THE 23 GREATEST MATHEMATICAL CHALLENGES AT THE THRESHOLD OF THE THIRD MILLENNIUM ON THE MODEL OF 1900’S HILBERT PROBLEMS

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In gratitude and homage to my master Jean DIEUDONNE

Abstract: We propose a concise, precise, fluent, historical, epistemological, motivated and original presentation of what we consider as "the 23 greatest mathematical challenges at the threshold of the third millennium", with the hope and the ambition that they will be received by the international mathematical community as a more faithful, more ambitious and more complete actualization of the 1900’s "Hilbert 23 Problems" than the 1998’s 18 "Smale Problems" and the 2000’s Clay Foundation 7 "Millennium Problems".

Keywords:

1. Introduction

According to its fecundity in the stimulation of mathematical researches in various branches of mathematics, from mathematical logic to differential equations passing through group theory, commutative algebra, number theory, and complex analysis, all along the 20th century, the 1900’s "Hilbert 23 Problems" are uncontestably the model for any ambitious list of "greatest mathematical problems", in particular for the 1998’s 18 "Smale Problems" and the 2000’s Clay Foundation 7 "Millennium Problems".

On the other hand, it is also uncontestable that Clay Foundation list of problems contains some problems, more precisely Birch and Swinnerton-Dyer and Yang-Mills problems, which are not representative of the intensive and top level research activities of the international mathematical community, while many more historical and more resisting problems of common greatest interest are missing in this list, like the problems on prime numbers mentioned in Hilbert list, or the fascinating Jacobian Conjecture, or the "in fashion" Langlands Program problems.

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Symetrically, it is also uncontestable that in Smale list of "mathematial problems for the next century", there are many thechnical problems which don’t have the statut of "greatest mathematical challenges".

So, taking in account our own multidisciplinary experiences of intensive, relentless and patient researches on various difficult mathematical challenges, in conformity with Nicolas Boileau’s principle formulated in his book "Poetic Art" according to which "every thing which is well conceived can be clairly stated, and the words to express it easily arrive", after fifteenth years of hesitations and maturation, we finally decide to propose to the international mathematical community a concise, precise, fluent, historical, epistemological, motived and original presentation of what we consider as "the 23 greatest mathematical challenges at the threshold of the third millennium", with the hope and the ambition to be received by this community as a more faithful, more ambitious and more complete actualization of the 1900's "Hilbert 23 Problems" than the 1998’s "Smale 18 Problems" and the 2000’s Clay Foundation 7 "Millennium Problems".

May it be so from now until and for long, in order to pay homage to the French architect Le Corbusier’s formula "tradition is the interupted chain of all innovations", and mainly to the intellectual tradition of mathematics, which is inherited, not from the European Ancien Greece as commonly believed, but the authentically African Ancient Egypt, according the unanimous testimony of all Ancient Greek intellectual leaders, in conformity with the African wisdom advising that "if you don’t know where you are going to, try to know where you coming from"!

2. List of problems

1) Twin Primes Conjecture: in consideration of the historical importance of the more than two millenniums old theorem of "Euclid Elements" (book 9, proposition 20) claiming that "there is an infinity of primes", one can naturally ask, since at least 1849, following A. de Polignac and D. Hilbert in his 8th problem from 1900, if "there is an infinity of twin primes", e.g. of couples of odd primes with distance two, as "Ishango Twin Primes" formed by 11 and 13 or by 17 and 19, the oldest written mentions of primes, hence of twin primes, which are inscribed on one of "Ishango bones" from the heart of Africa about 25 millenniums ago?

2) Goldbach Conjecture: in consideration of the simplicity and the antiquity of the question commented by Euler in 1742, in accordance with the breakthroughs of G. H. Hardy and J. E. Littlewood in 1923, I. M. Vinogradov in 1937, J.-M. Deshouillers, G. Effinger, H. te Riele, D. Zinivievi in 1997, and with the announcement by H. Helfgott in 2013, concerning the "Weak Goldbach Conjecture", one can naturally ask following Goldbach since 1742 and D. Hilbert in his 8th problem from 1900 if "any even natural greater than 2 is the sum of two, eventually equal, primes", as popularized by the 1992 Novel "Uncle Petros and Goldbach’s Conjecture" of Apostolos Doxiadis, and as illustrated by the famous "Ishango Twin primes", since 30 can be written as the sum of 11 and 19 or of 13 and 17?

3) Constructible prime regular polygons Conjecture: in consideration of the historical importance of the Circle
Squaring problem concerning the construction, only with a compass and a straightedge, of a square with the same area as any given circle, solved after more than two millenniums of researches by Lindemann in 1882, according to the proof by Euler in 1732 of the falseness of 1640’s Fermat conjecture concerning Fermat numbers, in order to fill the incredible lack of knowledge of human kind on the elementary geometry of the real plane, one can naturally try to refine both this answer of Euler and 1837’s Gauss-Wantzel theorem, completing 1800’s Gauss theorem, and claiming that a regular polygon with prime number of edges is constructible with a compass and a straightedge if and only if it is a prime Fermat number, by naturally asking if ”a regular polygon with an prime number of edges is constructible if and only if this number is one of the five first Fermat numbers 3, 5, 17, 257, 65537”? 

4) Riemann Hypothesis : in consideration of the historical importance of ”prime numbers theorem” independently proved in 1896 by Jacques Hadamard and Nicolas de La Vallée-Poussin, and knowing that the resolution by Pierre Deligne in 1973, thanks to his master and mentor Alexander Grothendieck published and unpublished works on the subject, of the ”Weil Conjectures” formulated by André Weil in 1949 as ”the analogue of Riemann Hypothesis for finite fields”, brought nothing for the resolution of Riemann Hypothesis, even more than fourthy years after Deligne’s breakthrough, one can naturally ask following Riemann since 1859 and D. Hilbert in his 8th problem from 1900 if the ”prime numbers theorem” could be refined by proving that ”the non trivial zeros of Zeta function are on the real line of the real vector space of the complex numbers defined by those with real part ”, or equivalently that ”there exists a constant such that the absolute value of the difference between the number of primes lower than any real number and its integral logarithm is not greater than the constant times the square root of this number and times its logarithm”? According to a testimony of Georges Polya in 1969, ”somebody allegedly asked Hilbert, ”if you revive, like Barbarossa, after five hundred years, what would you do?” ”I would ask”, said Hilbert, ”has somebody proved the Riemann Hypothesis?” ”. On the other hand, 8 years before this testimony of Polya and 18 years after Hilbert’s death, Richard Bellman wrote in his 1961 book on theta functions : ”Hilbert is reputed to have said that the first comment he would make after waking at the end of a thousand years sleep would be : is the Riemann Hypothesis established yet?”. In each one of these testimonies, Hilbert underlined by this answer both the importance he gave to the resolution of this conjecture and the time he estimed for human kind to overcome the difficulties of such resolution.

5) Poincaré Conjecture for smooth manifolds of dimension 4 : in consideration of the historical and epistimological importance for cosmology of the topological characterization of the smooth unit sphere of dimension four, with the hope that this characterization could be a breakthrough in our understanding of the famous ”space-time” and ”dark energy”, according to the American Defense Advanced Reasearch Projects Agency (DARPA), which is, after S. Smale, Michael Freedman and Perelman breakthroughs on the subject respectively in 1962 for dimension greater than 4, 1982 for topological case in dimension 4, and 2003 for dimension 3, the only unsolved cases up to now of Poincaré Conjectures for smooth, piecewise linear or topological manifolds of any dimension, according to the rectification by Witold Hurewics in 1935 of the 1900 formulation of these conjectures by Henri Poincaré at the very end of his paper, one can naturally ask, following the mathematical challenge 22 of the 2008 ”DARPA
Mathematical challenges”, if ”any smooth compact manifold of dimension four without boundary, which is connected and two simply connected, is diffeomorphic to the smooth unit sphere of dimension four”, knowing that such statement implies the smooth (resp. combinatorial) Schoenflies Conjecture in dimension 4, claiming that ”every smooth (resp. combinatorial) 3-sphere in the 4-sphere bounds a smooth (resp. combinatorial) standard 4-ball” ?

6) Jacobian Conjecture : in consideration of the historical and epistemological importance of the fascinating Jacobian conjecture formulated by O. Keller in 1939, celebrated by S. Smale in 1998 as one of the ”mathematical problems for the next century”, honored by the Chern Institute of Mathematics Conference in Tianjin on ”Affine Algebraic Geometry and the Jacobian Conjecture” on 21-25 July 2014, the deepest signification of which is the global inversion version for complex polynomial maps of the famous ”local inversion theorem”, illustrating by this way the very difficult problem in mathematical of ”passage from local to global”, and according to the equivalence of this conjecture with another famous conjecture, formulated by J. Dixmier in 1968, claiming surprisingly that ”any endomorphism of the complex algebra of differential operators with polynomial coefficients and with any number of variables is an automorphism”, as proved by Y. Tsuchimoto, A. Belov and M. Kontsevich in 2005, confirming by this way that ”Jacobian Conjecture” is a ”singular point” of the ”variety of mathematics” and an ”intersection point of many subvarieties of mathematics”, as complex analysis, complex analytic geometry, dynamical systems, partial differential equations, commutative algebra, non commutative algebra, algebraic geometry, non commutative algebraic geometry, algebraic symplectic geometry, model theory, quantum mechanics, and quantum field theory, one can naturally wonder if ”any locally injective polynomial map from a complex affine space to itself is bijective with a polynomial inverse” ? So, according to what G.H. Hardy wrote in his book ”A mathematician’s Apology”, ”a mathematical idea is significant if it can be connected, in a natural and illuminating way, with a large complex of other mathematical ideas”, it is uncontestable that the fascinating ”Jacobian Conjecture” is a ”significant mathematical idea”.

7) Desingularization Conjecture in positive characteristic : in consideration of the historical importance of the theory of desingularization of algebraic varieties, initiated by B. Riemann in 1862 with the proof of the desingularization of complex algebraic curves, and popularized by O. Zariski who proved in 1939 the desingularization by normalization of algebraic curves over algebraically closed fields of any characteristic, the desingularization of algebraic hypersurfaces of dimension two over algebraically closed fields of characteristic zero, before extending this last result in dimension three in 1944, before his thesis student Hironaka extended this last result to algebraic varieties of any dimension in characteristic zero in 1964, and before his other thesis student S. Abhyankar extended in 1966 Zariski’s result for any positive characteristic curves to hypersurfaces of dimension two of any positive characteristic and hypersurfaces of dimension three and characteristic greater than 5, and according to the increasing interest for applications of algebraic geometry over finite fields and fields of positive characteristic, specially in cryptography ans numerical security, one can naturally ask if ”any singular algebraic variety over a field of positive characteristic is the image of a non singular algebraic variety over this field by a proper morphism
inducing an isomorphism over the open set of the non-singular points of the singular variety”?

8) Sixteenth Hilbert Problem: in consideration of the historical importance of the dynamical systems defined by a polynomial field over the real plane, popularized by D. Hilbert in the 16th Problem of his 1900’s list of 23 problems, and in consideration of the refutation in 1981 of 1923’s claimed proof by Dulac of the finiteness of limit cycles, before a more reliable proof by J. Ecalle and Y. Ilyashenko in 1991-1992, one can naturally ask if “the number of limit cycles, i.e. periodic isolated periods of the dynamical system defined by a polynomial field of a given degree over the real plane, is bounded by a constant depending only on this degree”?

9) Navier-Stokes Conjecture: in consideration of the major epistemological importance of 1822’s Navier-Stokes equations in the modelisation of many and various physical phenomena as turbulence, weather, ocean and air currents, water flow in pipe, air flow around a wing or a car, air pollution, blood flow in human or animal bodies, magnetohydrodynamics, in consideration of the progresses in the numerical analysis of the solutions of this equation without enough quantitative and conceptual knowledge about them, in consideration of Leray’s breakthrough in his 1934 paper on the subject where he proved the existence and uniqueness of a local-time smooth solution of Navier-Stokes equations, and in consideration of the confirmed difficulty of problems of passage from local to global for smooth solutions of partial differential equations, one can naturally ask following S. Smale in his 1998’s list of ”mathematical problems for the next century” and Clay Foundation in its 2000’s list of ”Millennium Prize Problems”, if ”Navier-Stokes equations has any long term unique and smooth solution”?

10) P and NP Conjecture: in consideration of the status of this problem as the deepest one at the border of mathematics and computer science, concerning the measure of the ”complexity” or equivalently the ”speed of execution by a computer” of algorithms, posed by S. Cook in 1971, and popularized by 1998’s Smale list of ”mathematical problems for the next century”, and 2000’s Clay Foundation list of ”Millennium Prize Problems”, one can naturally ask if ”every problem whose solution can be verified by a computer in polynomial time can be solved by a computer in polynomial time”?

11) Zariski Cancellation problem: in consideration of the equivalent ”cancellation” form of the smooth Poincaré Conjecture in dimension n claiming that if the cartesian product of the real line with a smooth, compact and boundary free manifold of dimension n is diffeomorphic to the cartesian product of the real line with the standard smooth sphere of dimension n, then this manifold is diffeomorphic to this sphere, as if the affine lines in this first isomorphism could be ”cancelled”, and in consideration with the hidden unity of different branches of the three of mathematics, one could naturally ask following Zariski in his question, reported by a Nagata’s 1967 paper, about the ”uniqueness” of the ring of coefficients of a polynomial ring in one indeterminate over a commutative ring, if ”for any positive integer n, any smooth complex affine algebraic variety of dimension n whose cartesian product with complex affine line is isomorphic to the cartesian product of the complex affine space of dimension n with the complex affine line, is in fact isomorphic to the former affine space, as if the complex affine lines could be cancelled in the former isomorphism”?

12) Hilbert Inverse Galois Problem: in consideration of the historical importance of the Inverse Galois Theory,
initiated by D. Hilbert in 1892 by proving thanks to his famous "irreductibility theorem" that any symmetric group is the Galois group of an irreducible polynomial in one indeterminates with rational coefficients, and popularized by E. Noether since 1913 by proposing a method to generalize Hilbert cited result, by I. Shafarevitch by extending since 1954 Hilbert cited result to any resoluble finite group, by John Thompson since by 1984 proving Hilbert result for 25 of the 26 sporadic groups including the Monster, and by J.-P. Serre by his 1991 book "Topics in Galois Theory", one can naturally ask if "any finite group is the Galois group of an irreducible polynomial in one indeterminates with rational coefficients"?

13) Hodge Conjecture: In the wake of Poincaré Conjecture for smooth manifolds of any dimension, in order to prove a new striking illustration of the fecundity of Poincaré Algebraic topology, not only in differential geometry, but also in complex algebraic geometry, in particular in order to prove the fecundity of "the topological invariants of algebraic varieties", according to the title of the historical communication of W. Hodge at the 1950's International Congress of Mathematicians where he stated his celebrated conjecture, in order to solve deep and subtle problems of the theory of residus of multiple integrals on a projective complex variety, also initiated by Henri Poincaré before the breakthroughs of Emile Picard, Salomon Lefschetz, Jean Leray, Atiyah, Kodaira, Spencer, Hodge, and Griffiths on the subject, as explained by Jean Dieudonné in his historical introduction to Hodge conjecture, one can ask following W. Hodge if "the nullity of its topological invariant called its rotation number is sufficient for a topological cycle on a complex projective variety to be homologuous to a rational algebraic cycle on this variety"?

14) Langlands Reciprocity Conjecture: Defining a Dirichlet L-function as a function with is "analogous" to Riemann Zeta function, i.e. is meromorphic in the whole complex field and satisfies an "functional equation" similarly to one of the Zeta function, in consideration of the fascination exerted on mathematicians since Euler and Gauss by the Zeta function which is believed to know all the secrets of prime numbers, as suggested by the classical equivalent forms of "Primes Theorem" or "Rieman Hypothesis" in terms of primes and Zeta function, in consideration of the principle according to which "Dirichlet L-functions know every thing, we only have to make then speak", in consideration of "Artin Reciporcity Theorem", asserting since 1930 that the L-function associated to a complex linear representation of dimension one of an abelian Galois group of a number field is the Dirichlet L-function associated to a Hecke character, one can naturally ask following Langlands in his famous 1967 letter to André Weil if a faithful generalization of this last theorem is possible, more precisely if "the Artin L-function associated to any complex linear representation of finite dimension n of the Galois group of any number field is the automorphic Dirichlet L-function associated (by Langlands) to the n-general linear group over the field of rational numbers", knowing that the much more easy but highly non trivial analogous of this statement when this last field is replaced by a function field has been proved by Vladimir Drinfeld in 1978 when n=2 and by Laurent Lafforgue in 2002 for any n.

15) Inaccessible Cardinals and Category Theory Foundations Problem: saying following Alfred Tarski since 1930 that the cardinal of a set is inaccessible if this set is uncountable, and if his cardinal is greater than the cardinal of
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the set of all parts of any other set with lower cardinal, and greater than the cardinal of the union of any family of sets with lower cardinal which is indexed by a set of lower cardinal, knowing that the theory defined by Zermelo-Frankel and Choice axioms added to the statement "there is no inaccessible cardinal" is not in contradiction with the Zermelo-Frankel and Choice axioms, that according to the Second Gdel Incompletude Theorem it is impossible to prove the non contradiction of the theory defined by Zermelo-Frankel and Choice axioms, added to the statement "there is an inaccessible cardinal", that the existence of any "large cardinal" implies the existence of an inaccessible cardinal, that "Grothendieck Universes Axiom" is equivalent to the statement "any cardinal is lower than an inaccessible cardinal" under Zermelo-Frankel and Choice axioms, and that "Grothendieck Universes Axiom", added in 1963 by Alexander Grothendieck in the collective book SGA4 to the axioms of the original Category Theory initiated in 1945 by Eilenberg and Mac Lane, is useful for instance to prove that any locally small category has a Yoneda embedding, in order to avoid that researches in mathematical logic become a relentless pursuit of "large cardinal" and of fiction-science, good sense, intellectual honesty and mathematical rigor force mathematicians to naturally ask if "the existence of an inaccessible cardinal under the assumption of Zermelo-Frankel and Choice axioms could be denied by a contradiction to be proved", or equivalently if "the foundations of modern Category Theory could be shaked by a refutation to be proved of the inaccessible cardinals axiom under the assumption of Zermelo-Frankel and Choice axioms".

16) Jacobian Conjecture in positive characteristic : still in consideration of the increasing interest for applications of algebraic geometry over finite fields and fields of positive characteristic, specially in cryptography and numerical security, illustrating the new and irrefutable fact that such fields are more "real" for computers than the fields of "real" or "complex" numbers, in consideration of the equivalence of Jacobian, Dixmier and Poisson Conjecture in any characteristic proved by P.K. Adjamagbo and A. van den Essen in 2006, and by analogy with the Desingularization Conjecture in positive characteristic, one can naturally wonder following P. K. Adjamagbo since 1995 (see for instance Wikipedia on Jaacobian Conjecture) if "any endomorphism of an algebra of polynomials over a field of positive characteristic with a non zero constant jacobian and a geometric degree not divisible by the characteristic is an automorphism of this albegra", which is nothing more than the most natural extension of the basic determinantal characterisation of an automorphism of linear vector spaces of finite dimension over a field of positive characteristic, and which implies the Jacobian Conjecture in zero characteristic thanks to a theorem on the bound of the geometric degree of quasi-finite endomorphisms of an affine space and to a theorem from model theory on first order propositions ?

17) Polignac Conjecture : in consideration of the heuristical fecundity to embbed a problem to be solved into a good environment in conformity with the philosophy and the practise of Alexander Grothendieck, of the spectacular result of Zhang Yitang announced in 2013 and claming for the first time in the history that there exists at least one even integer n such that there are infinitely many couples of consecutives primes with difference n and that this number could be choosen lower than 70 millions, before the group of collaborators "Polymath8 project" proved with the help of computers in April 2014 that n could be choosen lower than 246, in other to strongly
generalize the "Twin Primes Conjecture", one could naturally ask following Alphonse de Polignac since 1849 if "for any even integer \( n \) there exists infinitely many ordered couples of consecutive primes with difference \( n \)?

18) Mersene Primes and perfect numbers conjecture: In consideration of the antiquity and historical interest of Mersenne Primes, i.e. a prime which is a power of two minus one, for the characterisation of even perfect numbers, i.e. even numbers which are the sum of all its lower divisors, as revealed by Euclide at the Proposition 36 of the book 9 of his "Elements", and in consideration the interest of Mersenne Primes in the research of greatest Primes, one can naturally ask if "there is an infinity of Mersenne Primes", or equivalently if "there is an infinity of even perfect numbers"?

19) Primes quadruplets conjecture: In consideration of the antiquity of the indubitable inscription of the quadruplet of primes \( (11, 13, 17, 19) \) on Ishango Bone in the heart of Africa from more than 25 millennium ago, in consideration of the remarkable mathematical property of this primes quadruplet as a kind of generator of all primes quadruplets other than the first one \( (5, 7, 11, 13) \), which can be obtained from the "Ishango’s primes quadruplet" by adding a good same multiple of 30 to all "Ishango primes", and according the finiteness of the sum of the inverse of all primes appearing in a primes quadruplet, one can naturally ask if "there is an infinity of primes quadruplets", which of course is a more general question than asking if "there is an infinity of twin primes"?

20) Euler Constant rationality conjecture: In consideration of the multiple occurrences in many formulas in mathematics and physics, concerning for instance Gamma or Zeta functions or condensed matter physics, of "Euler Constant", nowadays denoted "Gamma" and defined by Euler in 1735 as the limit of \( 1 + \frac{1}{n} \log(n) \), when \( n \) tends to infinity, and in consideration of our unknowing concerning the arithmetic nature of this useful "universal constant" since almost three centuries, one can naturally ask following David Hibert and G. H. Hardy if "Euler Constant is irrational"? According to J. Havil in his 2003’s book "Gamma: Exploring Euler’s Constant", Hilbert mentioned the irrationality of this constant as an unsolved problem that seems "unapproachable" and in front of which mathematicians are helpless, while Hardy is alleged to have offered to give his Savilian Chair at Oxford to anyone who proved that this constant is irrational. In a recent 2013’s ArXiv preprint "Euler’s Constant: Euler’s work an modern developpements", Jeffrey C. Lagarias reported the recent advances on the rationality and the transcendence problems of Euler Constant.

21) Generalized Riemann Hypothesis for Dirichlet L-functions: in consideration of the historical importance Riemann and David Hilbert gave since 1859 to the resolution of Riemann Hypothesis, of the increasing importance of Zeta like functions in the actual mathematical researches because of Langlands Program since 1967, of the heuristical fecundity to embed a problem to be solved into a good environment in conformity with the philosophy and the practise of Alexander Grothendieck, defining a Dirichlet character with period a positive integer \( p \) as a periodic homomorphism of period \( p \) from the additive monod of positive integers to the multiplicative monod of complex numbers which vanishes at any positive integer not prime with \( p \), the Dirichlet series at a complex number \( z \) with real part greater than 1 associated to a Dirichlet character as the series obtained from the Zeta
series at \( z \) by replacing the numerator of the fraction with denominator \( n \) power \( z \) by the value of the character at \( n \) for each positive integer \( n \), the Dirichlet function as the meromorphic function in the whole complex field with an eventual pole at 1, in consideration of the role played by Dirichlet series in the proof of Dirichlet’s theorem on arithmetic progressions in 1837, and in order to faithfully generalize the Riemann Hypothesis, one could naturally ask following Adolf Piltz since 1884 if "the real part of a zero of the Dirichlet function associated to any Dirichlet character with real part which which is between 0 and 1 is actually”? This conjecture implies a sharp refinement of both Dirichlet’s theorem on arithmetic progression and of Riemann Hypothesis by claiming that "for any coprime positive integers \( a \) and \( d \), there exists a constant such that for the absolute value of the number of primes in the arithmetic progression with first term \( a \) and reason \( d \) which is lower than any real number minus the integral logarithm of \( x \) divided by the value of Euler function at \( d \) is not greater than the constant times the square root of this last number and times its logarithm”.

22) Langlands Fonctoriality Conjecture : knowing that, for each reductive group \( G \) over a number field \( F \), Langlands defined the "automorphic representations of \( G \) over \( F \)” generalizing the complex "automorphic forms” on the upper half real plane, constructed a complex reductive Lie group called the "Lie-group of \( G \)”, the semi-direct product of which with the Galois group of the algebraic cloure of \( F \) is called the "L-group of \( G \)”, defined in a natural way a complex representation of dimension \( n \) of the L-group of \( G \) as an homomorphism from this group to the \( n \)-th general linear group over \( C \) such that its restriction to the Lie-group of \( G \) is analytic, and finally defined a complex "L-function” on some open sets of \( C \) associated to each automorphic representation of \( G \) and each complex representation of finite dimension of the L-group of \( G \), defined on the underlying Galois groups, any automorphic representation \( p \) of \( G \) over \( F \), and any complex representation \( r' \) of finite dimension of the L-group of \( G' \), there exists an automorphic representation \( p' \) of \( G \) over \( F \) such that the L-function associated to \( p \) and \( r' \) composed with \( h \) is equal to the L-function associated to \( p' \) and \( r' \). The fecundity of this conjecture is justified by the fact that it implies not only "Langlands Reciprocity Theoem”, but also another conjecture of Langlands claiming that "for any reductive group \( G \) over a number field \( F \), the L-function associated to an automorphic representation of \( G \) over \( F \) and a complex representation representation of finite dimension of the L-group of \( G \) is a Dirichlet L-function”. The proof by Ngô Bao Châu in 2008 of an auxiliary but difficult statement related to Langlands Fonctoriality Conjecture, called ”fundamental lemma” and which was conjectured by Langlands in 1983, is considered as a breakthrough in the direction of the proof of this ultimate conjecture of "Langlands Program”.

23) Set Theory and Mathematics Foundations Problem : In consideration of the 1931 Second Gdel Incompleitude Theorem, which, together with the impossibility to rigourously define what is called a set, justifies the provocative but irrefutable assertion of Bertrand Russell in his 1918 book ”Mysticism and Logic” that ”mathematics may be defined as the subject in which we never know what we are talking about, nor whether what we are saying is
true", by ruining any hope to prove that Zermelo-Frankel Set Theory, initiated by Zermelo in 1908 and adapted by Frankel in 1925, has a model, i.e. is non contradictory, under the only assumptions of Zermelo-Frankel axioms, in order to avoid to transform all mathematicians into blind or fanatic "believers" in the "religion" of some Set Theory, intellectual honesty and mathematical rigor force mathematicians, first to consider and to explicitely present the non-contradiction of Zermelo-Frankel Set Theory as a simple and working "Fundamental Hypothesis of Set Theory and of the whole modern mathematics", contrary to usual presentations of Set Theory, then to naturally ask, following Rolland Frasse in his paper in the 1982 collective book "penser les mathématiques", if the "Fundamental Hypothesis of Set Theory and of the whole modern mathematics could be in the balance again because of an experience to be discovered", or in equivalent terms if "the foundations of Zermelo-Frankel Set theory and of the whole modern mathematics could be shaked by a contradiction to be proved", which keep over the head of Set Theory and the whole mathematics a permanent "Sword of Damocles".