



## An $L^p - L^q$ -Version of Morgan's Theorem for the Fourier Schrodinger Transform

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abstract

In this note, we establish an  $L^p - L^q$ -version of Morgan's theorem for the generalized Fourier transform associated to the class of the Schrodinger operators on the real line.

**Key words:** Fourier Transform, Fourier-Schrodinger transform, Morgan's Theorem

### I. INTRODUCTION

A famous theorem of G.W.Morgan<sup>1</sup> asserts that for  $\alpha > 2$ ,  $\beta = \frac{\alpha}{\alpha-1}$ ,  $a > 0$  and  $b > 0$ , measurable function  $f$  on  $\mathbb{R}$  satisfying the conditions

$$e^{a|x|^\alpha} f \in L^\infty(\mathbb{R}),$$

and

$$e^{b|y|^\beta} \widehat{f} \in L^\infty(\mathbb{R}),$$

should vanish  $f = 0$  if and only if

$$(a\alpha)^{1/\alpha} (b\beta)^{1/\beta} > (\sin(\frac{\pi}{2}(\beta-1)))^{1/\beta}.$$

Here  $\widehat{f}$  is the classical Fourier transform of  $f$  on  $\mathbb{R}$  and the spacial case  $\alpha = 2$  is just Hardy's theorem. In this note, we study an extension of this theorem by considering the symmetric differential operator  $(L, D_0)$  defined by

$$D_0 = \mathcal{D}(\mathbb{R}) \quad \text{and} \quad Lu(x) = -\frac{d^2u(x)}{dx^2} + \phi(x)u(x), \quad u \in \mathcal{D}(\mathbb{R}).$$

$\mathcal{D}(\mathbb{R})$  is the space of  $C^\infty$ - function on  $\mathbb{R}$  with compact support.  $\phi$  is a measurable function satisfying

$$\int_{-\infty}^{+\infty} (1 + |x|) |\phi(x)| dx < +\infty.$$

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The operator  $(L, D_0)$  has a unique self-adjoint extension  $(L, D_L)$  with

$$D_L = \left\{ f \in L^2(\mathbb{R}) : f, f' \text{ are absolutely continuous and } L(f) \in L^2(\mathbb{R}) \right\}.$$

On the other hand, for

$$\mu \in \mathbb{C}_+ = \{ \lambda \in \mathbb{C} : (\text{Im}(\lambda) > 0) \text{ or } (\text{Im}(\lambda) = 0 \text{ and } (\text{Re}(\lambda) \geq 0)) \},$$

the differential equation  $Lu = \mu^2 u$  possesses two linear independent solutions  $E_{\pm}(\cdot, \mu)$  satisfying<sup>1</sup>

$$\lim_{x \rightarrow \pm\infty} e^{\mp i\mu x} E_{\pm}(x, \mu) = 1.$$

$E_{\pm}(\cdot, \mu)$  are called generalized eigenfunction of the self-adjoint operator  $(L, D_L)$ . We also have the generalized Fourier transforms defined by,

$$\mathcal{F}_{\pm}(f)(\mu) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(x) E_{\pm}(x, \mp\mu) dx, \mu \in \mathbb{R}, f \in \mathcal{D}(\mathbb{R}). \tag{1.1}$$

Moreover, for all  $\mu \in \mathbb{C}_+$  and  $x \in \mathbb{R}$ , there exist constants  $C_{\pm}$  independent of  $\mu$ , such that<sup>2,3</sup>

$$|E_{\pm}(x, \mu)| \leq C_{\pm} e^{\mp \text{Im}(\mu)x}.$$

**Lemma 1** Suppose that  $\rho \in ]1, 2[$ ,  $q \in [1, +\infty[$ ,  $\sigma > 0$ , and  $B > \sigma \sin \frac{\pi}{2}(\rho - 1)$ . If  $g$  is an entire function on  $\mathbb{C}$  satisfying the conditions

$$|g(x + iy)| \leq \text{const } e^{\sigma|y|^{\rho}} \text{ for any } x, y \in \mathbb{R} \tag{1.2}$$

and  $e^{B|x|^{\rho}} g|_{\mathbb{R}} \in L^q(\mathbb{R})$ ; then  $g = 0$ .

## II. MORGAN'S THEOREM FOR THE FOURIER-SCHRODINGER TRANSFORM

In this section we study the Morgans theorem for the Fourier-Schrodinger transform on the real line<sup>3</sup>.

**Theorem 2** Let  $1 \leq p, q \leq \infty$ ,  $\alpha > 2$ ,  $\beta = \frac{\alpha}{\alpha-1}$ ,  $a > 0$  and  $b > 0$ . Suppose that  $f$  is a measurable function on  $\mathbb{R}$  satisfying the conditions

$$e^{a|x|^{\alpha}} f \in L^p(\mathbb{R}) \text{ and } e^{b|\mu|^{\beta}} \mathcal{F}_{\pm}(f) \in L^q(\mathbb{R}) \tag{2.1}$$

If

$$(a\alpha)^{1/\alpha} (b\beta)^{1/\beta} > (\sin(\frac{\pi}{2}(\beta - 1)))^{1/\beta} \tag{2.2}$$

then  $f$  is null almost everywhere.

**Proof.** Let  $f$  be a measurable function on  $\mathbb{R}$  such that

$$e^{a|x|^{\alpha}} f \in L^p(\mathbb{R}) \tag{2.3}$$

and

$$e^{b|\mu|^{\beta}} F_{\pm}(f) \in L^q(\mathbb{R}) \tag{2.4}$$

In order to prove that  $f = 0$ , we use the Lemma 3. Namely, we show that the Fourier-Schrodinger transform of  $f$  satisfies the conditions of Lemma 3. First, by using the condition (2-3), the function

$$F_{\pm}(f)(\mu) = \frac{1}{2\pi} \int_{\mathbb{R}} f(x) E_{\pm}(x, \mp\mu) dx,$$

is well defined, holomorphic on  $\mathbb{C}_+$ , and satisfies the condition

$$\begin{aligned} |F_{\pm}(f)(\mu)| &\leq \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} |f(x)| |E_{\pm}(x, \mp\mu)| dx \\ &\leq C_{\pm} \cdot \int_{\mathbb{R}} |f(x)| \cdot e^{\mp \text{Im}(\mu)x} dx \text{ for any } \mu \in \mathbb{C}_+, \end{aligned}$$

by Holder's inequality

$$\begin{aligned} &\leq C_{\pm} \cdot \left[ \int_{\mathbb{R}} (|f(x)| \cdot e^{a|x|^{\alpha}})^p dx \right]^{1/p} \cdot \left[ \int_{\mathbb{R}} (e^{-a|x|^{\alpha}} \cdot e^{\mp \text{Im}(\mu)x})^q dx \right]^{1/q} \\ &\leq C'_{\pm} \left[ \int_{\mathbb{R}} e^{\mp q \text{Im}(\mu)x - aq|x|^{\alpha}} dx \right]^{1/q}, \end{aligned}$$

where  $q$  is such that  $\frac{1}{p} + \frac{1}{q} = 1$ . Let

$$C \in I = ](b\beta)^{-1/\beta} (\sin \frac{\pi}{2}(\beta - 1))^{1/\beta}, (a\alpha)^{1/\alpha} [.$$

Applying the convex inequality

$$|ty| \leq (1/\alpha) |t|^{\alpha} + (1/\beta) |y|^{\beta},$$

to the positive numbers  $C|t|$  and  $|y|/C$ , we obtain

$$|ty| \leq \frac{C^{\alpha}}{\alpha} |t|^{\alpha} + \frac{1}{\beta C^{\beta}} |y|^{\beta},$$

thus

$$\int_{\mathbb{R}} e^{\mp q \text{Im}(\mu)x - aq|x|^{\alpha}} dx \leq e^{q \frac{|\text{Im}(\mu)|^{\beta}}{\beta C^{\beta}}} \int_{\mathbb{R}} e^{-q(a - \frac{C^{\alpha}}{\alpha})|x|^{\alpha}} dx.$$

Since  $C \in I$ , it follows that  $a > \frac{C^{\alpha}}{\alpha}$ , thus the integral

$$\int_{\mathbb{R}} e^{-q(a - \frac{C^{\alpha}}{\alpha})|x|^{\alpha}} dx,$$

is finite. Moreover,

$$|F_{\pm}(f)(\mu)| \leq \text{Const} \cdot e^{\frac{|\text{Im}(\mu)|^{\beta}}{\beta C^{\beta}}} \text{ for any } \mu \in \mathbb{C}_+, \tag{2.5}$$

condition (2.4) and inequality (2.5) imply that the function

$$g(z) = F_{\pm}(f)(\mu)$$

satisfies the assumptions (1.4) and (1.5) of Lemma 3 with

$$\rho = \beta, \sigma = \frac{1}{\beta C^{\beta}} \text{ and } B = b.$$

Next, applying the condition  $C \in I$ , implies the inequality

$$b > \frac{1}{\beta C^{\beta}} \sin \frac{\pi}{2}(\beta - 1),$$

which imply that  $\mathcal{F}_{\pm}(\{) = \iota$ . Hencefore, by the injectivity of the Fourier-Schrodinger transform, we have  $f = 0$ . Thus, the desired result is establish. ■

**III. REFERENCES**

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