STRUCTURES ON PRINCIPAL S^1-BUNDLES

From now on, we will deal with the case where the bundle is of structural group the circle S^1 and the total space M is of dimension $k+1$; equipped with a codimension one foliation defined by a non-vanishing Pfaffian form α. The base space is a closed smooth differential manifold. On the one hand, we have :

$$\alpha = \pi^\star(f)\eta + \pi^\star(\psi), \tag{3.1}$$
$$\alpha \wedge d\alpha = 0 \tag{3.2}$$

Explicating these equations, one has:

$$[\pi^\star(f)\eta + \pi^\star(\psi)] \wedge [\pi^\star(df) \wedge \eta + \pi^\star(f\Omega + d\psi)] = 0$$

that is

$$\pi^\star[f^2\Omega + fd\psi - df \wedge \psi)] \wedge \eta + \pi^\star[f\Omega \wedge \psi + \psi \wedge d\psi] = 0$$

We obtain

$$f^2\Omega + fd\psi - df \wedge \psi = 0 \tag{3.3}$$

and

$$f\Omega \wedge \psi + \psi \wedge d\psi = 0 \tag{3.4}$$

On the other hand, the 2−form ω^a belongs to the ideal generated by α:

$$\omega^a \wedge \alpha = 0 \tag{3.5}$$

ω^a has the form:

$$\omega^a = \pi^\star(\varphi^a) \wedge \eta + \pi^\star(\theta^a) \tag{3.6}$$

Combining these last two equations we obtain

$$[\pi^\star(\varphi^a) \wedge \eta + \pi^\star(\theta^a)] \wedge [\pi^\star(f)\eta + \pi^\star(\psi)] = 0$$

then

$$\pi^\star(f\theta^a - \varphi^a \wedge \psi) \wedge \eta + \pi^\star(\theta^a \wedge \psi) = 0$$

Henceforth

$$f\theta^a - \varphi^a \wedge \psi = 0 \tag{3.7}$$

$$\theta^a \wedge \psi = 0 \tag{3.8}$$

Let's now introduce the set $f^{-1}(0)$ that we denote by $\Sigma(E)$. It's called the singular set associated to the invariant Pfaffian[8,9] form α . We will study the particular two cases $\Sigma(E) = \emptyset$ and $\Sigma(E) = B$.

A. The case $\Sigma(E) = \emptyset$

One should remark that in the case $\Sigma(E) = \emptyset$ the function f is non-vanishing on B. Using equations (3) and (7) we can write:

$$0 = \Omega + d(\frac{\psi}{f}), \tag{3.9}$$

$$\theta^a = \varphi^a \wedge \frac{\psi}{f} = 0 \quad \forall a \in \{1, \dots, k\} \tag{3.10}$$

For simplicity, we are going to change the connection form η by the well defined new one

$$\widetilde{\eta} = \eta + \pi^\star(\frac{\psi}{f})$$

whose curvature 2-form is

$$\widetilde{\Omega} = \Omega + d(\frac{\psi}{f}) = 0$$

It turns out that the bundle is flat. With this change of connection form, the expressions of ω^a and α become :

$$\omega^a = \pi^\star(\varphi^a) \wedge \widetilde{\eta}$$

where φ^a is a non-vanishing closed 1-form on B and

$$\alpha = \pi^\star(f)\widetilde{\eta}$$

Proposition III.1 *$(\omega^1, \omega^2, ..., \omega^k; E)$ is an invariant k-symplectic structure on the total space M for which $\Sigma(E) = \emptyset$ if and only if the product $\varphi^1 \wedge ... \wedge \varphi^k$ is a volume form on the base space B*

Proof. Recall that $(\omega^1, \omega^2, \dots, \omega^k; E)$ is an invariant k-symplectic structure on M if and only if $C_x(\omega^1) \cap \dots \cap C_x(\omega^k) = \{0\}$ for every $x \in M$. Further,

$$C_x(\omega^a) = \{X \in T_xM / i(X)(\pi^\star(\varphi^a) \wedge \widetilde{\eta}) = 0\}$$

and one can easily see that

$$C_x(\omega^a) = \ker \pi^\star(\varphi^a_x) \cap \ker \widetilde{\eta}_x$$

Consequently, $C_x(\omega^1) \cap \dots \cap C_x(\omega^k) = \{0\}$ if and only if

$$\ker \pi^\star(\varphi^1_x) \cap \dots \cap \ker \pi^\star(\varphi^k_x) \cap \ker \widetilde{\eta}_x = \{0\}$$

if and only if

$$\pi^\star(\varphi^1_x) \wedge \dots \wedge \pi^\star(\varphi^k_x) \wedge \widetilde{\eta}_x \neq 0$$

which is equivalent to

$$\pi^{\star}(\varphi_x^1 \wedge \ldots \wedge \varphi_x^k) \neq 0$$

or simply

$$\varphi_x^1 \wedge \ldots \wedge \varphi_x^k \neq 0$$

Corollary III.1 *If $(\omega^1, \omega^2, ..., \omega^k; E)$ is an invariant k-symplectic structure on M satisfying $\Sigma(E) = \emptyset$ then the base space B is a fibre bundle over the torus T^k*

Proof. If $(\omega^1, \omega^2, \ldots, \omega^k; E)$ is an invariant k-symplectic structure on M satisfying $\Sigma(E) = \emptyset$ then for every $x \in M$

$$\varphi_x^1 \wedge \ldots \wedge \varphi_x^k \neq 0$$

It means that the base space B admits k closed 1-forms $\varphi^1, \ldots, \varphi^k$ independent everywhere in B. The following lemma due to Tishler achieves the proof.

Lemma III.1 *If a closed r-dimensional manifold admits k non-vanishing closed 1-forms $\varphi^1, ..., \varphi^k (k \leq r)$ independent everywhere then it's a fibre bundle over the torus T^k*

The proposition below occurs an example of 2-symplectic structure on a 3-dimensional principal S^1- *bundle.*

Proposition III.2 *If B is a closed, without boundary, surface of genus g then M admits a 2-symplectic structure $(\omega^1, \omega^2; E)$ for which $\Sigma(E) = \emptyset$ if and only if $g = 1$(B is diffeomorphic to the torus T^2)*

Proof. If $(\omega^1, \omega^2; E)$ is an invariant 2-symplectic structure satisfying $\Sigma(E) = \emptyset$ then corollary (1) asserts that B is a fibre bundle over the torus T^2. Hence $g = 1$ and consequently the base space is diffeomorphic to T^2.
Inversely, if B is diffeomorphic to $T^2 (g = 1)$ then it possesses two non-vanishing closed Pfaffian forms φ^1 and φ^2 independent everywhere in B.

Let Ω be a 2-form on B such that $[\Omega] = 0$ where $[\Omega]$ denotes the class of Ω in $H^2(B, \mathbf{Z})$. There is $\beta \in \Lambda^1(B)$ such that $\Omega = d\beta$ and there is a connection 1-form η on M such that $\pi^{\star}(\Omega) = d\eta$. Let us put :

$$\widetilde{\eta} = \eta - \pi^{\star}(\beta), \quad \alpha = \pi^{\star}(f)\widetilde{\eta},$$
$$\omega^a = \pi^{\star}(\varphi^a) \wedge \widetilde{\eta} \quad \forall a \in \{1, 2\}$$

where f is a function on B everywhere non-zero. It's easy to see that the triple $(\omega^1, \omega^2; E)$ defines an invariant 2-symplectic structure satisfying $\Sigma(E) = \emptyset$.

B. The case $\Sigma(E) = B$

The function f vanishes on B. We have :

$$\alpha = \pi^{\star}(\psi)$$

where ψ is a non-singular 1-form on the base space. By equations (7) and (8) we have:

$$\varphi^a \wedge \psi = 0 \quad and \quad \theta^a \wedge \psi = 0 \quad \forall a \in \{1, \ldots, k\}$$

We deduce the existence of two differential forms $\lambda^a \in \Lambda^0(B)$ and $\mu^a \in \Lambda^1(B)$ such that:

$$\varphi^a = -\lambda^a.\psi \quad and \quad \theta^a = \mu^a \wedge \psi$$

Consequently, the form ω^a has the expression :

$$\omega^a = \xi^a \wedge \alpha$$

where:

$$\xi^a = \pi^\star(\lambda^a)\eta + \pi^\star(\mu^a)$$

is an invariant 1-form on the total space.

Proposition III.3 *With the expression of ω^a above, the following equivalence $(1) \Leftrightarrow (2)$ holds:*

1. *$(\omega^1, \omega^2, ..., \omega^k, E)$ is an invariant k-symplectic structure on the total space M for which $\Sigma(E) = B$*
2. *$\xi^1 \wedge ... \wedge \xi^k \wedge \alpha \neq 0$*

Proof. Hint: As we did in the previous proofs, it suffices to see that for every $x \in M$,

$$C_x(\omega^a) = \ker \xi^a_x \cap \ker \alpha_x$$

Remark that if $(\omega^1, \omega^2, \ldots, \omega^k; E)$ is an invariant k-symplectic structure for which $\Sigma(E) = B$ then the λ^as do not vanish simultaneously. Indeed, if not, the product $\mu^1 \wedge \ldots \wedge \mu^k \wedge \psi$ will be a non-vanishing differential form of degree $k+1$ on the k-dimensional space B, this fact is evidently impossible.

Consider now the particular case of a 3-dimensional total space. We are interested in 2-symplectic structures ($k = 2$). The preceding proposition asserts that the triple $(\omega^1, \omega^2; E)$ is an invariant 2-symplectic structure on M if and only if $\xi^1 \wedge \xi^2 \wedge \alpha \neq 0$

- If for $a \in \{1, 2\}$ one has $\lambda^a = 0$ then the form μ^a and the function $\lambda^b, b \neq a$ are non-vanishing on B.

Proposition III.4 *If the base space B is an oriented surface which is the total space of a fibre bundle over the torus T^2 then M admits an invariant 2-symplectic structure $(\omega^1, \omega^2; E)$ for which $\Sigma(E) = B$.*

Proof. Since the manifold B is a fibred over the torus T^2, one can deduce the existence of 2 non-vanishing closed Pfaffian forms μ^1 and ψ. Let's take:

$$\alpha = \pi^\star(\psi),$$
$$\omega^1 = \pi^\star(\mu^1 \wedge \psi)$$

and

$$\omega^2 = \pi^\star(\lambda) \wedge \eta + \pi^\star(\mu^2)$$

The triple $(\omega^1, \omega^2; E)$ determines on M a 2-symplectic structure (with $\Sigma(E) = B$).

- If for $a \in \{1, 2\}$ one has $\mu^a = 0$ then the form $\mu^b = 0, b \neq a$ and the function λ^b are non-vanishing on B. In particular B admits an orientable codimension one foliation which has no holonomy.

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