African Journal Of Mathematical Physics Volume **5** *(2007)43-49*

On a General Reduction Scheme of Fredholm Operators

Bekkai Messirdi[1,*] and Ghouti Djellouli[2,†]

1. University of Oran, Department of mathematics, Essenia 31000 Oran, Algeria
2. University of Saida, Department of Mathematics 20000 Saida, Algeria
* *bmessirdi@univ-oran.dz*, † *Gdjellouli@yahoo.fr*

Abstract

In this paper we present a new reduction scheme that we can applied to the study of Fredholm operators. We mainly show in a simple and original manner the stability results for Fredholm operators. We also get a fundamental characterization result of the spectrum of these operators.

Key words: Fredholm operators, reduction, stability theorems, spectrum.
AMS Classification Subject: 47A53, 47B48

I. INTRODUCTION

The Fredholm operators have two important properties. First of all the properties of stability under small perturbations, this result has been proved for the first time by F. V. Atkinson in 1951 [1]. It has been generalized successively in 1952 by M. G. Krein and M. A. Krasnoselsky in [4] and by B. Sg. Nagy in [10]. A more general form was introduced thirty five years after by J. P. Labrousse and B. Mercier in [5]. These operators have also another important property of decomposition known as Kato decomposition .

On the other hand, if $C(H)$ (resp. $\mathcal{L}(H)$) designates the closed linear operators space (resp. bounded) on a dense domain of a Hilbert space H, we know that $(A \sim B)$ if and only if $(A - B)$ is compact on H, is an equivalence relation in $\mathcal{L}(H)$. Many results have been obtained via this relation " $\sim$ ", we can mentioned in particular the spectral correction by compact perturbation, the Weyl theorem, the Calkin algebra, and the possibility of extending some results and spectral properties to the class of Fredholm operators of $C(H)$ (see eg.[5]). We have to note also the use of Fredholm theory in the essential spectrum definition of an operator of $C(H)$, a notion which has already been used in different contexts by many authors.

In this paper we present a new and original method adapted for the first time to Fredholm operators of class $C(H)$, some stability theorems and spectral results are easily obtained.

Let us briefly describe the essential ideas of our method inspired by the Feshbach method which is a well known technique of reduction, introduced by Feshbach in [2] also used in [6] and [7] to study the spectral properties and resonances of Schrödinger Hamiltonians for polyatomic molecules in the semiclassical limit. The way in which the results of [6] and [7] (see also [9]) have been proved relies on the construction of an operator acting on a grater space by means of eigenfunctions of the considered electronic Hamiltonian.

We present here this method in a slightly different way that can be applied to the class of Fredholm operators and prove our main results, namely the stability theorem and a result of spectral reduction for these operators. Our trick is to reduce the problem to the inversion of a general matrix operator using no spectral tools but including the classical case of the Feshbach method.

The main goal of the first section is the construction of a matrix operator associated to a continuous family $(P_z)_{z\in\mathbf{C}}$ of bounded linear operators between two complex Banach spaces E and F.

We assume that for a certain z_0, P_{z_0} (noted P) is a Fredholm operator with index $(n_0 - d_0)$, where n_0 is the nullity and d_0 is the deficiency of P.

One considers then the "Grushin operator" defined by

$$\mathcal{P}_z = \begin{pmatrix} P_z & R_0^- \\ R_0^+ & 0 \end{pmatrix}$$

and acts on $E \oplus \mathbf{C^{d_0}}$ to $F \oplus \mathbf{C^{n_0}}$.

R_0^+ and R_0^- are bounded linear operators respectively from E to $\mathbf{C^{n_0}}$ and from $\mathbf{C^{d_0}}$ to F, with maximum rank the ranges of P and R_0^- are transverse and R_0^+ is invertible on the kernel of P

Thus, $\mathcal{P}_z$ becomes invertible for z sufficiently near to z_0, its inverse is given by

$$\mathcal{P}_z^{-1} = \begin{pmatrix} A(z) & A^+(z) \\ A^-(z) & A^{-+}(z) \end{pmatrix}$$

where $A(z)$, $A^{\pm}(z)$ and $A^{-+}(z)$ are bounded and depends continuously on z.

We establish that the equation

$$P_z f = g$$

admits a solution f in E if and only if $A^-(z)g$ belongs to the range of $A^{-+}(z)$.

While using then $\mathcal{P}_z$ and its inverse, we obtain our first fundamental result that for z in a small complex neighborhood of z_0, P_z and $A^{-+}(z)$ are Fredholm operators with the same index equal to the index of P

Consequently, the property of being Fredholm and the index are stable under small perturbations. This theorem has long been known but such a result was proved by using the properties of closed operators with closed range (see eg. [3]). Nevertheless, it appears that the reduction that we have obtained enable us to find directly in a simple and original manner the stability results of Fredholm theory (the Fredholm operator class is open in $C(H)$, and the index remains constant on every connexe component of this class). In particular, this stability is new in the form given here.

In the second section we assume that $E \subset F$, $P_z = P - z$, where $z_0 = 0$, its shown that our method is also adapted for the spectral analysis of Fredholm operators.

In fact, another interest to consider the matrix $\mathcal{P}_z$ is that it reduces the spectral study of the Fredholm operator P to that of a simpler linear operator acting on $\mathbf{C^{n_0}}$ in the sense

$$z \in Sp(P) \iff z \in Sp(z - A^{-+}(z))$$

where Sp stands for the spectrum.

This reduction characterize the spectrum of P through a finite matrix of analytic compact operators with constant index defined on a finite dimensional complex euclidian space.

II. FREDHOLM THEORY

Let E and F be a complex Banach spaces, ω an open complex and $(P_z)_{z\in\omega}$ a continuous family on ω of bounded linear operators from E to F .

Suppose that for a certain z_0 of ω, the corresponding operator P_{z_0} noted P is a Fredholm operator. We denote $n_0 = \dim(\ker P) < +\infty$ and $d_0 = \mathrm{codim}_F(ImP) = \dim(F/ImP) < +\infty$, $(n_0 - d_0)$ is called the index of P.

By analogy with the Grushin operators introduced into [2], [6], and [7], we consider two linear operators R_0^+ from E to $\mathbf{C^{n_0}}$ and R_0^- from $\mathbf{C^{d_0}}$ to F, such that :

(H1) $R_0^{\pm}$ are bounded and of maximum rank,

$$\begin{cases} \dim(ImR_0^-) = d_0 \\ \dim(ImR_0^+) = n_0 \end{cases}$$

(H2) $R_0^-(\mathbf{C^{d_0}})$ is transverse with $P(E)$,

$$F = ImR_0^- \oplus ImP$$

(H3) $R_0 = R^+_{0|\ker P}$ is invertible

Remarks 1.1:

1) *The two nonorthogonal projections p_0 from F to ImP and p_0^- from F to ImR_0^- are bounded relatively to the decomposition* (**H2**). *In particular, $rg(p_0^-) = corg(p_0) = d_0$,* (here rg (resp. $corg$) stands for the rank (resp. corank or deficiency).

2) *If $\mathcal{E}$ is a complement of $\ker P$ in E, then $P' = P_{|\mathcal{E}}$ is an isomorphism of $\mathcal{E}$ onto ImP.*

3) R_0^- *is an isomorphism of* $\mathbf{C^{d_0}}$ *onto* ImR_0^-.

Example of operators $R_0^{\pm}$:

(1) If $\{\dot{e_1},...,\dot{e_{d_0}}\}$ is a fixed basis of the quotient space F/ImP of F by ImP, we put

$$R_0^-(\lambda_1,...,\lambda_{d_0}) = \sum_{k=1}^{d_0} \lambda_k e_k \in F, \quad (\lambda_1,...,\lambda_{d_0}) \in \mathbf{C^{d_0}}$$

We can easily see that $rg(R_0^-) = d_0$ and that ImR_0^- is transverse with ImP.

Indeed, $\forall g \in F$, $\exists$ $(\lambda_1,...,\lambda_{d_0}) \in \mathbf{C^{d_0}}$ such that $\dot{g} = \sum_{k=1}^{d_0} \lambda_k \dot{e_k} \in F/ImP$ and $(g - \sum_{k=1}^{d_0} \lambda_k e_k) \in ImP$. Then

$$g = \sum_{k=1}^{d_0} \lambda_k e_k + (g - \sum_{k=1}^{d_0} \lambda_k e_k) \in ImR_0^- \oplus ImP$$

because $ImR_0^- \cap ImP = \{0\}$.

(2) If $\{v_1,...,v_{n_0}\}$ is a basis of $\ker P$ and $\mathcal{E}$ is a complement of $\ker P$ in E, then every element f of E is written in a unique way in the form $f = \sum_{k=1}^{n_0} \mu_k v_k + f_{\mathcal{E}}$ where $f_{\mathcal{E}}$ is the projection of f on $\mathcal{E}$. We define R_0^+ from E to $\mathbf{C^{n_0}}$ by $R_0^+ f = (\mu_1,...,\mu_{n_0})$. By construction $rg(R_0^+) = n_0$, R_0^+ is also invertible on $\ker P$.

▶

We consider now the following matrix operator (the so-called Grushin operator) :

$$\mathcal{P}_0 = \begin{pmatrix} P & R_0^- \\ R_0^+ & 0 \end{pmatrix}$$

defined from $E \oplus \mathbf{C^{d_0}}$ to $F \oplus \mathbf{C^{n_0}}$.

Proposition 1.2:
The operator $\mathcal{P}_0$ defined above is invertible in $\mathcal{L}(E \oplus \mathbf{C^{d_0}}, \mathbf{F} \oplus \mathbf{C^{n_0}})$ and its inverse is bounded given by

$$\mathcal{P}_0^{-1} = \begin{pmatrix} A_0 & A_0^+ \\ A_0^- & A_0^{-+} \end{pmatrix}$$

Where

$$\begin{cases} A_0 = (1 - (R_0)^{-1} R_0^+)(P')^{-1} p_0 \in \mathcal{L}(F, E) \\ A_0^+ = (R_0)^{-1} \in \mathcal{L}(\mathbf{C^{n_0}}, \mathbf{E}) \\ A_0^- = (R_0^-)^{-1} p_0^- \in \mathcal{L}(F, \mathbf{C^{d_0}}) \\ A_0^{-+} = 0 \in \mathcal{L}(\mathbf{C^{n_0}}, \mathbf{C^{d_0}}) \end{cases}$$

Here we have denoted by $(P')^{-1}$ the inverse of $P_{|\mathcal{E}}$.

proof 1 *: Showing that $\mathcal{P}_0$ is invertible is equivalent to solving the following system where the unknown $f \oplus \alpha$ is in $E \oplus \mathbf{C^{d_0}}$:*

$$\begin{cases} Pf + R_0^- \alpha = g \\ R_0^+ f = \beta \\ g \in F, \ \beta \in \mathbf{C^{n_0}} \end{cases}$$

Since $E = \ker P \oplus \mathcal{E}$ we have $f = f_0 + f_{\mathcal{E}}$ with $f_0 \in \ker P$ and $f_{\mathcal{E}} \in \mathcal{E}$, then

$$\begin{cases} Pf_{\mathcal{E}} + R_0^- \alpha = g \\ R_0 f_0 + R_0^+ f_{\mathcal{E}} = \beta \end{cases}$$

*By the assumption **(H3)**, we can rewrite this system as*

$$\begin{cases} Pf_{\mathcal{E}} + R_0^- \alpha = g \\ f_0 = (R_0)^{-1}(\beta - R_0^+ f_{\mathcal{E}}) \end{cases}$$

or since $Pf_{\mathcal{E}} = p_0 g$ and $R_0^- \alpha = p_0^- g$, we obtain

$$\begin{cases} f_{\mathcal{E}} = (P')^{-1} p_0 g \\ f_0 = (R_0)^{-1}(\beta - R_0^+ (P')^{-1} p_0 g) \\ \alpha = (R_0^-)^{-1} p_0^- g \end{cases}$$

Thus $\mathcal{P}_0$ is invertible and its inverse is a bounded operator which can be written as the formulae given in the proposition.

To study the properties of the family $(P_z)_{z\in\omega}$, in a small enough neighborhood of z_0, we fixe $\varepsilon > 0$ such that $\varepsilon < 1/\left\|\mathcal{P}_0^{-1}\right\|_{\mathcal{L}(F\oplus\mathbf{C^{n_0}},\mathbf{E}\oplus\mathbf{C^{d_0}})}$ and we define the operator $\mathcal{P}_z$ from $E \oplus \mathbf{C^{d_0}}$ to $F \oplus \mathbf{C^{n_0}}$ by:

$$\mathcal{P}_z = \begin{pmatrix} P_z & R_0^- \\ R_0^+ & 0 \end{pmatrix}$$

By continuity of the family $(P_z)_{z\in\omega}$, there exist a complex neighborhood Ω of z_0 included in ω such that

$$\|P_z - P\|_{\mathcal{L}(E,F)} < \varepsilon \ \ , \ \forall z \in \Omega$$

Hence

$$\left\|\mathcal{P}_0^{-1}\right\|_{\mathcal{L}(F\oplus\mathbf{C^{n_0}},\mathbf{E}\oplus\mathbf{C^{d_0}})} \|\mathcal{P}_z - \mathcal{P}_0\|_{\mathcal{L}(E\oplus\mathbf{C^{d_0}},\mathbf{F}\oplus\mathbf{C^{n_0}})} < 1$$

Consequently, for all z in Ω

$$\mathcal{P}_z = \mathcal{P}_0[1 - \mathcal{P}_0^{-1}(\mathcal{P}_0 - \mathcal{P}_z)]$$

is invertible in $\mathcal{L}(E \oplus \mathbf{C^{d_0}}, \mathbf{F} \oplus \mathbf{C^{n_0}})$ and its inverse is a bounded operator given by the Neumann series:

$$\begin{aligned} \mathcal{P}_z^{-1} &= [1 - \mathcal{P}_0^{-1}(\mathcal{P}_0 - \mathcal{P}_z)]^{-1} \mathcal{P}_0^{-1} \\ &= \sum_{n=0}^{\infty} [1 - \mathcal{P}_0^{-1} \mathcal{P}_z]^n \mathcal{P}_0^{-1} \end{aligned}$$

We denote for $z \in \Omega$

$$\mathcal{E}(z)=\mathcal{P}_z^{-1}=\begin{pmatrix} A(z) & A^+(z)\\ A^-(z) & A^{-+}(z)\end{pmatrix}$$

Remark 1.3:
$A(z)\in\mathcal{L}(F,E)$, $A^+(z)\in\mathcal{L}(\mathbf{C^{n_o}},\mathbf{E})$, $A^-(z)\in\mathcal{L}(F,\mathbf{C^{d_o}})$ and $A^{-+}(z)\in\mathcal{L}(\mathbf{C^{n_o}},\mathbf{C^{d_o}})$ depends continuously on z in Ω and $A^{-+}(z_0)=A_0^{-+}=0$, in addition the operators $A^{\pm}(z)$ remains *with maximal rank in Ω,*

$$\begin{cases} rg(A^+(z))=n_0\\ rg(A^-(z))=d_0\end{cases}$$

For $z\in\Omega$, let us now study the equation $P_zf=g$ and observe the interest of the operator $\mathcal{P}_z$ in the solutions of this equation through the following result:

Proposition 1.4:
For a given $g\in F$, the operational equation $P_zf=g$, admits a solution f in E if and only if $A^-(z)g\in ImA^{-+}(z)$.

proof 2 *:*
By introducing the auxiliary variable $R_0^+f\in\mathbf{C^{n_o}}$, we have

$$\begin{aligned} P_zf &= g\\ &\Leftrightarrow \mathcal{P}_z(f\oplus 0)=(g\oplus R_0^+f)\\ &\Leftrightarrow \mathcal{E}(z)(g\oplus R_0^+f)=(f\oplus 0)\end{aligned}$$

Then
$$\begin{cases} A(z)g+A^+(z)R_0^+f=f & (1)\\ A^-(z)g+A^{-+}(z)R_0^+f=0 & (2)\end{cases}$$

If $A^-(z)g=\beta\in ImA^{-+}(z)\subset\mathbf{C^{d_o}}$, then there exist $\alpha\in\mathbf{C^{n_o}}$ such that $\beta=A^{-+}(z)\alpha$. Replacing this value of β in (2), we find

$$A^{-+}(z)[\alpha+R_0^+f]=0$$

We can choose f such that

$$R_0^+f=-\alpha$$

While substituting this expression in (1), one has the solution

$$f=A(z)g-A^+(z)\alpha$$

Conversely, if $f\in E$ is a solution of the equation $P_zf=g$ then (2) permits to write

$$A^-(z)g=-A^{-+}(z)R_0^+f\in ImA^{-+}(z)$$

On one hand, since $rg(A^-(z))=d_0$ if we assume that for $g\in F$, there exist $\varepsilon>0$ such that $B_F(g,\varepsilon)\subset F\setminus ImP_z$ (where $B_F(g,\varepsilon)$ indicates the open ball of F centered in g and of radius ε), then $\forall h\in B_F(g,\varepsilon)$ the equation $P_zf=h$ does not have any solution in E, consequently $A^-(z)h\notin ImA^{-+}(z)$, and $ImA^{-+}(z)\subsetneq\mathbf{C^{d_o}}$, which is absurd. We deduce that ImP_z is closed in F.

On the other hand, let us consider the following commutative diagram :

$$\begin{array}{ccc} F & \xrightarrow{A^-(z)} & \mathbf{C^{d_o}}\\ \downarrow s & t\searrow & \downarrow s'\\ F/ImP_z & \xrightarrow{\rho} & \mathbf{C^{d_o}}/\mathbf{ImA^{-+}(z)}\end{array}$$

Where $s(g)=\dot{g}$, $g\in F$; $s'(\alpha)=\widehat{\alpha}$, $\alpha\in\mathbf{C^{d_o}}$; $\rho(\dot{g})=\widehat{A^-(z)g}$, $t(g)=\widehat{A^-(z)g}$; $\dot{}$ is the equivalence class in F/ImP_z and $\widehat{}$ is the class in $\mathbf{C^{d_o}}/\mathbf{ImA^{-+}(z)}$.

$$\begin{aligned}\rho(\dot{g}) &= \rho(\dot{h})\Longleftrightarrow A^-(z)(g-h)\in ImA^{-+}(z)\\ &\Longleftrightarrow P_zf=(g-h)\text{ admits a solution } f \text{ in } E\\ &\Longleftrightarrow (g-h)\in ImP_z\Longleftrightarrow \dot{g}=\dot{h}\end{aligned}$$

ρ maps F/ImP_z onto $\mathbf{C^{d_o}}/\mathbf{ImA^{-+}(z)}$ because $\rho\circ s=t$ and $Imt=\mathbf{C^{d_o}}/\mathbf{ImA^{-+}(z)}$. ρ is then a linear isomorphism.

Consequently,

$$\dim(\ F/ImP_z)=\dim(\mathbf{C^{d_o}}/\mathbf{ImA^{-+}(z)})<+\infty \qquad \mathbf{(3)}$$

Similarly, if $\alpha\in\mathbf{C^{n_o}}$, then for all z in Ω

$$\begin{aligned}\beta &= A^{-+}(z)\alpha\Longleftrightarrow\mathcal{E}(z)(0\oplus\alpha)=(A^+(z)\alpha\oplus\beta)\\ &\Longleftrightarrow\mathcal{P}_z(A^+(z)\alpha\oplus\beta)=(0\oplus\alpha)\end{aligned}$$

and

$$\begin{cases} P_zA^+(z)\alpha+R_0^-\beta=0 & (4)\\ R_0^+A^+(z)\alpha=\alpha & (5)\end{cases}$$

From (5), we deduce that if $A^+(z)\ \alpha=0$ then $\alpha=0$, thus $\ker A^+(z)=\{0\}$ and $\dim(ImA^+(z))=n_0$ (remark 1.3).
Let l be the application defined by $l(\alpha)=A^+(z)\alpha$ from $\ker A^{-+}(z)$ to $\ker P_z$, then

- l is well defined because if $A^{-+}(z)\alpha=0$, we have from (4), $P_zA^+(z)\ \alpha=P_z(l(\alpha))=0$ and thus $l(\alpha)\in\ker P_z$.
- $\ker l=\{0\}$. l maps $\ker A^{-+}(z)$ onto $\ker P_z$ or $Iml=\ker P_z$, because the equation $A^+(z)\ \alpha=f$ has to be solved in $\ker A^{-+}(z)$ for a given f in $\ker P_z$.

We make according to equation (4), $\alpha = R_0^+ f$. By using (2) we find $A^{-+}(z)\alpha = -A^-(z)P_z f = 0$ and then $l(\alpha) = A^+(z)\alpha = f$, by virtue of (1).

Consequently,

$$\dim(\ker P_z) = \dim(\ker A^{-+}(z)) < +\infty \quad , \quad \forall z \in \Omega \qquad (6)$$

We then have established the following first main result relatively to the stability on Ω of the Fredholm character and the corresponding index.

Theorem 1.5:
For any z in Ω, P_z and $A^{-+}(z)$ are Fredholm operators of the same index independent of z.

$$index(P_z) = index(A^{-+}(z)) = n_0 - d_0 \quad , \quad \forall z \in \Omega$$

III. SPECTRAL REDUCTION OF FREDHOLM OPERATORS

Now let us suppose that Ω is an open connexe neighborhood of 0 in $\mathbf{C}$ verifying the conditions of the previous section.

$E \subset F$ with a dense inclusion, and suppose $P_z = P - z$, where P is the Fredholm operator such that $n_0 = \dim(\ker\ P) < +\infty$ and $d_0 = \dim(F/ImP) < +\infty$. We suppose also that P_z is invertible for at least a point of Ω different from 0. Then according to (3) and (6), $A^{-+}(z)$ is also invertible at the same point and thus $n_0 = d_0$.

A principal interest to reconsider the operator $\mathcal{P}_z$ is that it reduces in fact the spectral study of the Fredholm operator P to that of a simpler linear operator acting on $\mathbf{C}^{\mathbf{n_0}}$. Our second main result is

Theorem 2.1:
For any z in Ω, we have the following equivalence

$$z \in Sp(P) \Longleftrightarrow 0 \in Sp(A^{-+}(z)) \Longleftrightarrow \det(A^{-+}(z)) = 0$$

where det *stands for the determinant.*

proof 3 *: If $0 \notin Sp(A^{-+}(z))$, then the equations (1) and (2) gives the following equivalence*

$$P_z f = g \Longleftrightarrow \begin{cases} R_0^+ f = -(A^{-+}(z))^{-1}A^-(z)g \\ f = A(z)g - A^+(z)(A^{-+}(z))^{-1}A^-(z)g \end{cases}$$

Thus

$$P_z^{-1} = A(z) - A^+(z)(A^{-+}(z))^{-1}A^-(z) \qquad (7)$$

which proves that $z \notin Sp(P)$.
Conversely, if $z \notin Sp(P)$, then equation (4) and (5) permits to write :

$$A^{-+}(z)\alpha = \beta \Longleftrightarrow \alpha = -R_0^+ P_z^{-1} R_0^- \beta$$

Then, $0 \notin Sp(A^{-+}(z))$ and

$$(A^{-+}(z))^{-1} = -R_0^+ P_z^{-1} R_0^- \qquad (8)$$

Remarks 2.2:
1) *The formulaes (7) and (8) shows owing to the fact that by construction $A(z)$, $A^{\pm}(z)$ and $A^{-+}(z)$ depend analytically on z, that a singularity of P_z^{-1} is necessarily a singularity of $(A^{-+}(z))^{-1}$ and conversely.*

2) *Since $A^{-+}(0) = A_0^{-+} = 0$, then* $\det(A_0^{-+}) = 0$ *and* $0 \in Sp(P)$.

Since the zeros of $\det(A^{-+}(z))$ form a discrete set in Ω, if we reduce Ω we can assume that P_z is invertible (and thus $A^{-+}(z)$) with a bounded inverse for all z in $\Omega \setminus \{0\}$. 0 is then a pole of multiplicity N_0 of the matrix $(A^{-+}(z))^{-1}$.

Let γ be a simple complex closed contour around the singularity 0 in Ω. Note that :

$$\pi \overset{\text{def}}{=} \frac{-1}{2\pi i} \oint_\gamma P_z^{-1} dz \qquad (9)$$

The operator π gives some supplementary informations on P and permits to specify the spectral result of the Theorem 2.1.

Theorem 2.3:

(1) π is a projector of a finite rank
(2) $Im\pi = \pi(E) = \pi(F)$. $\ker\pi$ and $Im\pi$ are invariant by P
(3) $Sp(P) \cap \overset{\circ}{\gamma} = Sp(P\mid_{Im\pi}) \cap \overset{\circ}{\gamma}$, $\overset{\circ}{\gamma}$ is the interior of the contour γ
$Sp(P\mid_{\ker\pi}) \cap \overset{\circ}{\gamma} = \emptyset$ and $\#(Sp(P) \cap \overset{\circ}{\gamma}) = rg(\pi)$

proof 4 *:*
(1) Let γ' be an other simple closed complex contour around 0 included in γ. By analyticity of the resolvent P_z^{-1} in $\Omega \setminus \{0\}$, we have by the Cauchy theorem:

$$\pi^2 = -\frac{1}{4\pi^2}\oint_{\gamma}\oint_{\gamma'} P_z^{-1}P_t^{-1}dzdt$$
$$= -\frac{1}{4\pi^2}\oint_{\gamma}\oint_{\gamma'}\frac{1}{(t-z)}[P_t^{-1}-P_z^{-1}]dzdt$$
$$= \frac{1}{4\pi^2}\oint_{\gamma}[\oint_{\gamma'}\frac{dt}{(t-z)}]P_z^{-1}dz$$
$$-\frac{1}{4\pi^2}\oint_{\gamma'}[\oint_{\gamma}\frac{dz}{(t-z)}]P_t^{-1}dt$$
$$= \frac{-1}{2\pi i}\oint_{\gamma'} P_t^{-1}dt = \pi$$

From (7) we get

$$\pi = \frac{1}{2\pi i}\oint_{\gamma} A^+(z)(A^{-+}(z))^{-1}A^-(z)dz \qquad (10)$$

In particular, $(A^{-+}(z))^{-1}$ is a meromorphic function and possesses a Laurent series in Ω expressed as

$$(A^{-+}(z))^{-1} = \frac{R_{-N_0}}{z^{N_0}} + ... + \frac{R_{-1}}{z} + \widetilde{R}(z) \qquad (11)$$

where $\widetilde{R}(z)$ is analytical function in z.

$A^{\pm}(z)$ admits in Ω a Taylor series given by:

$$A^{\pm}(z) = \sum_{j=0}^{\infty}\frac{d^jA^{\pm}}{dz^j}(0)z^j \qquad (12)$$

Moreover, by using the residue formula, we also have

$$\pi = -\sum_{\substack{j+l-k=-1\\ 1\le k\le N_0}}\frac{d^jA^+}{dz^j}(0)R_{-k}\frac{d^lA^-}{dz^l}(0) \qquad (13)$$

Since the operators $\frac{d^jA^{\pm}}{dz^j}$ are of finite rank ($\le n_0$) , then the expression (13) shows that π is of finite rank ($\le n_0N_0$).

2) We also see in (9) that $Im\pi \subset E$. With E being dense in F, π is of finite rank and $\pi(E) \subset \pi(F)$, we have $\pi(E) = \pi(F) = Im\pi$.

$\forall f \in E, \forall z \in \gamma$, $PP_z^{-1}f = P_z^{-1}Pf = f + zP_z^{-1}f$, integrating on γ, we obtain

$$\pi Pf = P\pi f,\ \forall f \in E \qquad (14)$$

In particular, we deduce that $Im\pi$ and $\ker\pi$ are invariant by P.

3) Since π is a projector then $F = \ker\pi \oplus Im\pi$.

Let $P_1 = P\mid_{\ker\pi}$, $P_2 = P\mid_{Im\pi}$ and $R(z) = \frac{1}{2\pi i}\oint_{\gamma}(t-z)^{-1}P_t^{-1}dt$, $z \in \overset{\circ}{\gamma}$. We obtain

$$\pi R(z) = \frac{1}{4\pi^2}\oint_{\gamma}\oint_{\gamma'}(t-z)^{-1}P_\lambda^{-1}P_t^{-1}dtd\lambda$$
$$= \frac{1}{4\pi^2}\oint_{\gamma'}[\oint_{\gamma}\frac{dt}{(t-z)(\lambda-t)}]P_\lambda^{-1}d\lambda$$
$$-\frac{1}{4\pi^2}\oint_{\gamma}[\oint_{\gamma'}\frac{d\lambda}{(\lambda-t)}]\frac{P_t^{-1}}{(t-z)}dt$$
$$= \frac{1}{4\pi^2}\oint_{\gamma'}[2\pi i(\frac{1}{(\lambda-z)}+\frac{1}{(z-\lambda)})]P_\lambda^{-1}d\lambda = 0$$

Thus, $ImR(z) \subset \ker\pi$.

Consequently,

$$P_zR(z) = \frac{1}{2\pi i}\oint_{\gamma}(t-z)^{-1}P_zP_t^{-1}dt$$
$$= \frac{1}{2\pi i}\oint_{\gamma}\frac{dt}{(t-z)} + \frac{1}{2\pi i}\oint_{\gamma}P_t^{-1}dt = (1-\pi)$$

Hence

$$(P_1 - z)R(z)\mid_{\ker\pi} = 1\mid_{\ker\pi\cap F} \qquad (15)$$

Similarly

$$R(z)(P_1 - z)\mid_{\ker\pi} = 1\mid_{\ker\pi\cap E} \qquad (16)$$

Then, if $z \in \overset{\circ}{\gamma}$, $(P_1 - z)$ is invertible from $\ker\pi \cap E$ into $\ker\pi \cap F$ with bounded inverse equal to $R(z)\mid_{\ker\pi}$ and thus $z \notin Sp(P_1)$, or else $Sp(P) \cap \overset{\circ}{\gamma} = \emptyset$.

On the other hand, $\forall z \in \Omega \setminus \{0\}$,

$$P_z^{-1} = (P_1 - z)^{-1} \oplus (P_2 - z)^{-1} \qquad (17)$$

Thus

$$Sp(P) \cap \overset{\circ}{\gamma} = Sp(P_2) \cap \overset{\circ}{\gamma} \qquad (18)$$

However, $P_2 = \pi P\mid_{Im\pi}$ is defined on a finite dimension space, it possesses then a finite discrete spectrum of a cardinal equal to rank of π.

Remarks 2.4:

(1) If $z_1 \neq 0$ is a second point where $\det(A^{-+}(z_1)) = 0$, P_{z_1} is not invertible and if π and π_1 are the corresponding projectors given by (9), then $\pi\pi_1 = \pi_1\pi = 0$.

(2) The use of equation (2) with $\alpha = R_0^+f$ and $f = A^+(z)\alpha$, we have $A^-(z)P_zA^+(z) + A^{-+}(z) = 0$ on $\mathbf{C}^{\mathbf{n_0}}$. Then

$$\frac{dA^{-+}(z)}{dz} = A^-(z)A^+(z)$$

Consequently, if $\det(A^-(z)A^+(z)) \neq 0$, *the equation* $A^{-+}(z) = 0$ *admits the unique solution* $z = 0$ *in* Ω, *In this case,* 0 *is then a simple pole for* P_z^{-1}.

(3) According to equations (4) and (5), $A_0^- A_0^+$ *is invertible if and only if* $\ker P \cap ImP = \{0\}$, *which excludes the existence of generalized eigenvectors for* P.

(4) If P *is a pseudodifferential operator on* $\mathbf{R^n}$ *with an elliptical symbol* $\sigma(x, \xi)$ *for which there exist a constant* $\varepsilon > 0$, *such that* $\Omega_\varepsilon = \{(x, \xi) \in \mathbf{R^{2n}}; |\sigma(x, \xi)| < \varepsilon\}$ *is bounded in* $\mathbf{R^{2n}}$, P *is a Fredholm operator on* $L^2(\mathbf{R^n})$ (cf. [11], [8]) *, then it can be spectrally reduced by the previous methods.*

Acknowledgements:
This research work is supported by University of Oran Es-senia, Algeria. Research topics CNEPRU **B3101/02/03.**

IV. REFERENCES

[1] F.V. Atkinson: "La solvabilité normale d'équations linéaires dans des espaces normés" Math. Sbornik. N.S. 28/70, (1951), 3-14 .
[2] H. Feshbach: "Unified theory of nuclear reactions I and II" Ann. Physics 5, 1958, 363 and 19, (1962), 287.
[3] T. Kato: "Perturbation Theory for Linear Operators" Springer Verlag Berlin Heidelberg New York, (1976).
[4] M.G. Krein et M.A. Krasnoselski: "Stabilité de l'indice d'un opérateur non borné" Math. Sbornik N.S. 30(72), (1952), 219-224.
[5] J.P. Labrousse et B. Mercier: "Equivalence compacte entre deux opérateurs fermés sur un espace de Hilbert" Math. Nach. 133, (1987), 91-105.
[6] B. Messirdi: "Asymptotique de Born-Oppenheimer pour la prédissociation moléculaire (cas de potentiels réguliers)" Ann. Inst. Henri Poincaré 61(3), (1994), 255-292.
[7] B. Messirdi et A. Senoussaoui: "Méthode BKW formelle et spectre des molécules polyatomiques dans l'approximation de Born-Oppenheimer" Canadian Journal of Physics, CJP, Vol. 79(4), (2001), 757-771.
[8] B. Messirdi et A. Senoussaoui: "Parametrix du problème de Cauchy C^∞ hyperbolique muni d'un système d'ordres de Leray-Volevîc" Journal for Analysis and its Applications ZAA, Volume 24 (2005), N°3, 581-592.
[9] B. Messirdi and A. Senoussaoui: "Resonances of Polyatomic Molecules in the Born-Oppenheimer Approximation" Journal of Mathematical Physics, 46, 103506-1, (2005).
[10] B.Sg. Nagy: "On the stability of the index of unbounded linear transformations" Act. Math. Acad. Sci. Hung. 3, (1952), 49-52.
[11] F. Qihong and M.N. Wong: "A characterization of the Fredholm pseudo-differential operators" The Journal of the London Mathematical Society. 55(2), (1997), 139-145