



FQH Droplets Wave Function on Spheres

R. Ahl Laamara, L.B Drissi, E.H Saidi

1. *Lab/UFR-Physique des Hautes Energies, Faculté des Sciences de Rabat, Morocco.*
2. *Groupement National de Physique des Hautes Energies, GNPHE; Siege focal, Lab/UFR-HEP, Rabat, Morocco.*
3. *Virtual African Centre for Basic Science & Technology, Focal point, Lab/UFR-PHE, FSR.*

Abstract

Using matrix model formulation of Laughlin fluid as well as $su(2)$ representation theory, we consider fractional Quantum Hall system on 3d and 2d spheres and derive explicitly the ground state wave function of the corresponding fluid droplets.

Keywords: FQHE, Matrix model, Higher dimensions, Ground state wave function, Conifold.

I. INTRODUCTION

In the past few years; a great deal of interest was devoted to fractional quantum Hall (FQH) fluids in Laughlin state¹⁻⁶. These studies were carried out in the context of Susskind proposal as systems moving in $(2 + 1)$ real dimensions. Recently, they have been extended to higher dimensions⁷⁻¹⁶. These generalizations include quantum Hall solitons and involve non commutative Chern-Simons gauge theory as well as type II superstrings compactified on Calabi-Yau threefolds and topological string theory.

In particular we have studied in¹⁵, a FQH system in fluid approximation living in complex three dimension space in presence of a strong constant RR B-field. This system corresponds to FQH particles viewed, in the string theory language, as D0-

branes coming from D1 strings wrapped on S^1 . Moreover, our D strings fluid (DSF) proposal is thought of as enveloping Susskind description of FQH system, by taking appropriate parameter limits of DSF moduli space in such a way that the real geometry of FQH system is contained in conifold. This result yields us to consider in¹⁶ a second alternative using matrix model formulation for FQH droplets on the spheres S^3 and S^2 . This approach may be viewed as a linearized description of Morariu-Polychronakos model on S^2 because it avoids the high non linearity of the matrix field elaborated in⁶.

In this paper we return to these two works^{15,16} and review the basic lines of the ideas developed there. One of the principal purposes of this paper is to find the wave functions denoted $|\Phi\rangle$ which describe FQH droplets on S^2 . But, before passing on to this main business, it is interesting to treat briefly ideas useful for our analysis as we will see later on.

A crucial ingredient of our works^{15,16} is the constrained method using the Lagrange multipliers. This method has the advantage of building FQH extensions moving on S^2 and geometries beyond.

The paper is organized as follows: In section 2

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we start with a brief summary presenting conifold geometry and the 2, 3-spheres. After we derive the classical action of FQH fluid on S^3 . Section 3 focus on a new proposal of FQH droplets on the spheres. The later section is specialized even further to a quantum version of the systems. This, of course, leads to an explicit expression of the droplet ground state. Finally, we end with a conclusion summarizing the basic ideas of this paper.

II. FQH FLUID CLASSICAL FIELD ACTION

We are interested to find the action describing classical FQH fluid dynamics on the spheres realized as subconifolds.

A. Conifold as mother manifold

1. Conifold picture

We start by introducing special properties of conifold geometry. From conifold point of view; the D string system we are considering is a set of N D strings running in conifold in presence of a constant RR background field B that governs their dynamics. In this picture, all interactions are neglected by taking B field of type IIB very strong.

For this case of D string with world sheet variable $\xi = t + i\sigma$, the fields should satisfy the following projective transformations

$$(x, y, z, w) \rightarrow (\lambda x, \frac{1}{\lambda}y, \lambda z, \frac{1}{\lambda}w). \quad (2.1)$$

together with the local constraint equation

$$x(\xi)y(\xi) - z(\xi)w(\xi) = \mu \quad (2.2)$$

where μ is a complex constant. The role of the noncommutative parameter θ of usual FQH liquid is played here by this complex modulus μ of the conifold⁸.

From these formulas we reads that the string dynamics involves five complex holomorphic variables namely (ξ, x, y, z, w) obeying the two conditions (2.2, 2.1). So the resulting set, denoted (ξ, u, v) , parameterizes a complex three dimension projective hypersurface which is the deformed conifold geometry T^*S^3

$$T^*S^3 \simeq T^*S^1 \times T^*S^2. \quad (2.3)$$

Note that the fiber T^*S^1 describes the D-string world interacting RR magnetic field and moving on the base T^*S^2 .

Such D-strings, moving with complex holomorphic

coordinates, collapse to the usual point like particles parameterized by $(z(t), \bar{z}(t))$. In this spirit, the local coordinates (t, z, \bar{z}) of real three dimension space are analog to (ξ, u, v) that parameterize a complex three dimension space, which is just the conifold T^*S^3 defined in (2.3).

Armed with this observation, we could show easily that the spheres S^3 and S^2 appears then as a special real lower dimension slices of conifold.

Indeed, reducing the degrees of freedom down to three follows by restricting the conifold analysis to its Lagrangian sub-manifolds. This aim is achieved by identifying ξ with σ ; that is a periodic time. This leads to S^3 .

On the other hand; the second geometry S^2 is obtained by restricting world sheet variable $\xi = t + i\sigma$ to its real part. Explicitly; by setting $y = \bar{x}$ and $w = \bar{z}$, the eq(2.2) takes the form $|x|^2 + |z|^2 = Re\mu$, where the coordinates satisfy the condition $(x, z) \rightarrow e^{i\theta}(x, z)$, defining then a real two sphere S^2 embedded in complex space \mathbb{C}^2 parameterized by (x, z) .

2. Subconifolds coordinate frame

Among the different ways of parameterizing the spheres S^3 and S^2 , we choose the one that use manifest isometry by constructing them as hypersurfaces in \mathbb{C}^2 .

Starting from this complex space \mathbb{C}^2 with local holomorphic coordinates x and y , the 3-sphere S^3 of radius R is realized as a real hypersurface $H_{real} = H_{real}(x, y, \bar{x}, \bar{y})$. To do so; (x, y) and their complex conjugates must obey the following constraint relation,

$$H_{real}(x, y, \bar{x}, \bar{y}) : x\bar{x} + y\bar{y} = |x|^2 + |y|^2 = R^2, \quad (2.4)$$

Using representations of the $SU(2)$ isometry of S^3 ; in particular the isospinors,

$$z^i = (x, y), \quad \bar{z}_i = \overline{(z^i)} = (\bar{x}, \bar{y}), \quad i = 1, 2. \quad (2.5)$$

the eq(2.4) reads as follows,

$$H_{real}(z, z) : \sum_{i=1}^2 z^i \bar{z}_i = R^2. \quad (2.6)$$

On the other hand; to reduce S^3 down to S^2 ; we use the fibration $S^3 \simeq S^1 \times S^2$ corresponding to

$$SU(2) \rightarrow SU(2)/U_C(1). \quad (2.7)$$

which leads to the identification

$$x' \equiv e^{i\varphi}x, \quad y' \equiv e^{-i\varphi}y, \quad (2.8)$$

where φ is the group parameter of $U_C(1)$ Cartan abelian sub-symmetry of the $SU(2)$ isometry of the 3-sphere.

B. N FQH particles field action on S^3

Taking advantage from previous results; we focus our interest in what follows to build classical field action of N FQH particles moving on S^3 .

To implement dynamics, we should add time variable to the coordinate already introduced. For a set of N particles described by $x_a(t), y_a(t)$ and $\bar{x}_a(t), \bar{y}_a(t)$ with $a = 1, \dots, N$, moving in the complex space \mathbb{C}^{\neq} , the restriction of their dynamics on the 2-sphere is achieved by requiring a set of two constraint relations at every moment t , for $a=1, \dots, N$;

$$x_a(t)\bar{x}_a(t) + y_a(t)\bar{y}_a(t) = R^2, \quad (2.9)$$

and

$$x'_a(t) = e^{i\varphi(t)}x_a(t), \quad y'_a(t) = e^{-i\varphi(t)}y_a(t) \quad (2.10)$$

where the phase $\varphi = \varphi(t)$ is time dependent. As far as the classical field action of the N FQH particles systems moving on $S^3 \sim SU(2)$ is concerned; we proceed as follows:

1/ For the isodoublet coordinates $z_a^i = (x_a, y_a)$, the Lagrangian density $L_4 = L_4(z^{ai}, \bar{z}_{ai})$ of the system moving on the complex two dimension space $\mathbb{C}^2 \sim \mathbb{R}^4$ reads as,

$$dt L_4 = -\frac{iB}{2} \sum_{a=1}^N (\bar{z}_{ai} dz^{ai} - z^{ai} d\bar{z}_{ai}). \quad (2.11)$$

2/ We implement in L_4 , by using the Lagrange method, the constraint eq

$$B(x_a\bar{x}_a + y_a\bar{y}_a) = l, \quad a = 1, \dots, N, \quad (2.12)$$

to get the Lagrangian density L_3 describing the dynamics of the N FQH particles restricted to moving on S^3 embedded into \mathbb{C}^2 . So we get

$$L_3(z, \bar{z}, C) = -B \sum_{a=1}^N \left[i\bar{z}_{ia} \mathcal{D}_t z^{ia} - C_a \frac{1}{B} \right], \quad (2.13)$$

where the real field $C_a = C_a(t)$ is the Lagrange gauge field capturing the constraint eq(2.12) and $\mathcal{D}_t z^{ia} = \left(\frac{dz^{ia}}{dt} - iC_a z^{ia} \right)$ is the gauge covariant derivative.

We deduce from (2.13) that $\pi_{ia} = -iB\bar{z}_{ia}$ is the conjugate momenta of the field variable z^{ia} . The corresponding Poisson bracket is defined as $\{F, G\}_{PB} = \sum_{l,e} \left(\frac{\partial F}{\partial z^{le}} \frac{\partial G}{\partial \pi_{le}} - \frac{\partial F}{\partial \pi_{le}} \frac{\partial G}{\partial z^{le}} \right)$, so it leads to

$$\begin{aligned} \{z^{ia}, \bar{z}_{jb}\}_{PB} &= \frac{i}{B} \delta_j^i \delta_b^a, \\ \{z^{ia}, z^{jb}\}_{PB} &= \{\bar{z}_{ia}, \bar{z}_{jb}\}_{PB} = 0. \end{aligned} \quad (2.14)$$

C. FQH fluid approximation

Recall that we are interested in FQH fluid on the spheres S^3 and S^2 embedded in \mathbb{C}^2 . So besides extra constraint relations due to non flat geometries of these varieties eqs (2.9-2.10), we have the usual one dealing with fluid incompressibility. This condition is essential in the field theoretic modeling of FQH systems since it plays a central role in understanding the quantum dynamics.

In the fluid approximation requiring a large number N of particles, z_a^i and \bar{z}_{ia} are promoted to highest dimensional fields as shown below,

$$\begin{aligned} z_a^i(t) &\rightarrow z^i(t; \tau, \bar{\tau}, \sigma, \bar{\sigma}), \\ C_a(t) &\rightarrow C(t; \tau, \bar{\tau}, \sigma, \bar{\sigma}) \end{aligned} \quad (2.15)$$

So the incompressibility condition of the FQH fluid is translated into a condition on the Jacobian \mathcal{J}_{ac} of the transformation,

$$\tau \rightarrow x(\tau, \sigma; \bar{\tau}, \bar{\sigma}), \quad \sigma \rightarrow y(\tau, \sigma; \bar{\tau}, \bar{\sigma}), \quad (2.16)$$

together with similar transformations for the complex conjugate partners. This Jacobian is a polynomial quadratic in the Poisson brackets and it involves the sum of six terms.

Fluid incompressibility requires that the absolute value $|\mathcal{J}_{ac}| = \left| \frac{\partial^4(x, y; \bar{x}, \bar{y})}{\partial^4(\tau, \sigma; \bar{\tau}, \bar{\sigma})} \right|$ should satisfy $|\mathcal{J}_{ac}| = 1$. To deal with this condition; we simplify the \mathcal{J}_{ac} expression by restricting the coordinate transformations (2.16) to the sub-class,

$$\tau \rightarrow x(\tau, \bar{\tau}), \quad \sigma \rightarrow y(\sigma, \bar{\sigma}). \quad (2.17)$$

The fluid incompressibility condition reduces to the following relation,

$$\{x, \bar{x}\}_{PB} \{y, \bar{y}\}_{PB} = -\theta^2, \quad (2.18)$$

where θ is the usual non commutative coordinates parameters with $\theta B = k$. This constraint eq can be solved as $\{x, \bar{x}\}_{PB} = i\theta$ and $\{y, \bar{y}\}_{PB} = i\theta$ or equivalently by using the $SU(2)$ doublets as follows,

$$\{\bar{z}_i, z^j\}_{PB} = -i\theta \delta_i^j. \quad (2.19)$$

Implementing this constraint eq into eqs(2.13) by using the Lagrange method, we get the field action $S_3 = S_3(z, \bar{z}, C, A)$ for incompressible fluid running on the 3-sphere,

$$S_3 = -B \int dt d^2\tau d^2\sigma \left[i\bar{z}_i \nabla_t z^i - C \frac{1}{B} - A \frac{k}{B} \right], \quad (2.20)$$

where we have set,

$$\nabla_t z^i = \mathcal{D}_t z^i - i \{A, z^i\}_{PB}, \quad (2.21)$$

and where θ is the non commutative parameter considered previously. The gauge field $A = A(t, \tau, \sigma; \bar{\tau}, \bar{\sigma})$ is a real gauge field capturing the fluid incompressibility constraint eq.

III. MATRIX MODEL OF DROPLETS ON SUBCONIFOLDS

The main goal of this section is to develop a matrix description of the FQH droplets on the real spheres \mathbb{S}^3 and \mathbb{S}^2 . For this aim we proceed as follows:

To build the matrix model; we elevate the commuting coordinate position variables $\{z_a^i(t), \bar{z}_i^a, 1 \leq a \leq N\}$ of the ambient space \mathbb{C}^2 into diagonal $N \times N$ complex matrices X_0 and Y_0 ; or more generally, equivalent matrices X and Y related to them by using $U(N)$ similarity transformations

$$P : X_0^+ X_0 \rightarrow X^+ X = P^+ (X_0^+ X_0) P, \quad (3.1)$$

with the $N \times N$ matrix P satisfies $P^+ P = P P^+ = I_{\text{id}}$ to ensure the hermiticity property of $X^+ X$ and the same must be done for $Y^+ Y$.

The constraint relations (2.12), describing dynamics of droplet on \mathbb{S}^3 , become in the diagonal case

$$B (X_0^+ X_0 + Y_0^+ Y_0) = l I_{\text{id}}. \quad (3.2)$$

In terms of $N \times N$ matrices X and Y as well as their adjoints, that encode the coordinate positions of the FQH droplet restricted to move on the 3-sphere \mathbb{S}^3 ; the condition reads as,

$$B \text{Tr} (X^+ X + Y^+ Y) = l N, \quad (3.3)$$

Besides, the reduction down to the \mathbb{S}^2 geometry requires

$$B \text{Tr} (X^+ X - Y^+ Y) = m N, \quad (3.4)$$

where m is a relative integer. Combing eqs(3.3) and (3.4), one has

$$\begin{aligned} B \text{Tr} (Y^+ Y) &= \frac{l - m}{2} N \\ B \text{Tr} (X^+ X) &= \frac{l + m}{2} N. \end{aligned} \quad (3.5)$$

Positivity of these quantities leads to the condition,

$$m N \leq l N. \quad (3.6)$$

Since the matrix variables X and Y carry an $SU(2)$ charge; the relation (3.6) recalls the usual $SU(2)$ spin projection inequality,

$$j_z \leq j. \quad (3.7)$$

On the other hand; by implementing the matrix elevation $x \rightarrow X$ and $y \rightarrow Y$ and taking into account all the constraint eqs for droplet dynamics on \mathbb{S}^3 and on \mathbb{S}^2 in addition to the fluid incompressibility condition; we obtain a matrix field action denoted as $\mathcal{S}_2 = \mathcal{S}_2[X, Y, \mathcal{C}, \mathcal{B}, \mathcal{A}]$ which reads, up to a total derivative, as follows ,

$$\begin{aligned} \mathcal{L}_2 &= -i B \text{Tr} \left[X^+ \frac{dX}{dt} + Y^+ \frac{dY}{dt} \right] \\ &+ B \text{Tr} \left[\mathcal{A} \left([X^+, X] + [Y^+, Y] - \frac{k}{B} \right) \right] \\ &- B \text{Tr} \left[\mathcal{C} \left((X^+ X + Y^+ Y) - \frac{1}{B} \right) \right] \\ &- B \text{Tr} \left[\mathcal{B} \left((X^+ X - Y^+ Y) - \frac{m}{B} \right) \right]. \end{aligned} \quad (3.8)$$

The Lagrange gauge fields $\mathcal{C}(t)$, $\mathcal{B}(t)$, $\mathcal{A}(t)$ capture the three constraint relations also mentioned. Minimizing \mathcal{L}_2 with respect to these gauge fields lead to the explicit expression of matrix conditions,

$$\begin{aligned} B [X^+, X] + B [Y^+, Y] &= k I_{\text{id}}, \\ B (X^+ X + Y^+ Y) &= l I_{\text{id}}, \\ B (X^+ X - Y^+ Y) &= m I_{\text{id}}. \end{aligned} \quad (3.9)$$

It is clear that we have a problem when taking the trace of both sides of the first relation of eqs(3.9). To overcome this difficulty, we add the Polychronakos field $\Psi = (\Psi^a(t))$ transforming in the fundamental representation of $U(N)$ as a scalar field under the isometry of the spheres; so it has no coupling with the extra gauge fields \mathcal{B} and \mathcal{C} .

Then, the full classical Lagrangian density of the matrix model proposal for FQH droplet on the 2-sphere becomes $\mathcal{S} = \int dt \mathcal{L}$,

$$\mathcal{L} = \mathcal{L}_2 - \text{Tr} \left[\Psi^+ \left(i \frac{d\Psi}{dt} + \mathcal{A} \Psi \right) \right]. \quad (3.10)$$

Setting $T_0 = \sum_{a=1}^N \Psi_a^+ \Psi^a = k N$, the modified constraint relations are,

$$\begin{aligned} T_b^a &= B [X^+, X]_a^b + B [Y^+, Y]_a^b + \Psi_b^+ \Psi^a, \\ S_b^a &= T_b^a - T_0 \delta_b^a = 0 \\ E_0 &= B \text{Tr} (X^+ X + Y^+ Y) = l N, \\ I_0 &= B \text{Tr} (X^+ X - Y^+ Y) = m N. \end{aligned} \quad (3.11)$$

Another consequence of Lagrangian density variations gives the conjugate momenta Π_a^b , Γ_a^b and Υ_a

$$\Pi_a^b = -i B X_a^{*b}, \quad \Gamma_a^b = -i B Y_a^{*b}, \quad \Upsilon_a = -i \Psi_a^*.$$

Using the following Poisson bracket for functions F and G ,

$$\begin{aligned} \{F, G\}_{PB} = & +i \left(\frac{\partial F}{\partial \Psi_c^*} \frac{\partial G}{\partial \Psi^c} - \frac{\partial F}{\partial \Psi^c} \frac{\partial G}{\partial \Psi_c^*} \right) \\ & + \frac{i}{B} \left(\frac{\partial F}{\partial X_b^{*a}} \frac{\partial G}{\partial X_a^b} - \frac{\partial F}{\partial X_a^b} \frac{\partial G}{\partial X_b^{*a}} \right) \\ & + \frac{i}{B} \left(\frac{\partial F}{\partial Y_b^{*a}} \frac{\partial G}{\partial Y_a^b} - \frac{\partial F}{\partial Y_a^b} \frac{\partial G}{\partial Y_b^{*a}} \right), \end{aligned}$$

we have the following classical relations for the canonical variables,

$$\begin{aligned} \{\Psi_a^*, \Psi^b\} &= +i\delta_a^b, \\ \{X_a^{*d}, X_c^b\} &= +\frac{i}{B}\delta_a^b\delta_c^d, \\ \{Y_a^{*d}, Y_c^b\} &= +\frac{i}{B}\delta_a^b\delta_c^d, \end{aligned} \quad (3.12)$$

and zero for all remaining others.

IV. WAVE FUNCTIONS FOR DROPLETS

This part is devoted to study the quantization of above FQH droplets on the spheres \mathbb{S}^3 and \mathbb{S}^2 respectively. In other words; we derive the explicit form of the wave functions of their ground states $|\Phi\rangle$.

A. $\mathbb{D}_{a_1 \dots a_q}^{(n)\dagger}$ building blocks

To build the quantum states of the fluid droplets on the spheres, we develop some specific features of creation operators monomials.

For quantum matrix model, the canonical commutation relations (3.12) become for $\theta = \frac{1}{B}$

$$\begin{aligned} [\Psi_a^\dagger, \Psi^b] &= -\delta_a^b, \\ [X_a^{\dagger c}, X_b^d] &= -\theta \delta_a^d \delta_b^c, \\ [Y_a^{\dagger c}, Y_b^d] &= -\theta \delta_a^d \delta_b^c. \end{aligned} \quad (4.1)$$

Moreover, the quantum version of classical eqs(3.11) for FQH droplets on the 2-sphere, reads as;

$$\begin{aligned} S_b^a |\Phi\rangle &= 0, \\ \mathcal{T}_I |\oplus\rangle &= kN |\Phi\rangle, \\ \mathcal{E}_0 |\Phi\rangle &= lN |\Phi\rangle, \\ \mathcal{I}_0 |\Phi\rangle &= mN |\Phi\rangle, \end{aligned} \quad (4.2)$$

where the quantum operators \mathcal{T}_I , \mathcal{E}_0 and \mathcal{I}_0 , are given by,

$$\begin{aligned} \mathcal{T}_I &= \text{Tr}(\Psi^\dagger \Psi), \\ \mathcal{E}_0 &= B \text{Tr}(X^\dagger X + Y^\dagger Y), \\ \mathcal{I}_0 &= B \text{Tr}(X^\dagger X - Y^\dagger Y). \end{aligned} \quad (4.3)$$

They depend linearly on the creation/annihilation operators Ψ^\dagger/Ψ , X^\dagger/X and Y^\dagger/Y .

The wave functions describing droplets on the 2-sphere is

$$|\Phi\rangle \equiv |\Phi_{k,l,m}\rangle \equiv |\Phi_{k,j,j_z}\rangle, \quad (4.4)$$

with the isospin $j = lN$ is the highest weight and $|j_z| \leq j$.

Combining the creators Ψ^\dagger and X^\dagger , Y^\dagger as follows

$$\begin{aligned} A_{a_0}^\dagger &= \Psi_{a_0}^\dagger = B_{a_0}^\dagger, \\ A_{a_n}^\dagger &= \Psi_{b_0}^\dagger X_{b_1}^{\dagger b_0} \dots X_{a_n}^{\dagger b_{n-1}} \quad n \geq 1, \\ B_{a_n}^\dagger &= \Psi_{b_0}^\dagger Y_{b_1}^{\dagger b_0} \dots Y_{a_n}^{\dagger b_{n-1}}; \quad n \geq 1, \end{aligned} \quad (4.5)$$

allows us to define new specific building blocks,

$$\mathbb{A}_{a_1 \dots a_p}^\dagger = \prod_{n=1}^p A_{a_n}^\dagger, \quad \mathbb{B}_{a_1 \dots a_q}^\dagger = \prod_{n=1}^q B_{a_n}^\dagger. \quad (4.6)$$

The commutation relations of these hermitian operators with \mathcal{T}_I , \mathcal{E}_0 and \mathcal{I}_0 are,

$$\begin{aligned} [\mathcal{T}_0, \mathbb{A}_{a_1 \dots a_p}^\dagger] &= p \mathbb{A}_{a_1 \dots a_p}^\dagger, \\ [\mathcal{T}_0, \mathbb{B}_{a_1 \dots a_q}^\dagger] &= q \mathbb{B}_{a_1 \dots a_q}^\dagger \\ [\mathcal{E}_0, \mathbb{A}_{a_1 \dots a_p}^\dagger] &= \frac{p(p+1)}{2} \mathbb{A}_{a_1 \dots a_p}^\dagger, \\ [\mathcal{E}_0, \mathbb{B}_{a_1 \dots a_q}^\dagger] &= \frac{q(q+1)}{2} \mathbb{B}_{a_1 \dots a_q}^\dagger, \\ [\mathcal{I}_0, \mathbb{A}_{a_1 \dots a_p}^\dagger] &= \frac{p(p+1)}{2} \mathbb{A}_{a_1 \dots a_p}^\dagger, \\ [\mathcal{I}_0, \mathbb{B}_{a_1 \dots a_q}^\dagger] &= -\frac{q(q+1)}{2} \mathbb{B}_{a_1 \dots a_q}^\dagger. \end{aligned}$$

Since the matrix operators X^\dagger and Y^\dagger as well as their adjoint partners play a symmetric role; as they are rotated under $SU(2)$ isometry of the 3-sphere generated by the operators J_0 and J_\pm

$$\begin{aligned} J_0 &= \frac{1}{\theta} (X^\dagger X - Y^\dagger Y), \\ J_+ &= \frac{1}{\theta} X^\dagger Y, \quad J_- = \frac{1}{\theta} Y^\dagger X; \end{aligned} \quad (4.7)$$

the building blocks $\mathbb{A}_{a_1 \dots a_q}^\dagger$ and $\mathbb{B}_{a_1 \dots a_q}^\dagger$ are related via the formula

$$\mathbb{B}_{a_1 \dots a_q}^\dagger = \frac{1}{q!} ad^q J_- \left(\mathbb{A}_{a_1 \dots a_q}^\dagger \right), \quad q \geq 1. \quad (4.8)$$

We deduce that $\mathbb{A}_{a_1 \dots a_q}^\dagger$ and $\mathbb{B}_{a_1 \dots a_q}^\dagger$ are in fact just a particular condensate of more general operators which we denote as $\mathbb{D}_{a_1 \dots a_q}^{(n)\dagger}$. Involving both the

creators X^\dagger and Y^\dagger , this general building blocks are given by

$$\mathbb{D}_{a_1 \dots a_q}^{(n)\dagger} = \frac{(q-n)!}{q!} a d^n J_- \left(\mathbb{A}_{a_1 \dots a_q}^\dagger \right), \quad n = 0, 1, \dots, q. \quad (4.9)$$

Using the fact that $[J_0, X] = -X$ and $[J_0, Y^\dagger] = -Y^\dagger$ (which means Y^\dagger and X carries a spin $\frac{1}{2}$ charge), it follows that the $\mathbb{D}_{a_1 \dots a_q}^{(n)\dagger}$ s form an $SU(2)$ representation of isospin $j = \frac{q}{2}$.

With these tools at hand, we turn now to our goal: building the wave function for the ground state of the FQH fluids on \mathbb{S}^3 and \mathbb{S}^2 .

Since \mathbb{S}^2 is obtained from \mathbb{S}^3 as a fiber bundle section, we begin by studying the fundamental wave function for a FQH fluid droplet on the 3-sphere. Then we consider the computation of the ground state for the case of FQH droplets on the 2-sphere.

B. Ground state for droplets on \mathbb{S}^3

Introduce a vacuum state $|0\rangle$ describing a state with zero particle. This state is $U(N) \times SU(2)$ gauge invariant*

$$\mathcal{S}_b^a |0\rangle = 0, \quad J_{0,\pm} |0\rangle = 0, \quad (4.10)$$

and is also annihilated by the operators

$$\Psi^b |0\rangle = 0, \quad X_a^b |0\rangle = 0, \quad Y_a^b |0\rangle = 0. \quad (4.11)$$

A candidate for the wave function $|\Phi\rangle$ describing FQH droplet ground states on \mathbb{S}^3 carrying a $SU(2)$ isospin charge j is expressed as follows

$$|\Phi_{k,j,0}\rangle = \left[\left(\epsilon^{a_0 a_1 \dots a_{N-1}} \Psi_{a_0}^\dagger \mathbb{A}_{a_1 \dots a_{N-1}}^\dagger \right)^k \right] |0\rangle, \quad (4.12)$$

It is clear that we have $[kN(N-1) + 1]$ possible such wave functions invariant under $U(N)$ gauge symmetry and forming a $SU(2)$ representation. More precisely, it is easy to show that the functions satisfy

$$J_0 |\Phi_{k,j}\rangle = k \frac{N(N-1)}{2} |\Phi_{k,j}\rangle, \quad J_+ |\Phi_{k,j}\rangle = 0 \quad (4.13)$$

from which we deduce that $|\Phi_{k,j}\rangle$ belong to $SU(2)$ highest weight representation of a $(2j+1)$ component vector $(|\Phi_{k,j,j_z}\rangle)$, such as

$$|\Phi_{k,j}\rangle \equiv |\Phi_{k,j,j_z}\rangle, \quad j_z = j, \quad (4.14)$$

* \mathcal{S}_b^a and $J_{0,\pm}$ are the generators of $U(N)$ and $SU(2)$ respectively.

carry an isospin $j = \frac{kN(N-1)}{2} = lN$. The quantum number l describing the quantization of the radius squared of the 3-sphere ($l = BR^2$) is determined by computing the mean value $\langle \Phi | \mathcal{E}_0 | \Phi \rangle$ which leads to $l = k \frac{(N-1)}{2}$. The factor $\frac{(N-1)}{2}$ is a consequence of matrix elevation of the 3-sphere geometry.

Notice that by applying the monomials $(J_-)^p$, the states of the $SU(2)$ highest weight representation are obtained as in³

$$|\Phi_{k,j,j-p}\rangle = \frac{(j-p)!}{j!} (J_-)^p |\Phi_{k,j,j}\rangle, \quad (4.15)$$

where the integer p takes values as $p = 0, \dots, \frac{kN(N-1)}{2}$. Finally, the lowest state is given by

$$|\Phi_{k,j,-j}\rangle = \left[\epsilon^{a_0 a_1 \dots a_{N-1}} \left(\Psi_{a_0}^\dagger \mathbb{B}_{a_1 \dots a_{N-1}}^\dagger \right)^k \right] |0\rangle, \quad (4.16)$$

involving only monomials built of the creation operators Y^\dagger .

C. Droplets wave function on \mathbb{S}^2

In the case of FQH droplets on the 2-sphere, droplets are described by $SU(2)/U_C(1)$ representations. So the construction of the wave functions follows directly from the link between the isometries of \mathbb{S}^3 and \mathbb{S}^2 . More precisely, it is obtained from those of $SU(2)$ representations by projecting the isospin j highest weight vector on one of the $(2j+1)$ possible directions j_z .

We proceed as follows, we start from the solution (4.15)

$$\left\{ |\Phi_{k,j,j-p}\rangle, \quad 0 \leq p \leq \frac{1}{2} kN(N-1) \right\}, \quad (4.17)$$

obtained for the 3-sphere, then we restrict \mathbb{S}^3 down to \mathbb{S}^2 geometry by imposing the constraint eq $\mathcal{I}_0 |\Phi\rangle = m |\Phi\rangle$. This constraint eq corresponds just to singling out one of the state of the highest weight representation. Thus the wave function with spin projection m reads as follows,

$$|\Phi_{SU(2)/U_C(1)}\rangle = |\Phi_{k,j,j-m}\rangle, \quad (4.18)$$

where $m \in [-l, +l]$. Using the building block operators $\mathbb{D}_{a_1 \dots a_{N-1}}^{(n_i)\dagger}$ eq(4.9), the above fundamental wave function become,

$$|\Phi_{k,j,j-m}\rangle = \left(\prod_{i=1}^k \left[\epsilon^{a_0 a_1 \dots a_{N-1}} \Psi_{a_0}^\dagger \mathbb{D}_{a_1 \dots a_{N-1}}^{(n_i)\dagger} \right] \right) |0\rangle, \quad (4.19)$$

where k and l are same as before and the integer m is given by,

$$m = \sum_{i=1}^k \left(\frac{N(N-1)}{2} - 2n_i \right), \quad (4.20)$$

with $n_i = 0, \dots, \frac{N(N-1)}{2}$. In conclusion, it is illuminating to recognize that one FQH droplet moving on the 3-sphere manifests as,

$$2 \frac{kN(N-1)}{2} + 1 = kN(N-1) + 1 \quad (4.21)$$

FQH droplets on S^2 .

V. CONCLUSION

In this paper, we have derived a new matrix model proposal for FQH droplets on the subconifolds S^3 and S^2 . The isometries of the spheres play a central role in building the wave function for ground state of droplets that we have worked out explicitly. Constructing the real 3-sphere as the fibration $S^3 = S^2 \times S^1$ has yield to the following interesting result: a generic FQH droplet on S^3 carrying isospin j and described by the ground state,

$$|\Phi_{k,j}\rangle = \{ |\Phi_{k,j,j-p}\rangle, \quad |p| \leq j \}, \quad (5.1)$$

is made of $(2j + 1)$ droplets on the 2-sphere,

$$|\Phi_{k,j,j-p}\rangle, \quad p \in \{-j, \dots, j\}, \quad (5.2)$$

with fixed value of the relative integer p . So our proposal is a unified description of both real spheres S^3 and S^2 realized as hypersurfaces embedded in C^2 . More technical details can be found in^{15,16}.

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