African Journal Of Mathematical Physics Volume **5***(2007)51-56*

An Integral Involving Gegenbauer Polynomials

Fouzia El Wassouli

Department of Mathematics, Faculty of Sciences
University Ibn Tofail, Kenitra, Morocco.
f_elwassouli@yahoo.fr

abstract

In this paper we compute explicitly an integral involving the Gegenbauer polynomials $C_n^{\alpha}(x)$. Then we establish new addition formulas for $C_n^{\alpha}(x)$.

Keywords: Gegenbauer polynomial; Orthogonal polynomials; Integral; ultraspherical polynomials.
2000 Mathematics subject classification. 11S05;11S23;26B20.

I. INTRODUCTION.

In 1953 Courant and Hilbert considered the orthogonal polynomials associated with the weight function $w(x) = x^{q-1}(1-x)^{p-q}, q > 0, p - q > -1$, over the interval [0,1]. Also one can consider the orthogonal polynomials associated with the weight function $w(x) = (x-a)^{\alpha}(b-x)^{\beta}$ (with $\alpha > -1, \beta > -1$) on the interval [a,b]. The orthogonality formula for the Gegenbauer polynomial $C_l^{\lambda}(x)$ is explicitly by

$$\int_{-1}^{1} C_l^{\lambda}(x) C_m^{\lambda}(x)(1-x^2)^{\lambda-\frac{1}{2}} dx = \begin{cases} 0, & l \neq m \\ \dfrac{\pi 2^{1-2\lambda}\Gamma(m+2\lambda)}{\Gamma^2(\lambda)(m+\lambda)m!}, & l = m. \end{cases} \qquad (1.1)$$

Thus motivating the title of this paper; the most general we have found in the literature is the integral for two Gegenbauer polynomials $C_m^{\vartheta}(x)C_n^{\lambda}(x)$ over the interval [-1,1] with respect to the measure $(1-x)^{\beta}(1+x)^{\lambda-\frac{1}{2}}dx$, namely integral of the type:

$$\mathcal{N}_{\lambda,(\beta+\frac{1}{2})}(\vartheta;\lambda) := \int_{-1}^{1} C_l^{\vartheta}(x) C_m^{\lambda}(x)(1-x)^{\beta}(1+x)^{\lambda-\frac{1}{2}} dx. \qquad (1,2)$$

The evaluation of this integral is Gamma factors times ${}_4F_3$ of argument 1 with upper parameters $-l, l+2\vartheta, \alpha+1, \alpha-\lambda+\frac{3}{2}$ and lower parameters $\vartheta+\frac{1}{2}, \lambda+\alpha+m+\frac{3}{2}, \alpha-\lambda-m+\frac{3}{2}$.
See Erdelyi, Tables of integral transforms, Vol. 2, section 16.3, formula (16), or Gradshteyn Ryzhik, Tables of integrals, series and products, 7.314.7.
The first main result of this paper is the change the parameter λ in formula (1,1) by $\lambda = \vartheta$ and $\lambda = \mu$ with $\vartheta > -\frac{1}{2}$ and $\mu > -\frac{1}{2}$. Namely we calculate the integrals

$$\mathcal{N}_{\lambda,\lambda}(\vartheta;\mu) := \int_{-1}^{1} C_l^{\vartheta}(x) C_m^{\mu}(x)(1-x^2)^{\lambda-\frac{1}{2}} dx.$$

Theorem 1 . *For $\vartheta > -\frac{1}{2}$, $\mu > -\frac{1}{2}$ and $\lambda > -\frac{1}{2}$ we have*

$$\mathcal{N}_{\lambda,\lambda}(\vartheta;\mu) = \begin{cases} 0, \qquad l+m \text{ odd} \\ \dfrac{\pi 2^{1-2\lambda}}{\Gamma(\mu)\Gamma(\vartheta)(l-2k)!}\displaystyle\sum_{k=0}^{[\frac{l}{2}]}\dfrac{(l-2k+\lambda)}{k!s!} \\ \times\dfrac{\Gamma(l-\vartheta-k)}{\Gamma(l+\lambda-k+1)}\dfrac{\Gamma(l-2k+2\lambda)\Gamma(m+\mu-s)}{\Gamma(m+\lambda-s+1)} \\ \times(\vartheta-\lambda)_k(\mu-\lambda)_s, \quad m=l-2k+2s, \quad [\frac{m}{2}] \geq s \in \mathbb{N} \end{cases}$$

The proof is based on the following Lemma.

Lemma 2 . *For $\vartheta > -\frac{1}{2}$ and $\lambda > -\frac{1}{2}$ we have*

$$C_m^{\vartheta}(x) = \frac{\Gamma(\lambda)}{\Gamma(\vartheta)}\sum_{k=0}^{[\frac{m}{2}]} C_k C_{m-2k}^{\lambda}(x)$$

where

$$C_k = \frac{m-2k+\lambda}{k!}(\vartheta-\lambda)_k\frac{\Gamma(m+\vartheta-k)}{\Gamma(m+\lambda-k+1)}.$$

Here Γ being the usual Gamma Euler function on $\mathbb{C}$ and $(\alpha)_n$ is the Pochammer's symbol $(\alpha)_n = \alpha(\alpha+1).....(\alpha+n-1) = \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)}$.
The next main result of this paper is the following addition formula for the ultraspherical polynomials.

Theorem 3 . *For $\theta \in]0,\pi[$ we get*
1) for n even

$$C_l^{\frac{n-2}{2}}(\cos\theta) = \Pi_{j=2}^{\frac{n-2}{2}}(n-2j)\frac{\Gamma(n-2+l)}{2^{n-3}\Gamma^2(\frac{n-2}{2})l!}\frac{(-1)^{\frac{n-2}{2}}}{\sin^{2l}(\theta)}$$
$$\times\sum_{k=0}^{\frac{n-6}{2}} A_{k+1}^{\frac{n-4}{2}}(-1)^k\Big[\frac{2\sin 2\theta\cos\theta(l+n-4-k)}{(l+n-4-k)} - \frac{\sin\theta\cos\theta(l+n-3-k)}{(l+n-3-k)}\Big]\cos^k\theta.$$

2) for n odd

$$C_l^{\frac{n-2}{2}}(\cos\theta) = \pi\Pi_{j=2}^{\frac{n-1}{2}}(n-2j)\frac{\Gamma(n-2+l)}{2^{n-3}\Gamma^2(\frac{n-2}{2})l!}\frac{(-1)^{\frac{n-2}{2}}}{\sin^{2l}(\theta)}$$
$$\times\sum_{k=0}^{\frac{n-5}{2}} A_{k+1}^{\frac{n-3}{2}}(-1)^k\Big[\cos\theta P_{l+n-4-k}(\cos\theta) - P_{l+n-3-k}(\cos\theta)\Big]\cos^k\theta.$$

with $P_n(x) = C_n^{\frac{1}{2}}(x)$ being the Legendre polynomial and

$$\begin{cases} a_i = \frac{1}{l+i}, \qquad 1 \leq i \leq j \\ A_1^1 = a_1, \\ A_1^j = A_1^{j-1}a_{2j-1} \\ A_i^j = (A_i^{j-1} + A_{i-1}^{j-1})a_{2j-i}, \quad 1 < i < j \\ A_j^j = A_{j-1}^{j-1}a_j \end{cases}$$

II. PROOF OF THE NEW ADDITION FORMULAS.

A. Proof of theorem 1.

In this section we prove the formula stated in Theorem 1.

Proposition 4 . *For $\vartheta > -\frac{1}{2}$ and $\lambda > -\frac{1}{2}$ we have*

$$\mathcal{N}_{\lambda,\lambda}(\vartheta;\lambda) = \begin{cases} 0, \quad l < m \quad or\, l+m \ odd \\ \dfrac{\pi 2^{1-2\lambda}}{\Gamma(\lambda)\Gamma(\vartheta)k!m!}\dfrac{\Gamma(m+2\lambda)}{\Gamma(l+\lambda-k+1)} \\ \times\Gamma(l+\vartheta-k)(\vartheta-\lambda)_k, \quad l = m+2k, \ [\frac{l}{2}] \geq k \in \mathbb{N} \end{cases}$$

Proof. For $\vartheta > -\frac{1}{2}$ and $\lambda > -\frac{1}{2}$ the polynomial $C_l^{\vartheta}(x)$ can be written

$$C_l^{\vartheta}(x) = \frac{\Gamma(\lambda)}{\Gamma(\vartheta)}\sum_{k=0}^{[\frac{l}{2}]} C_k C_{l-2k}^{\lambda}(x)$$

where

$$C_k = \frac{l-2k+\lambda}{k!}(\vartheta-\lambda)_k\frac{\Gamma(l+\vartheta-k)}{\Gamma(l+\lambda-k+1)}.$$

Then the above integral $\mathcal{N}_{\lambda,\lambda}(\vartheta;\lambda)$ can be rewritten as

$$\mathcal{N}_{\lambda,\lambda}(\vartheta;\lambda) = \frac{\Gamma(\lambda)}{\Gamma(\vartheta)}\sum_{k=0}^{[\frac{l}{2}]} C_k \int_{-1}^{1} C_{l-2k}^{\lambda}(x)C_m^{\lambda}(x)(1-x^2)^{\lambda-\frac{1}{2}}dx.$$

Then from the orthogonality formula (1,1) the integral in the above sum vanishes if $l-2k \neq m, \quad k \in \{0, ..., [\frac{l}{2}]\}$ (ie $l+m$ odd or $l < m$) and equal to $\dfrac{\pi 2^{1-2\lambda}\Gamma(m+2\lambda)}{[\Gamma(\lambda)]^2(m+\lambda)m!}$ if $l-2k = m, \quad k \in \{0, ..., [\frac{l}{2}]\}$. Thus

$$\mathcal{N}_{\lambda,\lambda}(\vartheta;\lambda) = \begin{cases} 0, \quad l < m \ \text{ or } l+m \text{ odd} \\ \dfrac{\pi 2^{1-2\lambda}}{\Gamma(\lambda)\Gamma(\vartheta)k!m!}\dfrac{\Gamma(m+2\lambda)}{\Gamma(l+\lambda-k+1)} \\ \times\Gamma(l+\vartheta-k)(\vartheta-\lambda)_k, \quad l = m+2k, \quad [\frac{l}{2}] \geq k \in \mathbb{N} \end{cases}$$

Proof of Theorem 1. We have by Lemma 2

$$\mathcal{N}_{\lambda,\lambda}(\vartheta;\mu) = \frac{\Gamma(\lambda)}{\Gamma(\vartheta)}\sum_{k=0}^{[\frac{l}{2}]}\frac{(l-2k+\lambda)}{k!}(\vartheta-\lambda)_k \times \frac{\Gamma(l+\vartheta-k)}{\Gamma(l+\lambda-k+1)}\int_{-1}^{1} C_{l-2k}^{\lambda}(x)C_m^{\mu}(x)(1-x^2)^{\lambda-\frac{1}{2}}dx.$$

Using the Proposition 4 we can then compute the integral in the above sum

$$\int_{-1}^{1} C_{l-2k}^{\lambda}(x)C_m^{\mu}(x)(1-x^2)^{\lambda-\frac{1}{2}}dx = \begin{cases} 0, \quad m < l-2k \ \text{ or } \ l+m \text{ odd} \\ \dfrac{\pi 2^{1-2\lambda}(m+2s+\lambda)\Gamma(l-2k+2\lambda)\Gamma(m+\mu-s)}{\Gamma(\lambda)\Gamma(\mu)s!(l-2k)!(l-2k+\lambda)\Gamma(m+\lambda-s+1)} \\ \times(\mu-\lambda)_s, \qquad m = l-2k+2s, \quad [\frac{m}{2}] \geq s \in \mathbb{N}. \end{cases}$$

Next, since

$$\Big\{m \in \mathbb{N}/m < l-2k, \quad k \in \{0, ..., [\tfrac{l}{2}]\}\Big\} = \begin{cases} \{0\}, & \text{if } l \text{ odd} \\ \emptyset, & \text{if } l \text{ even} \end{cases}$$

we deduce that the condition $m < l-2k$ or $m+l$ odd is the same $l+m$ odd .

Finally the above expression becomes

$$\mathcal{N}_{\lambda,\lambda}(\vartheta;\mu)=\begin{cases}0, \qquad l+m\ odd\\ \dfrac{\pi 2^{1-2\lambda}}{\Gamma(\mu)\Gamma(\vartheta)(l-2k)!}\displaystyle\sum_{k=0}^{[\frac{l}{2}]}\dfrac{(l-2k+\lambda)}{k!s!}\\ \times\dfrac{\Gamma(l-\vartheta-k)}{\Gamma(l+\lambda-k+1)}\dfrac{\Gamma(l-2k+2\lambda)\Gamma(m+\mu-s)}{\Gamma(m+\lambda-s+1)}\\ \times(\vartheta-\lambda)_k(\mu-\lambda)_s, \quad m=l-2k+2s, \quad [\frac{m}{2}]\geq s\in\mathbb{N}\end{cases}$$

This finished the proof.

Remark 1 . *If we use the evaluation of integral (1,2) we will get that the integral $\mathcal{N}_{\lambda,\lambda}(\vartheta,\mu)$ is a double series but the result we have found here is a single series.*

B. Proof of Theorem 3.

For the proof of Theorem 3 we just combine the Lemma 6 below and the formula

$$\int_0^\pi(\cos\theta-i\sin\theta\cos\varphi)^l\sin^{n-3}\varphi d\varphi=\frac{2^{n-3}\Gamma^2(\frac{n-2}{2})l!}{\Gamma(n-2+l)}C_l^{\frac{n-2}{2}}(\cos\theta).$$

Putting $x=\cos\theta-i\sin\theta\cos\varphi$ and $J_l^n(\theta)=\int_0^\pi x^l\sin^{n-2}\theta\sin^{n-3}\varphi d\varphi$

Lemma 5 . *For every $m\in\{2,...,\frac{n-1}{2}\}$ we have*

$$\begin{aligned}J_l^n(\theta)=&\prod_{j=2}^m(n-2j)\sum_{k=0}^{m-2}(-1)^{m+k}A_{k+1}^{m-1}\cos^k\theta\\ &\times\Big[\cos\theta\int_0^\pi x^{l+2m-3-k}\sin^{n-2m}\theta\sin^{n-2m-1}\varphi d\varphi\\ &-\int_0^\pi x^{l+2m-2-k}\sin^{n-2m}\theta\sin^{n-2m-1}\varphi d\varphi\Big]\end{aligned}$$

with

$$\begin{cases}a_i=\frac{1}{l+i}, \quad 1\leq i\leq j\\ A_1^1=a_1\\ A_1^j=A_1^{j-1}a_{2j-1}\\ A_i^j=(A_i^{j-1}+A_{i-1}^{j-1})a_{2j-i}, \quad 1<i<j\\ A_j^j=A_{j-1}^{j-1}a_j\end{cases}$$

Proof of Lemma 6. Our proof will be done by induction.
For m=2 integrating by parts, we obtain

$$\begin{aligned}J_l^{n-2}(\theta)=&\frac{n-4}{l+1}\cos\theta\Big[\int_0^\pi x^{l+1}\sin^{n-4}\theta\sin^{n-5}\varphi d\varphi\Big]\\ &-\frac{n-4}{l+1}\int_0^\pi x^{l+2}\sin^{n-4}\theta\sin^{n-5}\varphi d\varphi\end{aligned}$$

So, for $m=2$ the Lemma 6 is true. Now suppose the lemma 6 is true for m. Then

$$\begin{aligned}J_l^{n-2}(\theta)=&\prod_{i=2}^m(n-2i)\sum_{k=0}^{m-2}(-1)^{m+k}A_{k+1}^{m-1}\cos^k\theta\\ &\times\Big[\cos\theta\int_0^\pi x^{l+2m-3-k}\sin^{n-2m}\theta\sin^{n-2m-1}\theta\varphi d\varphi\\ &-\int_0^\pi x^{l+2m-2-k}\sin^{n-2l}\theta_1\sin^{n-2m-1}\theta_2 d\theta_2\Big].\end{aligned}$$

Integrating by parts, we get

$$J_l^n(\theta) = \prod_{i=2}^{m+1}(n-2i)[B_1+B_2]$$

with

$$\begin{aligned}
B_1 = {} & \sum_{k=0}^{m-2}(-1)^{m+k}A_{k+1}^{l-1}\cos^{k+1}\theta \\
& \times \Big[a_{2m-2-k}\cos\theta\int_0^\pi x^{l+2m-2-k}\sin^{n-2m-2}\theta\sin^{n-2m-3}\varphi d\varphi \\
& - a_{2m-1-k}\int_0^\pi x^{l+2m-1-k}\sin^{n-2l}\theta\sin^{n-2m-1}\varphi d\varphi\Big].
\end{aligned}$$

and

$$\begin{aligned}
B_2 = {} & \sum_{k=0}^{m-2}(-1)^{m+k}A_{k+1}^{m-1}\cos^{k}\theta \\
& \times \Big[a_{2m-1-k}\int_0^\pi x^{l+2m-k}\sin^{n-2m-2}\theta\sin^{n-2m-3}\varphi d\varphi \\
& - a_{2m-2-k}\cos\theta\int_0^\pi x^{l+2m-1-k}\sin^{n-2m}\theta\sin^{n-2m-1}\theta\varphi d\varphi\Big].
\end{aligned}$$

Then, evidently

$$\begin{aligned}
B_1 = {} & \sum_{k=0}^{m-2}(-1)^{m+k+1}(A_k^{m-1}+A_{k+1}^{m-1})a_{2m-1-k}\cos^{k+1}\theta \\
& \times \int_0^\pi x^{l+2m-k-1}\sin^{n-2m-2}\theta\sin^{n-2m-3}\varphi d\varphi \\
& + (-1)^{m+1}A_1^{l-1}a_{2m-1}\cos\theta\int_0^\pi x^{l+2m-1}\sin^{n-2m-2}\theta\sin^{n-2m-3}\varphi d\varphi \\
& + A_{m-1}^{m-1}a_m\cos^m\theta\int_0^\pi x^{l+m}\sin^{n-2m-2}\theta\sin^{n-2m-3}\varphi d\varphi \\
= {} & \sum_{k=0}^{m-2}(-1)^{m+k+1}A_{k+1}^{m}\cos^{k+1}\theta \\
& \times \int_0^\pi x^{l+2m-k-1}\sin^{n-2m-2}\theta\sin^{n-2m-3}\varphi d\theta \\
& + (-1)^{m+1}A_1^{m}\cos\theta\int_0^\pi x^{l+2m-1}\sin^{n-2m-2}\theta\sin^{n-2m-3}\varphi d\varphi \\
& + A_m^m\cos^m\theta\int_0^\pi x^{l+m}\sin^{n-2m-2}\theta\sin^{n-2m-3}\varphi d\varphi.
\end{aligned}$$

We do the same for B_2, we have

$$\begin{aligned}
B_2 = {} & \sum_{k=0}^{m-2}(-1)^{m+k}A_{k+1}^{l}\cos^{k}\theta \\
& \times \int_0^\pi x^{l+2m-k}\sin^{n-2m-2}\theta\sin^{n-2m-3}\varphi d\varphi \\
& + (-1)^m A_1^m\int_0^\pi x^{l+2m}\sin^{n-2m-2}\theta\sin^{n-2m-3}\varphi d\varphi \\
& - A_m^m\cos^{m-1}\theta\int_0^\pi x^{l+m+1}\sin^{n-2m-2}\theta\sin^{n-2m-3}\varphi d\varphi.
\end{aligned}$$

Therefore, we obtain

$$\begin{aligned} J_l^n(\theta) = {} & \prod_{i=2}^{m+1}(n-2i)\sum_{k=0}^{m-1}(-1)^{m+k+1}A_{k+1}^m\cos^k\theta \\ & \times\Big[\cos\theta\int_0^\pi x^{l+2m-3-k}\sin^{n-2(m+1)}\theta\sin^{n-2(m+1)-1}\varphi d\varphi \\ & -\int_0^\pi x^{l+2(m+1)-2-k}\sin^{n-2(m+1)}\theta\sin^{n-2(m+1)-1}\varphi d\varphi\Big]. \end{aligned}$$

This finishes the proof of Lemma 6.

III. REFERENCES

---

[1] W. Magnus, F. Oberhettinger, R. Pal Son: *Formulas and Theorems for the Special Function of Mathematical Physics*, Springer-Verlag Berlin Heidelberg New York,1966.